The Weinstein argument and the Darboux-Weinstein lemma

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Abstract

The aim of this project is to provide an in depth look into the statement and proof of [1, Prop A.1], which uses ideas from the paper [5] by Weinstein. We will also look at a proof of a generalisation of the well-known Darboux-Weinstein lemma, in particular we are interested in the Moser trick involved, which is used in many proofs. Most terms and theorems used will be defined explicitly, both for the convenience of the reader and to solidify my own understanding.

1 The Weinstein argument

Proposition 1 ([1, Prop A.1]). *Let X be a smooth manifold. Let Δ be a pseudo-differential operator defined in some cone $C \subset T^*X$. We denote by p the principal symbol of Δ, and we assume that the sub-principal symbol of Δ vanishes. Let $\chi : C \to C' \subset T^*Y$ be a canonical transformation, where Y is another smooth manifold. Then, there exists a microlocally unitary Fourier Integral Operator U_χ, associated with χ, such that $U_\chi \Delta U_\chi^* = B$, where B is a pseudo-differential operator in C' whose principal symbol is $p \circ \chi^{-1}$ (general Egorov theorem) and whose sub-principal symbol vanishes.*

First let us define some of the terms used above. Many of the definitions, but not all, comes from [4].

Definition (Differential operator). Let $A : \mathcal{F}_1 \to \mathcal{F}_2$ be a map between function spaces, and $f = A(u) \in \mathcal{F}_2$ is the image of $u \in \mathcal{F}_1$. A *differential operator* is represented as a linear combination, finitely generated by u and its derivatives containing higher degrees such as

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

where α is a multi-index, $a_\alpha(x)$ are functions on some open domain in n-dimensional space.

Definition (Pseudo-differential operator)**.** A *pseudo-differential operator* $P(x, D)$ on $\mathbb{R}^n$ is an operator that satisfies

$$P(x,D)u(x) = \frac{1}{(a\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} P(x,\xi)\hat{u}(\xi)d\xi$$

where $\hat{u}(\xi)$ is the Fourier transform of u.

Let M be a smooth compact manifold of dimension d equipped with a smooth non-vanishing density μ.

Definition (The spaces $S^m(\Omega, \mathbb{R}^n)$ and $\Psi^m(M)$)**.** The space $S^m(\Omega, \mathbb{R}^n)$ of *symbols of order* m, where $\Omega \subset \mathbb{R}^p$ and $\Omega \subset \mathrm{cl}(\mathrm{int}(\Omega))$, consists of the functions $a \in \mathcal{C}^\infty(\mathrm{int}(\Omega) \times \mathbb{R}^n)$ satisfying all the estimates

$$\left| D_z^\alpha D_\xi^\beta a(z,\xi) \right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \text{ on } \mathrm{int}(\Omega) \times \mathbb{R}^n \qquad \forall \alpha \in \mathbb{N}_0^p, \beta \in \mathbb{N}_0^n.$$

The space $\mathcal{S}^m(M)$ are smooth homogeneous functions in $S^m(\Omega, \mathbb{R}^n)$ defined on the cone $\Omega := T^*M\backslash\{0\}$

The space $\Psi^m(M)$ of *pseudo-differential operators of order* m is the space of all pseudo-differential operator where the total symbol is in $S^m(\Omega, \mathbb{R}^n)$. We set

$$\Psi^{-\infty}(M) := \bigcup_{m\in\mathbb{R}} \Psi^m(M).$$

There is a graded algebra $\Psi(M)$ of classical pseudo-differential operators on M via $\Psi_{-\infty}(M) \cdots \subset \Psi_m(M) \subset \Psi_{m+1}(M) \subset \cdots$, where m is called the order. There is also the notion of principal symbol σ_p and of sub-principal symbol σ_{sub} where there is a bijective map

$$(\sigma_p, \sigma_{\mathrm{sub}}) : \Psi^m(M)/\Psi^{m-2}(M) \to \mathcal{S}^m(M) \oplus \mathcal{S}^{m-1}(M).$$

