# EQUIVARIANT BIRATIONAL TYPES

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## 1. Introduction

This thesis concern the study of new invariants of complex algebraic varieties equipped with actions of abelian groups. These invariants, called *birational types*, were introduced in [2]. Our focus here is on arithmetic and algebraic properties of birational types; their first geometric applications can be found in [1]. Informally, consider an $n$-dimensional variety $X$ with an action of a finite abelian group $G$. The locus of points in $X$ fixed by $G$ is a finite union of disjoint subvarieties $Y_\alpha \subset X$, possibly of different dimensions. For each such subvariety $Y_\alpha$ one records the weights of the $G$-action in the tangent space at some point $y_\alpha \in Y_\alpha$, i.e., an *unordered* $n$-tupel

$$[a_{1,\alpha}, \dots, a_{n,\alpha}]$$

of *characters* of $G$:

$$a_{i,\alpha} \in A = G^\vee = \mathrm{Hom}(G, \mathbb{C}^\times).$$

Then one forms the sum

$$\beta(X) := \sum_\alpha [a_{1,\alpha}, \dots, a_{n,\alpha}].$$

One would like these classes to be *invariant* under a basic operation, called an *equivariant blowup*. This operation replaces $X$ by another variety $\tilde{X}$, which coincides with $X$ on the complement to a $G$-stable subvariety on $X$. In short, one introduces relations of the type

$$\beta(\tilde{X}) - \beta(X) = 0,$$

for all such blowups.

It turns out that this geometric picture admits a nice algebraic description. Let

$$\mathcal{S}_n(G)$$

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be the $\mathbb{Z}$-module generated by $n$-tupels

$$[a_1, \dots, a_n], \quad a_i \in A,$$

subjection to the following

(G) **Generation**: $a_1, \dots, a_n$ generate $A$, i.e.,

$$\sum_i \mathbb{Z} a_i = A,$$

(S) **Symmetry**: for all permutations $\sigma \in \mathfrak{S}_n$ and all $a_1, \dots, a_n \in A$ we have

$$[a_{\sigma(1)}, \dots, a_{\sigma(n)}] = [a_1, \dots, a_n].$$

Consider the quotient

$$\mathcal{S}_n(G) \to \mathcal{B}_n(G)$$

by the additional relation

(B) **Blowup**: for all $[a_1, a_2, b_1, \dots, b_{n-2}] \in \mathcal{S}_n(G)$, the symbol

$$[a_1, a_2, b_1, \dots, b_{n-2}]$$

equals

$$= \begin{cases} [a_1 - a_2, a_2, b_1, \dots, b_{n-2}] + [a_1, a_2 - a_1, b_1, \dots, b_{n-2}] & \text{if } a_1 \neq a_2 \\ [a_1, 0, b_1, \dots, b_{n-2}] & \text{if } a_1 = a_2 \end{cases}$$

The first main result states that the class $\beta(X) \in \mathcal{B}_n(G)$ does not change under equivariant blowups [2]. Here we will not discuss the applications of $\mathcal{B}_n(G)$ to algebraic geometry, as in [1]. Instead, we explore arithmetic and algebraic properties of these groups, focusing on examples.

It is helpful to define the quotient

$$\mathcal{B}_n(G) \to \mathcal{B}_n^-(G)$$

by the relation

$$[-a_1, \dots, a_n] = -[a_1, \dots, a_n] \tag{1.1}$$

which will play an important role in this thesis. Note that

$$[a_1, \dots, a_n]^- \in \mathcal{B}_n^-(G)$$

denotes the image of $[a_1, \dots, a_n]$ under this quotient.

We also have another quotient

$$\mathcal{S}_n(G) \to \mathcal{M}_n(G)$$

by the relation

(M) **Modular Blowup**: for all $[a_1, a_2, b_1, \dots, b_{n-2}] \in \mathcal{S}_n(G)$,

$$\langle a_1, a_2, b_1, \dots, b_{n-2}\rangle = \langle a_1 - a_2, a_2, b_1, \dots, b_{n-2}\rangle + \langle a_1, a_2 - a_1, b_1, \dots, b_{n-2}\rangle$$

where $\langle a_1, \dots, a_n\rangle$ is the image of $[a_1, \dots, a_n]$ under this quotient.

Note that we can also define, similar to above

$$\mathcal{M}_n(G) \to \mathcal{M}_n^-(G)$$

which is a quotient by the relation

$$\langle -a_1, \dots, a_n\rangle = -\langle a_1, \dots, a_n\rangle. \tag{1.2}$$

The introduction of $\mathcal{M}_n(G), \mathcal{M}_n^-(G)$ were motivated by experimentations with relations in $\mathcal{B}_n(G)$, and various constructions connected to $\mathcal{M}_n(G)$ were shown in [2, Sections 4,5].

There is also a map

$$\mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G),$$

first introduced in [2], defined on symbols as follows:

$$[a_1, \dots, a_n] \mapsto \langle a_1, \dots, a_n\rangle, \quad \text{if all } a_1, \dots, a_n \neq 0,$$
$$[0, a_2, \dots, a_n] \mapsto 2\langle 0, a_2, \dots, a_n\rangle, \quad \text{if all } a_2, \dots, a_n \neq 0,$$
$$[0, 0, a_3, \dots, a_n] \mapsto 0, \quad \text{for all } a_3, \dots, a_n,$$

and extend by $\mathbb{Z}$-linearity. This is a well-defined homomorphism from [2, Section 3]. It is also an isomorphism modulo torsion from [1, Prop. 7.1], which settles Conjecture 8 and 9 in [2].

