Kähler manifolds

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Abstract

The aim of this project is to investigate the geometric interpretation of the Kähler condition of a Kähler manifold. The Kähler condition in precise terms says that the fundamental form associated with a Riemannian structure of a complex manifold is closed. It turns out this is equivalent to saying the complex structure commutes with parallel transport. Most terms and theorems used will be defined explicitly, and for some theorems a proof will be provided.

1 Kähler manifolds

We will first define what a Kähler manifold is, referencing [2, Section 3.1].

Definition (Kähler manifold)**.** A *Kähler manifold* X is a complex manifold endowed with a Kähler structure.

I will use X to denote a Kähler manifold from now on. It is possible for a complex manifold to have a Kähler structure without fixing it. Which means it is more accurate to speak of a complex manifold of Kähler type in this case. A Kähler manifold also has several compatible structures that can be described from different points of view. We will focus on a particular interplay between the complex viewpoint and the Riemannian viewpoint, which is known as the Kähler metric (or Kähler structure). We will use the terms Kähler metric and Kähler structure interchangeably in this writeup. Some of the terms in italic will be defined later, and almost all of the contents in this writeup come from [2].

The three main viewpoints of a Kähler manifold X are

- **Symplectic viewpoint:** This is not the focus of this project, but I will give a brief explanation. There is a *symplectic structure* (X, ω) on X which is equipped with an *integrable almost complex-structure* J which is compatible with the *symplectic form* ω, i.e. for all u, v

  $$\omega(u, v) = \omega(Ju, Jv), \quad \omega(u, Jv) = \omega(v, Ju), \quad \omega(u, Ju) > 0.$$

  This gives us a bilinear form

  $$g(u, v) = \omega(u, Jv)$$

which on the tangent space of X at each point is symmetric and positive definite, and hence g is a Riemannian metric on X.

- **Complex viewpoint:** X is a complex manifold (X, ω) with a *hermitian metric* h whose associated 2-form ω, called the *fundamental form*, is closed. This is the Kähler metric. This is explained in more details in the next section.

- **Riemannian viewpoint:** X is a Riemannian manifold (X, g) of even dimension $2n$, and there is a *complex structure* J on the tangent space of X at each point p, such that J preserves the metric g and J is preserved by parallel transport. This is also explained in more details in the next section. Another more concise way of saying this is the *holonomy group* of X is contained in the unitary group $U(n)$.

I will now define some of terms used in the symplectic viewpoint. For the main point of this writeup please skip to the next section.

Definition (Almost symplectic structure)**.** An *almost symplectic structure* on a differentiable manifold X is a 2-form ω that is everywhere non-singular. i.e. ω does not vanish at any point.

Now we define a symplectic form.

Definition (Closed form)**.** A differential form ω is *closed* if $d\omega = 0$.

Definition (Exact form)**.** A differential ω is *exact* if there exists η such that $\omega = d\eta$.

In other words, a closed form is in the kernel of d, and an exact form is in the image of d.

Definition (Nondegenerate form)**.** A differential form ω on a manifold N is *nondegenerate* if for every $p \in N$, $\omega_p(x, y) = 0$ for all $y \in T_pN \backslash \{0\}$ implies $x = 0 \in T_pN$.

Definition (Symplectic form)**.** A *symplectic form* is a closed nondegenerate differential 2-form.

I will also give a brief explanation for the Holonomy group here, since it is not relevant to the later sections.

Definition (Holonomy group of X)**.** Given a piecewise smooth closed loop $\gamma : [0, 1] \to X$ based at a fixed point $x \in X$, the parallel transport along this curve give rise to an automorphism of T_xX. Note that this automorphism is just a linear map. We can compose and invert closed loops in the obvious way, which means that the set of automorphism of T_xX is a group, which we call the *holonomy group*.

2 The fundamental form is closed

In this section we will go into more details about the complex viewpoint. We will first define what a hermitian structure is on X, then deduce the Kähler form from it, which gives us the Kähler metric. We define complex manifolds again for convenience.

Definition (Complex manifold)**.** An n-dimensional complex manifold is a manifold with an atlas of charts to the open unit disc in $\mathbb{C}^n$, such that the transition maps are holomorphic in the complex sense.