Definition (Total symbol, principal symbol and sub-principal symbol)**.** The *total symbol* of a differential operator $P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ is the polynomial

$$P(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha.$$

The *principal symbol* is the highest degree component of the total symbol

$$\sigma_p(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha.$$

The *sub-principal symbol* is the principal symbol of the degree (m-1) operator

$$Q := \frac{1}{2}(P - (-1^m)P^t)$$

where

$$P^t g = \sum (-1)^{|\alpha|} D^\alpha (a_\alpha g)$$

which has the formula

$$\sigma_{\text{sub}}(P)(x,\xi) = \sigma_Q(x,\xi) = p_{m-1}(x,\xi) + \frac{i}{2}\sum_i \frac{\delta^2}{\delta x_i \delta \xi_i} p_m(x,\xi)$$

where p_k is the k-th order terms of the total symbol of P.

Definition (Quantization)**.** A *quantization* is a continuous linear mapping

$$\mathrm{Op} : \mathcal{S}^0(M) \to \Psi^0(M)$$

satisfying $\sigma_p(\mathrm{Op}(a)) = a$.

Definition (Wave-front set)**.** To each distribution T on M is associated its wave-front set $WF(T)$, which is a closed subcone of $T^*M\backslash\{0\}$, whose projection onto M is the singular support of T. More precisely, in local coordinates, we have $(q,p) \notin WF(T)$ iff there exists $\chi \in C_0^\infty(M)$ with $\chi(q) \neq 0$ such that the Fourier transform of χT is rapidly decaying in some conical neighbourhood of p. For every operator $A : C^\infty(M) \to \mathcal{D}'(N)$, of Schwartz kernel $k_A \in \mathcal{D}'(M \times N)$, we define

$$WF'(A) = \{(q,p;q'-p') \in T^*M \times T^*N | (q,q',p,p') \in WF(k_A)\}.$$

Let $\chi : V \to W$ be a symplectic diffeomorphism from an open cone $V \subset T^*M$ to an open cone $W \subset T^*N$, where M and N are manifolds with the same dimension, respectively endowed with smooth non-vanishing measures μ and ν. We can associate a family of linear operators $U : L^2(M,\mu) \to L^2(N,\nu)$, called the quantizations of χ, with the following properties

- *Microlocally unitary:* $\{(z,\chi(z))|z \in V\} \cup WF'(U^*U - \mathrm{id}) = \emptyset$.
- *Egorov theorem:* If $A \in \Psi^m(N)$, then $B = U^*AU \in \Psi^m(M)$ and the principal symbols satisfy on V the relationship $\sigma_p(B) \circ \chi = \sigma_p(A)$.
- If $\sigma_{\text{sub}}(A) = 0$ then the same holds for B.

Definition (Fourier integral operator)**.** A *Fourier integral operator* is a linear operator $U : L^2 \to L^2$ which satisfy

$$(Uf)(x) = \int_{\mathbb{R}^n} e^{2\pi i \Phi(x,\xi)} a(x,\xi) \hat{f}(\xi)\, d\xi$$

where $a(x,\xi)$ is a standard symbol which is compactly supported in x and Φ is real valued and homogenous of degree 1 in ξ. We also need $\det\left(\frac{\partial^2\Phi}{\partial x_i \partial \xi_j}\right) \neq 0$ on the support of a. Note that this is a quantization.

Now we will proceed with the proof of the main proposition, [1, Prop A.1]. The main step of the proof uses the argument of Weinstein given in [5], which also uses a theorem from [2]. Other details of the proof are referred from the book [3].

Proof of Proposition 1. We will choose the Fourier Integral Operator U_χ associated with the canonical transformation χ such that its principal symbol is constant of modulus 1 and U_χ is microlocally unitary, i.e. $U_\chi^* U_\chi = \mathrm{id}$ in the cone C. To do this, first we choose U_0 such that its principal symbol is constant of modulus 1, then we have $U_0^* U_0 = \mathrm{id} + A$ where A is a self-adjoint pseudo-differential operator in $\Psi^{-1}(C)$. Let $D = (\mathrm{id}+A)^{-1/2}$ in C, which is self-adjoint since A is self-adjoint. We can now set $U_\chi = U_0 D$, so that

$$\begin{aligned} U_\chi^* U_\chi &= (U_0 D)^* U_0 D \\ &= D^* U_0^* U_0 D \\ &= D U_0^* U_0 D \\ &= (\mathrm{id} + A)^{-1/2}(\mathrm{id} + A)(\mathrm{id} + A)^{-1/2} \\ &= \mathrm{id} \end{aligned}$$

in C as promised.