The main result of section 2 is

**Proposition 7.** *For $p$ prime, $\mathcal{B}_n((\mathbb{Z}/p\mathbb{Z})^n) \neq 0$ for all $n \in \mathbb{N}$.*

which is shown by induction on $n$. The idea is to use a surjective map

$$\eta_{p,n}^- : \mathcal{B}_n^-((\mathbb{Z}/p\mathbb{Z})^n) \to \mathcal{B}_{n-1}^-((\mathbb{Z}/p\mathbb{Z})^{n-1})$$

to show that $\mathcal{B}_n^-((\mathbb{Z}/p\mathbb{Z})^n)$ is non-trivial given the induction hypothesis. The base case of $n = 1$ can be found explicitly, and some further calculations were also carried out in MAGMA.

In [2, Section 4] we have a lattices and cones construction for $\mathcal{M}_n(G) \otimes \mathbb{Q}$, each element is identified with an equivalence class of triples

$$(\mathbf{L}, \chi, \Lambda)$$

where $\mathbf{L} = \mathbb{Z}^n$ is a lattice, $\chi$ is an element in $\mathbf{L} \otimes A$ and $\Lambda \in \mathbf{L}_{\mathbb{R}}$ is a strictly convex cone spanned by a basis of $\mathbf{L}$. Explicitly we can choose a basis $e_1, \ldots, e_n$ of $\mathbf{L}$ that spans $\Lambda$, and write

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i$$

and identify

$$\psi(\mathbf{L}, \chi, \Lambda) = \langle a_1, \ldots, a_n \rangle \in \mathcal{M}_n(G) \otimes \mathbb{Q}.$$

The relation (M) is then equivalent to decomposing the cone $\Lambda$ into smaller subcones. Section 3 of this thesis uses the map $\mu$ to generalise the above construction to $\mathcal{B}_n(G) \otimes \mathbb{Q}$ and show explicitly that these structures are compatible with the defining relations in $\mathcal{B}_n(G)$.

Section 4 again uses $\mu$ to generalises [2, Section 5] to $\mathcal{B}_n(G) \otimes \mathbb{Q}$, which defined the following multiplication and co-multiplication maps: given a short exact sequence of finite abelian groups

$$0 \to G' \to G \to G'' \to 0$$

we get a short exact sequence of their character groups

$$0 \to A'' \to A \to A' \to 0.$$

This gives a $\mathbb{Z}$-bilinear multiplication map

$$\nabla : \mathcal{M}_{n'}(G') \otimes \mathcal{M}_{n''}(G'') \to \mathcal{M}_{n'+n''}(G)$$

where $n', n'' \geq 1$, defined by

$$\langle a'_1, \ldots, a'_{n'} \rangle \otimes \langle a''_1, \ldots, a''_{n''} \rangle \mapsto \sum \langle a_1, \ldots, a_{n'}, a''_1, \ldots, a''_{n''} \rangle$$

where the sum is over all lifts from $a_i \in A$ to $a'_i \in A'$. There is also a $\mathbb{Z}$-bilinear co-multiplication map, noting that there is a minus on the second factor of the image,

$$\Delta : \mathcal{M}_{n'+n''}(G) \to \mathcal{M}_{n'}(G') \otimes \mathcal{M}^-_{n''}(G''),$$

defined by

$$\langle a_1, ..., a_n \rangle \mapsto \sum \langle a_{I'} \text{ mod } A'' \rangle \otimes \langle a_{I''} \rangle^-$$

where the sum is over all ways to split

$$\{1, \ldots, n\} = I' \sqcup I'', \text{ with } \#I' = n', \#I'' = n'',$$

such that for all $j \in I''$ we have $a_j \in A'' \subset A$ and the elements $a_j$ span $A''$.

Section 5 explains the chain complexes induced from the maps $\nabla$ and $\Delta$ via simplicial complexes and the proofs of [2, Theorems 12, 14] in more detail, both of which are found in [2, Section 5]. In particular [2, Theorems 12, 14] state that the cohomology of the chain complexes defined is concentrated in degree 0. The chain complex is constructed more explicitly and the non-trivial steps of the proofs are written out in more detail.

Each section will also have a few specific examples at the end, showcasing the details of each section.

sect:exam

## 2. Examples

We know that

$$\mathcal{B}_1(\mathbb{Z}/N\mathbb{Z}) = \mathbb{Z}^{\phi(N)},$$

where $\phi = |(\mathbb{Z}/N\mathbb{Z})^\times|$ is the Euler function; in particular,

$$\mathcal{B}_1(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}^{\phi(2)} = \mathbb{Z}.$$

