Indefinite orthogonal groups

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Theorem 1. *We have an isomorphism*

$$\pi_1(SO^+(p,q)) \cong \pi_1(SO(p)) \times \pi_1(SO(q))$$

induced by the map

$$SO(p) \times SO(q) \to SO^+(p,q).$$

Proof. Let $n = p + q$ and

$$X^{\pm}_{p,q} := \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = \pm 1\}$$

then via the Serre fibration we get the following short exact sequence

$$0 \to S^{p-1} \hookrightarrow X^+_{p,q} \xrightarrow{\pi} \mathbb{R}^q \to 0$$

where π^{-1} is given by

$$\pi^{-1}(x) = (\sqrt{1 + x_{p+1}^2 + \cdots + x_{p+q}^2}, 0, \ldots, 0, x_{p+1}, \ldots, x_n).$$

From the above we get the long exact sequence on homotopy groups

$$0 \to \pi_2(S^{p-1}) \to \pi_2(X^+_{p,q}) \to \pi_2(\mathbb{R}^q) = 0 \to \pi_1(S^{p-1}) \to \pi_1(X^+_{p,q}) \to 0$$

and

$$\pi_0(X^+_{p,q}) = \pi_0(S^{p-1}) = \begin{cases} 2 \text{ points}, & p = 1 \\ 1 \text{ point}, & p \neq 1. \end{cases}$$

Now using the fact that

$$\pi_1(S^{1+m}) = 0, \quad m \geq 0$$
$$\pi_2(S^n) = 0, \text{ unless } \pi_2(S^2) = \mathbb{Z}$$

the long exact sequence becomes, for $p \neq 2, 3$,

$$0 \to 0 \to \pi_2(X^+_{p,q}) \to 0 \to 0 \to \pi_1(X^+_{p,q}) \to 0$$

from the exactness of the sequence we get

$$\pi_2(X^+_{p,q}) = \pi_1(X^+_{p,q}) = 0, \quad p \neq 2, 3.$$

In the case of $p=2$ the long exact sequence becomes

$$0 \to 0 \to \pi_2(X^+_{2,q}) \to 0 \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \pi_1(X^+_{2,q}) \xrightarrow{h} 0.$$

Since $\ker(g) = \mathrm{im}(f) = 0$ we have that g is an injection, so

$$\pi_1(X^+_{2,q}) = \ker(h) = \mathrm{im}(g) = \mathbb{Z}$$

and

$$\pi_2(X^+_{2,q}) = 0.$$

In the case of $p=3$ the long exact sequence becomes

$$0 \to \mathbb{Z} \to \pi_2(X^+_{p,q}) \to 0 \to 0 \to \pi_1(X^+_{p,q}) \to 0,$$

by the same logic as above we get

$$\pi_2(X^+_{3,q}) = \mathbb{Z}, \qquad \pi_1(X^+_{3,q}) = 0.$$

Putting everything together we have

$$\pi_1(X^+_{p,q}) = \begin{cases} 0 & \text{if } p \neq 2 \\ \mathbb{Z} & \text{if } p = 2 \end{cases}, \qquad \pi_2(X^+_{p,q}) = \begin{cases} 0 & \text{if } p \neq 3 \\ \mathbb{Z} & \text{if } p = 3. \end{cases}$$

Lemma 2. *The following are fibrations*

$$0 \to SO(p-1,q) \hookrightarrow SO(p,q) \to X^+_{p,q} \to 0$$
$$0 \to SO(p,q-1) \hookrightarrow SO(p,q) \to X^-_{p,q} = X^+_{q,p} \to 0.$$

Proof. Elements of $SO(p,q)$ are of the form

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

where B, C, D, E are $p \times p, q \times p, p \times q, q \times q$ matrices respectively. So we have, from the definition of $SO(p,q)$,

$$A^{-1} = I_{p,q} A^\top I_{p,q} = \begin{pmatrix} B^\top & -D^\top \\ -C^\top & E^\top \end{pmatrix}.$$

Write the first column of A as

$$A_1 = \begin{pmatrix} b_1 \\ \vdots \\ b_p \\ d_1 \\ \vdots \\ d_q \end{pmatrix}$$

and the first row of A^{-1} as

$$(A^{-1})_1 = \begin{pmatrix} b_1 & \dots & b_p & -d_1 & \dots & -d_q \end{pmatrix}$$

then the entry in the first row and first column of $A^{-1}A = I$ is

$$1 = (I)_{11} = (A^{-1}A)_{11} = (A^{-1})_1 A_1 = b_1^2 + \cdots + b_p^2 - d_1^2 - \cdots - d_q^2$$

which implies that $A_1 \in X^+_{p,q}$ as a vector in $\mathbb{R}^n$.