Denote the Schwartz kernel of U_χ by $K(x, y)$, i.e. the unique distribution K satisfying

$$\langle K, u \otimes v\rangle = \langle U_\chi v, u\rangle$$

which is a Lagrangian distribution associated with a submanifold of $C \times C'$ which is the graph of χ. Define $B := U_\chi \Delta U_\chi^*$, noting that we have $\sigma_p(B) = \sigma_p(\Delta) \circ \chi^{-1}$ by Egorov theorem, since $U_\chi^{**} = U_\chi$. Then the relation

$$BU_\chi - U_\chi \Delta = U_\chi \Delta U_\chi^* U_\chi - U_\chi \Delta = U_\chi \Delta - U_\chi \Delta \sim 0$$

is written as

$$(\mathrm{id}_X \otimes B_y - \Delta_x \otimes \mathrm{id}_Y)K \sim 0.$$

Now the following steps are due to Weinstein in the proof of [5, Theorem 4.1]. We assume that the principal symbol of U_χ is a constant of modulus 1, i.e.

$$\sigma_p(U_\chi) = a, \quad |a| = 1.$$

Let H be the Hamilton field of $\sigma_p(\mathrm{id}_X \otimes B_y)$ lifted to a function on $(T^*X\backslash 0) \times (T^*Y\backslash 0)$, so H is tangential to C, and $\mathcal{L}_H$ is the corresponding Lie derivative. Define $H', \mathcal{L}_{H'}$ similarly for $\sigma_p(\Delta_x \otimes \mathrm{id}_Y)$. Then we have, by [2, Theorem 5.3.1] which gives $(*)$, and the fact that $\sigma_p(K) = \sigma_p(U_\chi)$,

$$\begin{aligned} \sigma_p(0) &= \sigma_p\Big((\mathrm{id}_X \otimes B_y - \Delta_x \otimes \mathrm{id}_Y)K\Big) \\ &= \sigma_p\Big((\mathrm{id}_X \otimes B_y)K\Big) - \sigma_p\Big((\Delta_x \otimes \mathrm{id}_Y)K\Big) \\ &= \mathcal{L}_H \sigma_p(K) + \sigma_{\mathrm{sub}}(\mathrm{id}_X \otimes B_y)\sigma_p(K) \qquad (*) \\ &\qquad - \mathcal{L}_{H'}\sigma_p(K) - \sigma_{\mathrm{sub}}(\Delta_x \otimes \mathrm{id}_Y)\sigma_p(K) \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\mathcal{L}_H \sigma_p(U_\chi)}_{=0} + \sigma_{\text{sub}}(\text{id}_X \otimes B_y)\sigma_p(U_\chi) \\
&\qquad - \underbrace{\mathcal{L}_{H'}\sigma_p(U_\chi)}_{=0} - \underbrace{\sigma_{\text{sub}}(\Delta_x \otimes \text{id}_Y)}_{=0}\sigma_p(U_\chi) \\
&= \sigma_{\text{sub}}(\text{id}_X \otimes B_y)\underbrace{\sigma_p(U_\chi)}_{\neq 0} \\
0 &= \sigma_{\text{sub}}(B)
\end{aligned}
$$

where $\mathcal{L}_H\sigma_p(U_\chi), \mathcal{L}_{H'}\sigma_p(U_\chi)$ vanishes due to $\sigma_p(U_\chi)$ being constant and $\sigma_{\text{sub}}(\Delta_x \otimes \text{id}_Y)$ vanishes due to $\sigma_{\text{sub}}(\Delta) = 0$. This implies that the sub-principal symbol of B vanishes, as required. □

2 The Darboux-Weinstein lemma

We will now present a proof of a generalisation of the Darboux-Weinstein lemma, found in [6].