We also have [1, Section 3.2]:

$$\mathcal{B}_2((\mathbb{Z}/2\mathbb{Z})^2) = (\mathbb{Z}/2\mathbb{Z})^2$$

We now calculate $\mathcal{B}_3((\mathbb{Z}/2\mathbb{Z})^3)$. Write

$$\left((\mathbb{Z}/2\mathbb{Z})^3\right)^\vee = \{0, \chi_1, \chi_2, \chi_3, \chi_1+\chi_2, \chi_2+\chi_3, \chi_3+\chi_1, \chi_1+\chi_2+\chi_3\}$$

using numbers to simplify notation, the possible symbols are

$$\begin{gathered}
[1,2,3]\\
[1,2,2+3],[1,2,1+3],[1,2,1+2+3]\\
[2,3,3+1],[2,3,2+1],[2,3,1+2+3]\\
[3,1,1+2],[3,1,3+2],[3,1,1+2+3]\\
[1,1+2,2+3],[1,2+3,3+1],[1,3+1,1+2]\\
[2,1+2,2+3],[2,2+3,3+1],[2,3+1,1+2]\\
[3,1+2,2+3],[3,2+3,3+1],[3,3+1,1+2]\\
[1,1+2,1+2+3],[1,3+1,1+2+3]\\
[2,1+2,1+2+3],[2,2+3,1+2+3]\\
[3,2+3,1+2+3],[3,3+1,1+2+3]\\
[1+2,3+1,1+2+3],[2+3,3+1,1+2+3]\\
[1+2,2+3,1+2+3].
\end{gathered}$$

Using MAGMA to calculate this $\mathbb{Z}$-module, we get

$$\mathcal{B}_3((\mathbb{Z}/2\mathbb{Z})^3) = (\mathbb{Z}/2\mathbb{Z})^8 \neq 0.$$

Now consider the surjective map, for $n \geq 3$,

$$(\mathbb{Z}/2\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})^{n-1},$$

where the $n$-th generator is mapped to zero, which induces an inclusion map between character groups

$$A_{n-1} := \big((\mathbb{Z}/2\mathbb{Z})^{n-1}\big)^\vee \hookrightarrow A_n := \big((\mathbb{Z}/2\mathbb{Z})^n\big)^\vee.$$

This allows us to define a map

$$\eta : \mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n) \to \mathcal{B}_{n-1}((\mathbb{Z}/2\mathbb{Z})^{n-1})$$

given by the formula

$$[a_1, \dots, a_n] \mapsto \sum [a_{i_1}, \dots, a_{i_{n-1}}]$$

where $i_1, \dots, i_{n-1} \in I \subset \{1, \dots, n\}$ with $\#I = n-1$. The sum is over all $I$ such that $a_j \in A_{n-1} \hookrightarrow A_n$ for all $j \in I$ and $\{a_{i_1}, \dots, a_{i_{n-1}}\}$ span $A_{n-1}$.

**Lemma 1.** *The map $\eta$ is well-defined.*

*Proof.* We have the following two facts about $\mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n)$:

- Since the rank of the group $A_n$ is $n$, we need $n$ distinct, non-zero entries in each symbol to satisfy the generating condition (G). This means the relation (B) reduces to

$$[a_1, a_2, \dots, a_n] = [a_1 - a_2, a_2, \dots, a_n] + [a_1, a_2 - a_1, \dots, a_n]$$

  which is essentially the relation (M).
- Since every element of $(\mathbb{Z}/2\mathbb{Z})^n$ has order 2, so does $A_n$. So for every

$$[a_1, a_2, \dots, a_n] \in \mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n)$$

  we have

$$[a_1 + a_2, a_2, \dots] = [a_1, a_2, \dots] + [a_1 + a_2, a_1, \dots]$$
$$[a_1, a_1 + a_2, \dots] = [a_1, a_2, \dots] + [a_2, a_1 + a_2, \dots]$$

  which when added together gives

$$2[a_1, a_2, \dots, a_n] = 0.$$

  So we have

$$[-a_1, \dots, a_n] = [a_1, \dots, a_n] = -[a_1, \dots, a_n].$$

These two facts combined gives

$$\mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n) = \mathcal{B}_n^-((\mathbb{Z}/2\mathbb{Z})^n) = \mathcal{M}_n^-((\mathbb{Z}/2\mathbb{Z})^n) = \mathcal{M}_n((\mathbb{Z}/2\mathbb{Z})^n).$$

Now we can just reuse the argument of [2, Prop. 11], the details of which is shown explicitly in the proof of Lemma 4 below. □

**Lemma 2.** *The map $\eta$ is surjective.*

*Proof.* Each

$$[b_1, b_2, \ldots, b_{n-1}] \in \mathcal{B}_{n-1}((\mathbb{Z}/2\mathbb{Z})^{n-1})$$

is the image of

$$[b_1, b_2, \ldots, b_{n-1}, b_n] \in \mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n)$$

under $\eta$, where $b_n$ satisfies $b_n \notin A_{n-1}$ and $\sum_{j=1}^n \mathbb{Z}b_j = A_n$. □

**Proposition 3.** *$\mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n) \neq 0$ for all $n \in \mathbb{N}$.*

*Proof.* We want to show this via induction, i.e. $\mathcal{B}_{n-1}((\mathbb{Z}/2\mathbb{Z})^{n-1}) \neq 0$ implies $\mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n) \neq 0$. The base case $n = 1, 2, 3$ is shown above.