Definition (Almost complex structure)**.** Let V be a finite-dimensional real vector space, then an endomorphism $I : V \to V$ with $I^2 = -\mathrm{Id}$ is called an *almost complex structure* on V.

Note that if V has an almost complex structure on V, then its real dimension is even, due to the following lemma.

Lemma 1. *If I is an almost complex structure on a real vector space V, then V admits in a natural way the structure of a complex vector space.*

Proof. The $\mathbb{C}$-module structure on V is given by $(a + ib) \cdot v = a \cdot v + b \cdot I(v)$, where $a, b \in \mathbb{R}$. Since I is linear in $\mathbb{R}$, we have

$$((a + ib)(c + id)) \cdot v = (a + ib)((c + id) \cdot v)$$

and

$$i(i \cdot v) = -v.$$

□

We are now ready to define the hermitian structure.

Definition (Hermitian structure)**.** A Riemannian metric g on X is an *hermitian structure* on X if for any point $x \in X$ the scalar product g_x on T_xX is compatible with the almost complex structure I_x, which is defined on T_xX by treating it as a real vector space. Here compatible means $g_x(u, v) = g_x(I_xu, I_xv)$.

We should consider the hermitian structure as an inner product defined on the holomorphic(complexified) tangent bundle.

Definition (Fundamental form)**.** The *fundamental form* of a hermitian structure is a real $(1, 1)$-form ω defined by

$$\omega(u, v) := g(Iu, v).$$

A $(1, 1)$-form is an element of $\Lambda^{(1,1)}X \subset \Lambda^2X$, via the decomposition

$$\Lambda^2X \cong \Lambda^{(2,0)}X \oplus \Lambda^{(1,1)}X \oplus \Lambda^{(0,2)}X.$$

For more information please refer to section 1.3 of [2].

We also have the following lemma

Lemma 2. *For g, ω, I defined as above, we have*

$$g(u, v) = \omega(u, Iv).$$

Proof. Let $u' = Iu$, then we have $Iu' = I^2 u = -u$, which gives

$$\begin{aligned} g(Iu, v) &= \omega(u, v) \\ g(u', v) &= \omega(-Iu', v) \\ g(u', v) &= \omega(v, Iu') \end{aligned}$$

as required. □

Note that this implies the hermitian structure g is uniquely determined by the almost complex structure I and the fundamental form ω.

In local coordinates the fundamental form ω is

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\bar{z}_j$$

where the matrix $h_{ij}(x)$ is a positive definite hermitian matrix for any $x \in X$.

Definition (Hermitian manifold)**.** A *hermitian manifold* is a complex manifold endowed with an hermitian structure g.

We are now ready to talk about the Kähler structure.

Definition (Kähler metric and Kähler form)**.** A *Kähler metric (or Kähler structure)* is an hermitian structure g for which the fundamental form ω is closed, i.e. $d\omega = 0$. In this case the fundamental form ω is called the *Kähler form.*

The above condition that the fundamental form is closed *is* the Kähler condition formulated by Kähler in [3], which was published in 1933. We can define a variety of linear and differential operators on X, and it turns out with the Kähler condition they all behave especially well. In other words, the Kähler condition makes everything "nice", and we shall focus on a particular aspect of this in the next section.

3 The complex structure commutes with parallel transport

We will now take a look at the Riemannian viewpoint.

Definition (Complex structure of the Riemannian viewpoint)**.** Let (X, g) be a Riemannian manifold, then for a point $x \in X$ we can have a *complex structure* $J_x : T_x X \to T_x X$, which is an almost complex structure on the tangent space $T_x X$. i.e. it satisfies $J_x^2 = -1$.

Note that an almost complex structure is a linear map between vector spaces, but a complex structure is something defined on the tangent space of a Riemannian manifold.

Definition (Preserves the metric)**.** We say J preserves the metric g if for all u, v we have

$$g(u, v) = g(Ju, Jv).$$

Definition (Preserved by parallel transport)**.** We say J is preserved by a parallel transport $Y : T_{\gamma(t)}X \to T_{\gamma(t+s)}X$ along a smooth curve γ if we have $J_{\gamma(t)} \circ Y = Y \circ J_{\gamma(t+s)}$.

