Weyl Formula via Orthogonality

Xiangjia Kong

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Let V_λ be an irreducible representation of $\mathfrak{g}$ with the highest weight $\lambda = \sum_{i=1}^{k} a_i\omega_i$, where ω_i are the fundamental weights and $a_i \in \mathbb{Z}_+$. Then the character of V_λ is

$$\chi_\lambda(H) = \mathrm{Trace}(e^H) = \sum_\mu n_\mu e^{\mu(H)}$$

where the sum is over all weights μ of V_λ and n_μ is the dimension of the "eigenspace" $V_\mu = \{v \in V_\lambda | H(v) = \mu v\}$, known as the multiplicity of the weight μ. Now restrict $\chi_\lambda(H)$ to $H_c \in \oplus_{i=1}^{n} \iota\mathbb{R}\omega_i$, then e^{H_c} will be from $G_c = \exp(\mathfrak{g}_c)$ for the compact form $\mathfrak{g}_c$ of $\mathfrak{g}$. Here $\exp : \mathfrak{g} \to G$ is the map defined by $\exp(H) = \gamma_H(1)$ where γ_H is the unique one parameter subgroup on G with $\gamma_H(0) = \mathrm{id}_G$ and $\gamma'_H(0) = H$. Note that G_c is a compact group.

We will now show

$$\int_{G_c} \overline{\chi_V(g)}\chi_W(g)dg = 0 \quad \text{if } V \neq W$$

where dg is the Haar measure on $G_c \ni g$, and χ_V, χ_W are irreducible characters of G_c with underlying spaces V, W respectively. We have

$$\int_{G_c} \overline{\chi_V(g)}\chi_W(g)dg = \int_{G_c} \chi_{V^*\otimes W}(g)dg$$

since $\overline{\chi_V(g)} = \chi_{V^*}(g)$ and $\chi_V\chi_W = \chi_{V\otimes W}$. There is a natural isomorphism $V^* \otimes W \cong \mathrm{Hom}(V, W)$ via the functions

$$V^* \otimes W \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{Hom}(V, W)$$

defined by

$$f(\phi \otimes w)(v) = \phi(v)w$$
$$g(u) = \sum_i e_i^* \otimes u(e_i)$$

where $\{e_i\}$ is a basis of V. One can easily check f and g are inverse to each other. The action of g on this space is by conjugation, which means a G-invariant element is just a G-homomorphism $V \to W$. We can see this by looking at the diagram below for $f \in \mathrm{Hom}(V, W)$,

$$\begin{array}{ccc} V & \xrightarrow{g} & V \\ f\downarrow & & \downarrow g\circ f\circ g^{-1} \\ W & \xrightarrow{g} & W \end{array}$$

where if $g\circ f\circ g^{-1} = f$ for all $g \in G$ then the above diagram commutes. Therefore we can decompose $\mathrm{Hom}(V, W) = \mathrm{Hom}_G(V, W) \oplus U$ for some G-space U, where U has no G-invariant element. So in the decomposition of $\mathrm{Hom}(V, W)$ into irreducibles, we know there are exactly $\dim \mathrm{Hom}_G(V, W)$ copies of the trivial representation. But

$$\dim \mathrm{Hom}_G(V, W) = 0$$

by Schur's lemma since $V \ncong W$, so we can conclude $\mathrm{Hom}(V, W)$ is an irreducible representation. So it suffices to show that for a non-trivial irreducible character χ, we have

$$\int_{G_c} \chi(g) dg = 0.$$

Now if ρ affords χ, then any element in the image of $\int_{G_c} \rho(g)dg$ is fixed by G, i.e. the image is a G-subspace. Irreducibility then implies it is trivial. So $\int_{G_c} \rho(g)dg = 0$ and we are done.