So suppose $\mathcal{B}_{n-1}((\mathbb{Z}/2\mathbb{Z})^{n-1}) \neq 0$. The lemmas above tell us that

$$\eta : \mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n) \to \mathcal{B}_{n-1}((\mathbb{Z}/2\mathbb{Z})^{n-1})$$

is a well-defined surjective map, from which we can conclude that $\mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n)$ is also nontrivial. □

As an aside every element of $\mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n)$ has order 2, so by the fundamental theorem of finite abelian groups we must have

$$\mathcal{B}_n((\mathbb{Z}/2\mathbb{Z})^n) = (\mathbb{Z}/2\mathbb{Z})^{k_n}$$

for some integer $k_n$.

Now we extend this result to $\mathcal{B}_n((\mathbb{Z}/p\mathbb{Z})^n)$ where $p$ is a prime. We have, for $p$ prime,

$$\mathcal{B}_1(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}^{\phi(p)} = \mathbb{Z}^{p-1} \neq 0$$

as above.

We now calculate $\mathcal{B}_2((\mathbb{Z}/3\mathbb{Z})^2)$, write

$$\left((\mathbb{Z}/3\mathbb{Z})^2\right)^\vee = \{0, \chi_1, \chi_2, 2\chi_1, 2\chi_2, \chi_1+\chi_2, 2\chi_1+\chi_2, \chi_1+2\chi_2, 2\chi_1+2\chi_2\},$$

the possible symbols and relations are

$$\begin{aligned}
[\chi_1,\chi_2] &= [\chi_1,2\chi_1+\chi_2]+[\chi_1+2\chi_2,\chi_2]\\
[\chi_1,2\chi_2] &= [\chi_1,2\chi_1+2\chi_2]+[\chi_1+\chi_2,2\chi_2]\\
[\chi_1,\chi_1+\chi_2] &= [\chi_1,\chi_2]+[2\chi_2,\chi_1+\chi_2]\\
[\chi_1,2\chi_1+\chi_2] &= [\chi_1,\chi_1+\chi_2]+[2\chi_1+2\chi_2,2\chi_1+\chi_2]\\
[\chi_1,\chi_1+2\chi_2] &= [\chi_1,2\chi_2]+[\chi_2,\chi_1+2\chi_2]\\
[\chi_1,2\chi_1+2\chi_2] &= [\chi_1,\chi_1+2\chi_2]+[2\chi_1+\chi_2,2\chi_1+2\chi_2]\\
[\chi_2,2\chi_1] &= [\chi_2,2\chi_1+2\chi_2]+[\chi_1+\chi_2,2\chi_1]\\
[\chi_2,\chi_1+\chi_2] &= [\chi_2,\chi_1]+[2\chi_1,\chi_1+\chi_2]\\
[\chi_2,2\chi_1+\chi_2] &= [\chi_2,2\chi_1]+[\chi_1,2\chi_1+\chi_2]\\
[\chi_2,\chi_1+2\chi_2] &= [\chi_2,\chi_1+\chi_2]+[2\chi_1+2\chi_2,\chi_1+2\chi_2]\\
[\chi_2,2\chi_1+2\chi_2] &= [\chi_2,2\chi_1+\chi_2]+[\chi_1+2\chi_2,2\chi_1+2\chi_2]\\
[2\chi_1,2\chi_2] &= [2\chi_1,\chi_1+2\chi_2]+[2\chi_1+\chi_2,2\chi_2]\\
[2\chi_1,\chi_1+\chi_2] &= [2\chi_1,2\chi_1+\chi_2]+[\chi_1+2\chi_2,\chi_1+\chi_2]\\
[2\chi_1,2\chi_1+\chi_2] &= [2\chi_1,\chi_2]+[2\chi_2,2\chi_1+\chi_2]\\
[2\chi_1,\chi_1+2\chi_2] &= [2\chi_1,2\chi_1+2\chi_2]+[\chi_1+\chi_2,\chi_1+2\chi_2]\\
[2\chi_1,2\chi_1+2\chi_2] &= [2\chi_1,2\chi_2]+[\chi_2,2\chi_1+2\chi_2]\\
[2\chi_2,\chi_1+\chi_2] &= [2\chi_2,\chi_1+2\chi_2]+[2\chi_1+\chi_2,\chi_1+\chi_2]\\
[2\chi_2,2\chi_1+\chi_2] &= [2\chi_2,2\chi_1+2\chi_2]+[\chi_1+\chi_2,2\chi_1+\chi_2]\\
[2\chi_2,\chi_1+2\chi_2] &= [2\chi_2,\chi_1]+[2\chi_1,\chi_1+2\chi_2]\\
[2\chi_2,2\chi_1+2\chi_2] &= [2\chi_2,2\chi_1]+[\chi_1,2\chi_1+2\chi_2]\\
[\chi_1+\chi_2,2\chi_1+\chi_2] &= [\chi_1+\chi_2,\chi_1]+[2\chi_1,2\chi_1+\chi_2]\\
[\chi_1+\chi_2,\chi_1+2\chi_2] &= [\chi_1+\chi_2,\chi_2]+[2\chi_2,\chi_1+2\chi_2]\\
[2\chi_1+\chi_2,2\chi_1+2\chi_2] &= [2\chi_1+\chi_2,\chi_2]+[2\chi_2,2\chi_1+2\chi_2]\\
[\chi_1+2\chi_2,2\chi_1+2\chi_2] &= [\chi_1+2\chi_2,\chi_1]+[2\chi_1,2\chi_1+2\chi_2].
\end{aligned}$$

Using MAGMA to calculate this, we get

$$\mathcal{B}_2((\mathbb{Z}/3\mathbb{Z})^2)=\mathbb{Z}^7.$$

This suggests that we can use the same idea as $p=2$ but with the base case of the induction as $n=1$. The difference is that we will first show

$$\mathcal{B}_n^-((\mathbb{Z}/p\mathbb{Z})^n)\neq 0,$$

which will then imply

$$\mathcal{B}_n((\mathbb{Z}/p\mathbb{Z})^n) \neq 0$$

since the former is a quotient of the latter.