Now the inclusion map $i := SO(p-1,q) \hookrightarrow SO(p,q)$ can be given explicitly via considering any element $M \in SO(p-1,q)$ as an element

$$i(M) = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \in SO(p,q).$$

So given any $A \in SO(p,q)$, $M \in SO(p-1,q)$, we can see that $A \cdot i(M)$ has the same first column as A, which means the coset

$$A \cdot SO(p-1,q) \in SO(p,q)/SO(p-1,q)$$

can be represented by $[A_1]$, i.e. the first column of A. Indeed any two element in the same coset will have the same first column. So we have

$$SO(p,q)/SO(p-1,q) \cong X^+_{p,q}$$

from by the map

$$f : SO(p,q) \to X^+_{p,q}$$

given by mapping each $A \in SO(p,q)$ to the corresponding first column $A_1 \in X^+_{p,q}$. The kernel of f is exactly $SO(p-1,q) \hookrightarrow SO(p,q)$, and given an element $x \in X^+_{p,q}$ we can find a matrix $A \in SO(p,q)$ with x as the first column by using the Gram-Schmidt process. So f is surjective and the above follows from the isomorphism theorem. This gives exactly the first short exact sequence

$$0 \to SO(p-1,q) \hookrightarrow SO(p,q) \to X^+_{p,q} \to 0.$$

The second short exact sequence can be shown by the same argument but embedding $M \in SO(p,q-1)$ via

$$\begin{pmatrix} M & 0 \\ 0 & -1 \end{pmatrix} \in SO(p,q)$$

and looking at the last column. We just need to show $X^-_{p,q} = X^+_{q,p}$, which is given by the fact that

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = -1$$
$$\Leftrightarrow x_{p+1}^2 + \cdots + x_{p+q}^2 - x_1^2 - \cdots - x_p^2 = 1.$$

□

Lemma 3. *$\pi_1(SO(n)) = \pi_1(SO(n+1))$ for $n \geq 3$.*

Proof. For $n \geq 3$, $\pi_2(S^n) = \pi_1(S^n) = 0$, so the fibration

$$0 \to SO(n) \to SO(n+1) \to S^n \to 0$$

gives the long exact sequence

$$0 \to \pi_2(S^n) = 0 \to \pi_1(SO(n)) \to \pi_1(SO(n+1)) \to \pi_1(S^n) = 0 \to 0$$

which gives the desired result. □

We now assume $p \geq q$, since we have $SO^+(p,q) \cong SO^+(q,p)$. The first short exact sequence in lemma 0.2 reduces to

$$0 \to SO^+(p-1,q) \hookrightarrow SO^+(p,q) \to X^+_{p,q} \to 0, \quad p \geq 2.$$

This gives a long exact sequence on homotopy groups

$$0 \to \pi_2(X^+_{p,q}) \to \pi_1(SO^+(p-1,q)) \to \pi_1(SO^+(p,q)) \to \pi_1(X^+_{p,q}) \to 0.$$

We will first use this to deal with the cases of $p \leq 3$:

<u>If $p = 1$:</u> Since $\pi_1(X^+_{1,q}) = \pi_2(X^+_{1,q}) = 0$, we get the exact sequence

$$0 \to \pi_1(SO^+(0,q)) = \pi_1(SO(q)) \to \pi_1(SO^+(1,q)) \to 0.$$

From the exactness of the sequence above we get

$$\begin{aligned}\pi_1(SO^+(1,q)) &= \pi_1(SO^+(0,q)) = \pi_1(SO(q)) \\ &= 0 \times \pi_1(SO(q)) = \pi_1(SO(1)) \times \pi_1(SO(q))\end{aligned}$$

as required. Note that since $SO(1)$ consist of only one element we must have $\pi_1(SO(1)) = 0$.

<u>If $p = 2$:</u> Since $\pi_1(X^+_{2,q}) = \mathbb{Z}, \pi_2(X^+_{2,q}) = 0$, we get the short exact sequence

$$0 \to \underbrace{\pi_1(SO^+(1,q)) = \pi_1(SO(q))}_{\text{from } p=1 \text{ case above}} \hookrightarrow \pi_1(SO^+(2,q)) \to \mathbb{Z} \to 0$$

which gives

$$\pi_1(SO^+(2,q)) = \mathbb{Z} \times \pi_1(SO^+(1,q)) = \pi_1(SO(2)) \times \pi_1(SO(q))$$

since $SO(2) \cong S^1$ implies $\pi_1(SO(2)) = \mathbb{Z}$.