This preservation of parallel transport is equivalent to the Kähler condition shown above. I do not understand this connection fully, but a reference can be found at [4, Part 2 Section 5 Theorem 5.5]. Intuitively we want to say that parallel transport is a first order operation, since the covariant derivative is first order, and the Kähler metric g can be written as

$$g = \sum dx_k \otimes dx_k + dy_k \otimes dy_k + O(|z|^2)$$

which is also first order.

4 Why we should care about Kähler manifolds

Something I am very interested in is mirror symmetry. This is a relation between Calabi-Yau manifolds which originated from string theory. In essence, it refers to a situation where two Calabi-Yau manifolds looks very different geometrically, but when employed as extra dimensions of string theory they are equivalent physically. This implies there is a mathematical relationship between them, and mathematicians have been trying to prove this relation rigorously.

There are currently two main ways of attack, homological mirror symmetry proposed by Maxim Kontsevich and the SYZ conjecture proposed in a paper by Strominger, Yau and Zaslow titled "Mirror symmetry is T-duality". Homological mirror symmetry tries to use homological algebra to attempt to resolve mirror symmetry, however due to the difficulty only a in a few examples have mathematicians been able to verify this conjecture. This is an ongoing area of research.

The SYZ conjecture is not something I understand fully, and my understanding is that one of the difficulties is correctly formulating the conjecture in precise mathematical terms, since there is no agreed upon precise statement of the conjecture. However there is a general statement expected to be the correct formulation of the conjecture, found in [1], and I will present it here.

Conjecture 3 (SYZ Conjecture)**.** *Every 6-dimensional Calabi-Yau manifold X has a mirror 6-dimensional Calabi-Yau manifold $\hat{X}$ such that there are continuous surjections $f : X \to B$, $\hat{f} : \hat{X} \to B$ to a compact topological manifold B of dimension 3, such that*

1. *There exists a dense open subset $B_{reg} \subset B$ on which the maps $f, \hat{f}$ are fibrations by nonsingular special Lagrangian 3-tori. Furthermore for every point $b \in B_{reg}$, the torus fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ should be dual to each other in some sense, analogous to duality of Abelian varieties.*
2. *For each $b \in B \backslash B_{reg}$, the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ should be singular 3-dimensional special Lagrangian submanifolds of X and $\hat{X}$ respectively.*

For convenience I will give a quick definition of Calabi-Yau manifolds from [1, Part I Def 4.3].

Definition (Calabi-Yau manifolds)**.** Let $m \geq 2$. A *Calabi-Yau m-fold* is a quadruple (M, J, g, Ω) such that (M, J) is a compact m-dimensional complex manifold, g a Kähler metric on (M, J) with holonomy group $\mathrm{Hol}(g) = \mathrm{SU}(m)$, and Ω a nonzero constant $(m, 0)$-form on M called the *holomorphic volume form*, which satisfies

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \Omega,$$

where ω is the Kähler form of g. The constant factor in above is chosen to make $\mathrm{Re}(\Omega)$ a *calibration*.

However there are several *different* definitions of Calabi-Yau manifolds in use in the literature. The important thing here is that Calabi-Yau manifolds *are* Kähler manifolds, and we should think of it as a Kähler manifold with extra structure. Indeed, we just need the condition on the holonomy group to get a Calabi-Yau manifold from a Kähler manifold, which is the very next lemma in [1], which states

Lemma 4. *Let (M, J, g) be a compact Kähler manifold with $Hol(g) = SU(m)$. Then M admits a holomorphic volume form Ω, unique up to change of phase $\Omega \mapsto e^{i\theta}\Omega$, such that (M, J, g, Ω) is a Calabi-Yau manifold.*

Which means Kähler manifolds are an fundamental object of study in mirror symmetry. It is also used extensively in other parts of algebraic geometry and theoretical physics, but I think this is enough details for this writeup for now. Thank you for reading!

References

[1] M Gross, D Huybrechts, and D Joyce. *Calabi-Yau manifolds and related geometries*. Springer, New York, 2003.

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[3] Erich Kähler. über eine bemerkenswerte hermitesche metrik. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 1933.

[4] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry Volume II*. Wiley, 1969.