So consider the surjective map similar to above, this time for $n \geq 2$,

$$(\mathbb{Z}/p\mathbb{Z})^n \to (\mathbb{Z}/p\mathbb{Z})^{n-1}$$

where the $n$-th generator is mapped to zero, which induces an inclusion map between character groups

$$A_{p,n-1} := \left((\mathbb{Z}/p\mathbb{Z})^{n-1}\right)^\vee \hookrightarrow A_{p,n} := \left((\mathbb{Z}/p\mathbb{Z})^n\right)^\vee.$$

This allows us to define a map analogous to $\eta$ above

$$\eta^-_{p,n} : \mathcal{B}^-_n((\mathbb{Z}/p\mathbb{Z})^n) \to \mathcal{B}^-_{n-1}((\mathbb{Z}/p\mathbb{Z})^{n-1})$$

given by the formula

$$[a_1, \dots, a_n]^- \mapsto \sum [a_{i_1}, \dots, a_{i_{n-1}}]^-$$

where $i_1, \dots, i_{n-1} \in I \subset \{1, \dots, n\}$ with $\#I = n-1$. The sum is over all $I$ such that $a_j \in A_{p,n-1} \hookrightarrow A_{p,n}$ for all $j \in I$ and $\{a_{i_1}, \dots, a_{i_{n-1}}\}$ span $A_{p,n-1}$. Note that there can only be at most one term in the image on the right.

**Lemma 4.** *The maps $\eta^-_{p,n}$ are well-defined for primes $p \geq 2$.*

*Proof.* We modify the proof of $p = 2$ so that it works for any $n \geq 2$ and any prime $p$. Again, since the rank of the group $A_{p,n}$ is $n$, we need $n$ distinct, non-zero entries in each symbol to satisfy the generating condition (G). So the relation (B) reduces to

$$[a_1, a_2, \dots, a_n]^- = [a_1 - a_2, a_2, \dots, a_n]^- + [a_1, a_2 - a_1, \dots, a_n]^- \tag{2.1}$$

which is essentially the relation (M). This means

$$\mathcal{B}^-_n((\mathbb{Z}/p\mathbb{Z})^n) = \mathcal{M}^-_n((\mathbb{Z}/p\mathbb{Z})^n).$$

The rest of the proof is the same idea as in [2, Prop. 11], but written out explicitly. We just have to show the relation (2.1) holds under $\eta^-_{p,n}$. We have the following cases:

(1) $a_1, a_2 \in A_{p,n-1}$
(2) WLOG $a_1 \notin A_{p,n-1}$, $a_2 \in A_{p,n-1}$
(3) $a_1, a_2 \notin A_{p,n-1}$

In case (1), we must have

$$[a_1, a_2, \dots, a_n]^- \mapsto [a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n]^-$$

for some $i$ such that $a_i \notin A_{p,n-1}$, or everything is mapped to zero on both sides of the relation (2.1). Then this works with

$$\begin{aligned}[][a_1 - a_2, a_2, \dots, a_n]^- + [a_1, a_2 - a_1, \dots, a_n]^- \\ \mapsto [a_1 - a_2, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n]^- \\ + [a_1, a_2 - a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n]^-\end{aligned}$$

as required.

In case (2), we must have

$$[a_1, a_2, \dots, a_n]^- \mapsto [a_2, \dots, a_n]^-$$

or again everything is mapped to zero on both sides of the relation (2.1). Then we have

$$[a_1 - a_2, a_2, \dots, a_n]^- + [a_1, a_2 - a_1, \dots, a_n]^- \mapsto [a_2, \dots, a_n]^- + 0$$

since $a_2 - a_1 \notin A_{p,n-1}$.

In case (3), we have

$$[a_1, a_2, \dots, a_n]^- \mapsto 0$$

and

$$\begin{aligned}[][a_1 - a_2, a_2, \dots, a_n]^- + [a_1, a_2 - a_1, \dots, a_n]^- \\ \mapsto [a_1 - a_2, a_3, \dots, a_n]^- + [a_2 - a_1, a_3, \dots, a_n]^- = 0\end{aligned}$$

so everything maps to zero no matter what. □

**Lemma 5.** *The maps $\eta_{p,n}^-$ are surjective for primes $p \geq 2$.*

*Proof.* Exactly the same as the case $p = 2$ in Lemma 2 above. □

**Proposition 6.** *For $p$ prime, $\mathcal{B}_n^-((\mathbb{Z}/p\mathbb{Z})^n) \neq 0$ for all $n \in \mathbb{N}$.*

*Proof.* Exactly the same induction as $p = 2$, but with the base case $n = 1$:

$$\mathcal{B}_1^-(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}^{\frac{p-1}{2}} \neq 0.$$