<u>If $p = 3$:</u> We know

$$\pi_1(SO^+(3,0)) = \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$$

so in the case of $q = 1$ we get the following from the second short exact sequence in lemma 0.2,

$$0 \to \pi_2(X^+_{1,3}) = 0 \to \pi_1(SO^+(3,0)) \to \pi_1(SO^+(3,1)) \to \pi_1(X^+_{1,3}) = 0 \to 0$$
$$0 \to \pi_1(SO^+(3,0)) \to \pi_1(SO^+(3,1)) \to 0$$

which gives us

$$\pi_1(SO^+(3,1)) = \pi_1(SO^+(3,0)) = \pi_1(SO(3)) = \pi_1(SO(3)) \times \pi_1(SO(1))$$

as required. We do the same for $q = 2$

$$0 \to \pi_2(X^+_{2,3}) = 0 \to \pi_1(SO^+(3,1)) \to \pi_1(SO^+(3,2)) \to \pi_1(X^+_{2,3}) = \mathbb{Z} \to 0$$
$$0 \to \pi_1(SO^+(3,1)) \to \pi_1(SO^+(3,2)) \to \mathbb{Z} \to 0$$

to get

$$\pi_1(SO^+(3,2)) = \pi_1(SO^+(3,1)) \times \mathbb{Z} = \pi_1(SO(3)) \times \pi_1(SO(2))$$

as required. Now if $q > 3$ then we have

$$0 \to \pi_2(X^+_{q,3}) = 0 \to \pi_1(SO^+(3,q-1)) \to \pi_1(SO^+(3,q)) \to \pi_1(X^+_{q,3}) = 0 \to 0$$
$$0 \to \pi_1(SO^+(3,q-1)) \to \pi_1(SO^+(3,q)) \to 0$$

to get $\pi_1(SO^+(3,q)) = \pi_1(SO^+(3,q-1)) = \pi_1(SO^+(3,3))$ for $q > 3$, and since lemma 0.3 gives us $\pi_1(SO(q-1)) = \pi_1(SO(q))$ for $q > 3$ we get the desired

$$\begin{aligned}\pi_1(SO^+(3,q)) &= \pi_1(SO^+(3,3)) \\ &= \pi_1(SO(3)) \times \pi_1(SO(3)) = \pi_1(SO(3)) \times \pi_1(SO(q))\end{aligned}$$

if we can prove the claim for $\pi_1(SO^+(3,3))$, which is shown below.

<u>If $p > 3$:</u> We induct on p. Since $\pi_1(X^+_{p,q}) = \pi_2(X^+_{p,q}) = 0$ for $p > 3$, we get the exact sequence

$$0 \to \pi_1(SO^+(p-1,q)) \to \pi_1(SO^+(p,q)) \to 0$$

which means

$$\begin{aligned}\pi_1(SO^+(p,q)) &= \pi_1(SO^+(p-1,q)) \\ &= \pi_1(SO(p-1)) \times \pi_1(SO(q)) = \pi_1(SO(p)) \times \pi_1(SO(q))\end{aligned}$$

by the induction hypothesis. The last equality is from lemma 0.3.

Now we deal with the case of $\pi_1(SO^+(3,3))$. First we show $SO^+(3,3) = SL(4,\mathbb{R})/\{\pm 1\}$.

Now since $\pi_1(X^+_{3,q}) = 0, \pi_2(X^+_{3,q}) = \mathbb{Z}$, we get the following short exact sequence for $q = 0$,

$$0 \to \pi_2(X^+_{3,0}) \to \pi_1(SO^+(2,0)) \to \pi_1(SO^+(3,0)) \to \pi_1(X^+_{3,0}) \to 0$$
$$0 \to \pi_2(S^2) = \mathbb{Z} \xrightarrow{\times 2} \pi_1(SO(2)) \to \pi_1(SO(3)) \to \pi_1(S^2) = 0 \to 0$$
$$0 \to \mathbb{Z} \xrightarrow{\times 2} \pi_1(SO(2)) \to \pi_1(SO(3)) \to 0 \qquad (*)$$