Lemma 2. *Let N be a manifold endowed with two symplectic forms ω_1, ω_2, and let P be a compact submanifold of N along which $\omega_1 = \omega_2 + O_P(k)$, for some $k \in \mathbb{N}^* \cup \{+\infty\}$. Then there exists open neighbourhoods U and V of P in N and a diffeomorphism $f : U \to V$ such that $f = id_N + O_P(k+1)$ and $f^*\omega_2 = \omega_1$.*

Once again we will define the terms used above.

Definition (Closed form)**.** A differential form ω is **closed** if $d\omega = 0$.

Definition (Exact form)**.** A differential ω is **exact** if there exists η such that $\omega = d\eta$.

In other words, a closed form is in the kernel of d, and an exact form is in the image of d.

Definition (Nondegenerate form)**.** A differential form ω on a manifold N is **nondegenerate** if for every $p \in N$, $\omega_p(x, y) = 0$ for all $y \in T_pN \backslash \{0\}$ implies $x = 0 \in T_pN$.

Definition (Symplectic form)**.** A **symplectic form** is a closed nondegenerate differential 2-form.

Definition. We define $g \in O_P(k)$ if the function g vanish on P to the order k.

We will now present the proof.

Proof. Define the 2-form $\omega(t) = \omega_1 + t(\omega_2 - \omega_1)$ for $t \in [0, 1]$, which is closed since both ω_1 and ω_2 are closed, so that $\omega(0) = \omega_1$ and $\omega(1) = \omega_2$. Let U be a neighbourhood of P so that $\omega(t)$ is nondegenerate for all t, which exists since ω_1 and ω_2 agree along P. By the relative Poincaré lemma, which states that all closed differential k-forms are locally exact, since P is compact and $\omega_1 - \omega_2$ is closed, there must exists a 1-form η such that $\omega_1 - \omega_2 = d\eta$, with $\eta_x = 0$ for every $x \in P$. We can actually choose $\eta = O_P(k+1)$, since $\omega_1 - \omega_2 = O_P(k)$. The construction is given explicitly by $\eta = Q(\omega_1 - \omega_2)$, where $Q\omega = \int_0^1 F(t)^* \iota_{Y(t)} \omega \, dt$. Here $Y(t)$, at the point $y = F(t,x)$, is the vector tangent to the curve $F(s,x)$ at $s = t$, and $(F(t))_{0 \leq t \leq 1}$ is a smooth homotopy from the local projection onto P to the identity, fixing P.

Now we can construct the diffeomorphism f using the Moser trick. The time-dependent vector field $X(t)$ defined for every t by $\iota_{X(t)}\omega(t) = \eta$ generates the time-dependent flow $f(t)$, satisfying $\dot{f}(t) = X(t) \circ f(t), f(0) = \mathrm{id}_N$. This gives us, noting that ω is closed, i.e. $d\omega = 0$,

$$\begin{aligned}
\frac{\partial}{\partial t} f(t)^* \omega(t) &= f(t)^* \mathcal{L}_{X(t)} \omega(t) + f(t)^* \dot{\omega}(t) \\
&= f(t)^* \mathcal{L}_{X(t)} \omega(t) + f(t)^* (\omega_2 - \omega_1) \\
&= f(t)^* \iota_{X(t)} d\omega(t) + f(t)^* d(\iota_{X(t)} \omega(t)) - f(t)^* d\eta \\
&= f(t)^* d(\underbrace{\iota_{X(t)} \omega(t) - \eta}_{=0}) = 0
\end{aligned}$$

which means $f(t)^*\omega(t)$ is constant with respect to t, so we get

$$\begin{aligned}
f(0)^* \omega(0) &= f(1)^* \omega(1) \\
\mathrm{id}_N \omega_1 &= f(1)^* \omega_2 \\
\omega_1 &= f(1)^* \omega_2.
\end{aligned}$$

We set $f = f(1)$ to give us our diffeomorphism satisfying $f = \mathrm{id}_N + O_P(k+1)$ and $f^*\omega_2 = \omega_1$. □

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