□

**Proposition 7.** *For $p$ prime, $\mathcal{B}_n((\mathbb{Z}/p\mathbb{Z})^n) \neq 0$ for all $n \in \mathbb{N}$.*

*Proof.* By definition, $\mathcal{B}_n^-((\mathbb{Z}/p\mathbb{Z})^n)$ is a quotient of $\mathcal{B}_n((\mathbb{Z}/p\mathbb{Z})^n)$, this combined with Prop. 6 give the required result. □

Everything done above can be generalized to prove $\mathcal{B}_n((\mathbb{Z}/N\mathbb{Z})^n) \neq 0$ for composite $N$ as well, we just need to show

$$\mathcal{B}_1^-(\mathbb{Z}/N\mathbb{Z}) \neq 0,$$

since we did not use the fact that $p$ is prime apart from proving the base case $n = 1$. This can be shown by the fact that

$$\mathcal{B}_1(\mathbb{Z}/N\mathbb{Z}) = \mathbb{Z}^{\phi(N)} \neq 0$$

and the quotient (1.1) will not annihilate the whole module.

## 3. Lattice and cones

sect:lat

This section is heavily linked with [2, Section 4], which gave a geometric interpretation of the elements and defining relations of $\mathcal{M}_n(G)$. The initial problem with trying to find analogues of this lattice and cone structure for $\mathcal{B}_n(G)$ is that the decomposition $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$ always produces $k$ subcones, whereas the relation (B) does not have a consistent number of terms on the right hand side. This problem was resolved by changing the way we identify the triples $(\mathbf{L}, \chi, \Lambda)$ with symbols in $\mathcal{B}_n(G) \otimes \mathbb{Q}$, inspired by the map $\mu$ as discussed above, so that the excess cones are identified with either a torsion element or a zero element in $\mathcal{B}_n(G)$.

We have $n \geq 2$ is an integer, $G$ is a finite abelian group and $A = \mathrm{Hom}(G, \mathbb{C}^\times)$ is the character group of $G$. We can consider equivalence class of triples

$$(\mathbf{L}, \chi, \Lambda),$$

up to isomorphism, where

- $\mathbf{L} \simeq \mathbb{Z}^n$ is a (torsion-free) lattice of rank $n$,
- $\chi$ is an element of $\mathbf{L} \otimes A$, such that the following induced homomorphism is a surjection:

$$\mathbf{L}^\vee \to A, \tag{3.1}$$

- $\Lambda$ is a strictly convex cone in $\mathbf{L}_\mathbb{R}$ spanned by a basis of $\mathbf{L}$, i.e. it is isomorphic to the standard octant $\mathbb{R}^n_{\geq 0}$ for $\mathbf{L} = \mathbb{Z}^n \subset \mathbb{R}^n$.

Now for each symbol

$$[a_1, \ldots, a_n] \in \mathcal{B}_n(G) \otimes \mathbb{Q}$$

we can identify it in terms of

$$\phi(\mathbf{L}, \chi, \Lambda)$$

by choosing a basis $e_1, \dots, e_n$ of $\mathbf{L}$ spanning $\Lambda$, and writing

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i \tag{3.2}$$

we can identify

$$\phi(\mathbf{L}, \chi, \Lambda) = \begin{cases} [a_1, ..., a_n] & \text{if } a_1, ..., a_n \neq 0 \\ \frac{1}{2}[a_1, ..., a_n] & \text{if exactly one of } a_1, ..., a_n \text{ is } 0 \\ [a_1, ..., a_n] & \text{if two or more of } a_1, ..., a_n \text{ is } 0. \end{cases}$$

Here the condition (G) is satisfied via the surjectivity of (3.1), and since the order of the basis does not matter we satisfy (S). We just need to check the blowup condition (B), i.e. it is obeyed by

$$\phi(\mathbf{L}, \chi, \Lambda) = \phi(\mathbf{L}, \chi, \Lambda_1) + \phi(\mathbf{L}, \chi, \Lambda_2) \tag{3.3}$$

where

$$\Lambda_1 := \mathbb{R}_{\geq 0}(e_1 + e_2) + \mathbb{R}_{\geq 0}e_2 + ... + \mathbb{R}_{\geq 0}e_n$$
$$\Lambda_2 := \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(e_1 + e_2) + ... + \mathbb{R}_{\geq 0}e_n$$

and so

$$\Lambda = \Lambda_1 \cup \Lambda_2.$$

Note that (3.3) comes from the fact that we have

$$\begin{cases} \text{in the basis of } \Lambda_1 & : \quad \chi = (e_1 + e_2) \otimes a_1 + e_2 \otimes (a_2 - a_1) + \dots \\ \text{in the basis of } \Lambda_2 & : \quad \chi = e_1 \otimes (a_1 - a_2) + (e_1 + e_2) \otimes a_2 + \dots \end{cases}$$

There are the following cases:

(1) $a_1 \neq a_2$
   (a) $a_1, a_2 \neq 0$
      (i) exactly one of $a_3, ..., a_n$ is 0
      (ii) else
   (b) WLOG $a_1 = 0$
      (i) $a_3, ..., a_n \neq 0$
      (ii) exactly one of $a_3, ..., a_n$ is 0
      (iii) two or more of $a_3, ..., a_n$ is 0
(2) $a_1 = a_2 = a$
   (a) $a \neq 0$
      (i) $a_3, ..., a_n \neq 0$
      (ii) exactly one of $a_3, ..., a_n$ is 0
      (iii) two or more of $a_3, ..., a_n$ is 0
   (b) $a = 0$

Each case is checked below:

In Case 1(a)i we have

$$\begin{aligned}[a_1, a_2, ...] &= 2\phi(\mathbf{L}, \chi, \Lambda) \\ &= 2\phi(\mathbf{L}, \chi, \Lambda_1) + 2\phi(\mathbf{L}, \chi, \Lambda_2) \\ &= [a_1, a_2 - a_1, ...] + [a_1 - a_2, a_2, ...]\end{aligned}$$

In Case 1(a)ii this is the same as the case for $\mathcal{M}_n(G)$.

In Case 1(b)i we have

$$\begin{aligned}[0, a_2, ...] &= 2\phi(\mathbf{L}, \chi, \Lambda) \\ &= 2\phi(\mathbf{L}, \chi, \Lambda_1) + 2\phi(\mathbf{L}, \chi, \Lambda_2) \\ &= [0, a_2, ...] + 2[-a_2, a_2, ...]\end{aligned}$$

which holds since $[-a_2, a_2, ...] = 0$ by the lemma below.

**Lemma 8.** $[-a, a, ...] = 0$ *for any* $a \in A$.

*Proof.* We have the relation

$$[0, a, ...] = [-a, a, ...] + [0, a, ...]$$

by the blowup condition (B), subtract $[0, a, ...]$ from both sides to get the required result. □

In Case 1(b)ii we have

$$\begin{aligned}[0, a_2, ..., 0, ...] &= \phi(\mathbf{L}, \chi, \Lambda) \\ &= \phi(\mathbf{L}, \chi, \Lambda_1) + \phi(\mathbf{L}, \chi, \Lambda_2) \\ &= [0, a_2, ..., 0, ...] + \frac{1}{2}[-a_2, a_2, ..., 0, ...]\end{aligned}$$

which holds since $[-a_2, a_2, ...] = 0$ by Lemma 8.

In Case 1(b)iii this is the same as the case for $\mathcal{M}_n(G)$.

In Case 2(a)i we have

$$\begin{aligned}[a, a, ...] &= \phi(\mathbf{L}, \chi, \Lambda) \\ &= \phi(\mathbf{L}, \chi, \Lambda_1) + \phi(\mathbf{L}, \chi, \Lambda_2) \\ &= \frac{1}{2}[0, a, ...] + \frac{1}{2}[a, 0, ...] \\ &= [0, a, ...]\end{aligned}$$

as required.

In Case 2(a)ii we have

$$\begin{aligned}[a,a,...,0,...] &= 2\phi(\mathbf{L},\chi,\Lambda)\\ &= 2\phi(\mathbf{L},\chi,\Lambda_1)+2\phi(\mathbf{L},\chi,\Lambda_2)\\ &= 2[0,a,...,0,...]+2[a,0,...,0,...]\\ &= 4[0,a,...,0,...]\end{aligned}$$

this holds since $[a,a,...,0,...] = [0,a,\ldots,0,...] = [0,0,a,...]$ so both sides are torsion elements.

In Case 2(a)iii this is the same as the case for $\mathcal{M}_n(G)$.

In Case 2(b) this is the same as the case for $\mathcal{M}_n(G)$.

We will now give a few explicit examples of this structure. Consider the group

$$G = \mathbb{Z}/3\mathbb{Z}$$

which has characters

$$\{0, a, 2a\}.$$

In the case of $n = 3$, let

$$\begin{aligned}&\mathbf{L} = \mathbb{Z}^3, \quad e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}\\ &\Lambda = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}e_3.\end{aligned}$$

Now if we let

$$\chi_1 = e_1 \otimes a + e_2 \otimes 2a + e_3 \otimes a$$

then we have

$$\phi(\mathbf{L},\chi_1,\Lambda) = [a,2a,a] \in \mathcal{B}_3(\mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Q}.$$

The relation

$$[a,2a,a] = [2a,2a,a] + [a,a,a]$$

exactly corresponds to

$$\phi(\mathbf{L},\chi_1,\Lambda) = \phi(\mathbf{L},\chi_1,\Lambda_1) + \phi(\mathbf{L},\chi_1,\Lambda_2)$$

where

$$\begin{aligned}\Lambda_1 &:= \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(e_1+e_2) + \mathbb{R}_{\geq 0}e_3\\ \Lambda_2 &:= \mathbb{R}_{\geq 0}(e_1+e_2) + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}e_3.\end{aligned}$$