and similarly the following for $q = 3$

$$0 \to \mathbb{Z} \to \pi_1(SO^+(3,2)) \to \pi_1(SO^+(3,3)) \to 0.$$

There is a commutative diagram

$$\begin{array}{ccc} SO^+(3,2) & \hookrightarrow & SO^+(3,3) \\ \wr\Big\uparrow & & \Big\uparrow \\ SO(2) \times SO(3) & \hookrightarrow & SO(3) \times SO(3) \end{array}$$

since every arrow is an inclusion, which gives us from above

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(SO^+(3,2)) & \xrightarrow{\gamma} & \pi_1(SO^+(3,3)) & \longrightarrow & 0 \\ & & & & \wr\Big\uparrow & & \Big\uparrow & & \\ & & & & \pi_1(SO(2)) \times \pi_1(SO(3)) & \xrightarrow{g} & \pi_1(SO(3)) \times \pi_1(SO(3)) & \longrightarrow & 0. \end{array}$$

From $(\star)$ we see that g must necessarily be

$$g : (z, x) \mapsto (z \mod 2, x)$$

since the map in the first coordinate comes from $(\star)$. Therefore the image of γ is some image of $\mathbb{Z}/2\mathbb{Z}$. From the sequence

$$0 \to \mathbb{Z} \xrightarrow{f} \pi_1(SO^+(3,2)) = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \pi_1(SO^+(3,3)) = \mathrm{im}(\gamma) \to 0$$

there are 3 scenarios

- $f : 1 \mapsto (2, 0) \Rightarrow \mathrm{im}(\gamma) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $f : 1 \mapsto (1, 1) \Rightarrow \mathrm{im}(\gamma) = \mathbb{Z}/2\mathbb{Z}$
- $f : 1 \mapsto (2, 1) \Rightarrow \mathrm{im}(\gamma) = \mathbb{Z}/4\mathbb{Z}$.

Lemma 4. $SO^+(3,3) = SL(4,\mathbb{R})/\{\pm 1\}$.

Proof. We have $SL(4,\mathbb{R})$ acts on $V = \mathbb{R}^4$ via rotations, which means it also acts on $U = \Lambda^2 V$ of dimension $\binom{4}{2} = 6$. We can define a symmetric form on U via

$$(v_1 \wedge v_2, v_3 \wedge v_4) = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)/(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$$

where $(v_1 \wedge v_2 \wedge v_3 \wedge v_4) \in \Lambda^4 V$ which is a 1-dim vector space over $\mathbb{R}$. By definition we have for all $g \in SL(4, \mathbb{R})$,

$$
\begin{aligned}
1 = \det g &= g(v_1 \wedge v_2 \wedge v_3 \wedge v_4)/(v_1 \wedge v_2 \wedge v_3 \wedge v_4) \\
\implies \quad & g(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)
\end{aligned}
$$

and so $\Lambda^4 V$ is invariant under the action by $SL(4, \mathbb{R})$, i.e. the above symmetric form $(\cdot, \cdot)$ is preserved under $SL(4, \mathbb{R})$. Now we compute the signature of $(\cdot, \cdot)$ by looking at the canonical basis of $\Lambda^2 V$,

$$
\begin{aligned}
|e_1 \wedge e_2 \pm e_3 \wedge e_4|^2 &= \pm 2(e_1 \wedge e_2, e_3 \wedge e_4) = \pm 2 \\
|e_1 \wedge e_3 \pm e_2 \wedge e_4|^2 &= \pm 2(e_1 \wedge e_3, e_2 \wedge e_4) = \mp 2 \\
|e_1 \wedge e_4 \pm e_2 \wedge e_3|^2 &= \pm 2(e_1 \wedge e_4, e_2 \wedge e_3) = \pm 2
\end{aligned}
$$

and so the signature is $(3, -3)$. This means $SO(3,3)$ is also exactly the linear maps that preserves the symmetric form $(\cdot, \cdot)$ of determinant 1. Since $\dim SO(3,3) = 15 = \dim SL(4, \mathbb{R})$, the connected component $SO^+(3,3)$ must be isomorphic to the connected component $SL(4, \mathbb{R})/\{\pm 1\}$, as required. □

So from the above lemma we know $\mathrm{im}(\gamma) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$, but since it is the image of $\mathbb{Z}/2\mathbb{Z}$ under a surjective map we conclude that

$$
\pi_1(SO^+(3,3)) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \pi_1(SO(3)) \times \pi_1(SO(3))
$$

as required. □