Geometrically this represents the cone $\Lambda$ splitting into two cones along the plane spanned by $(e_1 + e_2)$ and $e_3$. Now in the basis of $\Lambda_1$ we have

$$\begin{aligned}\chi_1 &= e_1 \otimes (a - 2a) + (e_1 + e_2) \otimes 2a + e_3 \otimes a \\ &= e_1 \otimes 2a + (e_1 + e_2) \otimes 2a + e_3 \otimes a\end{aligned}$$

which gives

$$\phi(\mathbf{L}, \chi_1, \Lambda_1) = [2a, 2a, a]$$

and similarly we have in the basis of $\Lambda_2$

$$\begin{aligned}\chi_1 &= (e_1 + e_2) \otimes a + e_2 \otimes (2a - a) + e_3 \otimes a \\ &= (e_1 + e_2) \otimes a + e_2 \otimes a + e_3 \otimes a\end{aligned}$$

which gives

$$\phi(\mathbf{L}, \chi_1, \Lambda_2) = [a, a, a]$$

as required. So far this is exactly the same as the case for $\mathcal{M}_3(\mathbb{Z}/3\mathbb{Z})$, so we will look at the relation

$$[a, a, 2a] = [0, a, 2a]$$

which does not hold in $\mathcal{M}_3(\mathbb{Z}/3\mathbb{Z})$, this should correspond to

$$\phi(\mathbf{L}, \chi_1, \Lambda) = \phi(\mathbf{L}, \chi_1, \Lambda_3) + \phi(\mathbf{L}, \chi_1, \Lambda_4)$$

where

$$\begin{aligned}\Lambda_3 &:= \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0}(e_1 + e_3) \\ \Lambda_4 &:= \mathbb{R}_{\geq 0}(e_1 + e_3) + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3.\end{aligned}$$

This time the cone $\Lambda$ is split along the plane spanned by $(e_1 + e_3)$ and $e_2$. In the basis of $\Lambda_3$, we get

$$\begin{aligned}\chi_1 &= e_1 \otimes (a - a) + e_2 \otimes 2a + (e_1 + e_3) \otimes a \\ &= e_1 \otimes 0 + e_2 \otimes 2a + (e_1 + e_3) \otimes a\end{aligned}$$

since there is a 0 in the above expansion, this gives

$$\phi(\mathbf{L}, \chi_1, \Lambda_3) = \frac{1}{2}[0, 2a, a]$$

and in the basis of $\Lambda_4$

$$\begin{aligned}\chi_1 &= (e_1 + e_3) \otimes a + e_2 \otimes 2a + e_3 \otimes (a - a) \\ &= (e_1 + e_3) \otimes a + e_2 \otimes 2a + e_3 \otimes 0\end{aligned}$$

which gives

$$\phi(\mathbf{L}, \chi_1, \Lambda_4) = \frac{1}{2}[a, 2a, 0].$$

Using the symmetry condition (S) we have

$$\phi(\mathbf{L}, \chi_1, \Lambda_3) + \phi(\mathbf{L}, \chi_1, \Lambda_4) = \frac{1}{2}[a, 2a, 0] + \frac{1}{2}[0, 2a, a] = [0, a, 2a]$$

exactly as expected.

Now let's have look at a case where there is a 0 in the symbol

$$\chi_2 = e_1 \otimes 0 + e_2 \otimes a + e_3 \otimes a$$

which is identified with

$$\phi(\mathbf{L}, \chi_2, \Lambda) = \frac{1}{2}[0, a, a].$$

There is a few relations we can check, first consider

$$[0, a, a] = [2a, a, a] + [0, a, a]$$

we expect this to correspond to

$$2\phi(\mathbf{L}, \chi_1, \Lambda) = 2\phi(\mathbf{L}, \chi_1, \Lambda_1) + 2\phi(\mathbf{L}, \chi_1, \Lambda_2)$$

where $\Lambda_1$ and $\Lambda_2$ is as above. We have in the basis of $\Lambda_1$

$$\begin{aligned}\chi_2 &= e_1 \otimes (0 - a) + (e_1 + e_2) \otimes a + e_3 \otimes a \\ &= e_1 \otimes 2a + (e_1 + e_2) \otimes a + e_3 \otimes a\end{aligned}$$

which gives

$$2\phi(\mathbf{L}, \chi_2, \Lambda_1) = 2[2a, a, a] = [2a, a, a]$$

where the second equality is from the fact that $[2a, a, a] = [-a, a, a] = 0$ by Lemma 8, and in the basis of $\Lambda_2$

$$\begin{aligned}\chi_2 &= (e_1 + e_2) \otimes 0 + e_2 \otimes (a - 0) + e_3 \otimes a \\ &= (e_1 + e_2) \otimes 0 + e_2 \otimes a + e_3 \otimes a\end{aligned}$$

which gives

$$2\phi(\mathbf{L}, \chi_2, \Lambda_2) = [0, a, a]$$

as required. Now consider the relation

$$[a, a, 0] = [0, a, 0]$$

which also tells us that this is a torsion element in $\mathcal{B}_3(\mathbb{Z}/3\mathbb{Z})$ and therefore trivial in $\mathcal{B}_3(\mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Q}$, this should correspond to

$$2\phi(\mathbf{L}, \chi_1, \Lambda) = 2\phi(\mathbf{L}, \chi_1, \Lambda_5) + 2\phi(\mathbf{L}, \chi_1, \Lambda_6)$$

where

$$\begin{aligned}\Lambda_5 &:= \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0}(e_2 + e_3) \\ \Lambda_6 &:= \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0}(e_2 + e_3) + \mathbb{R}_{\geq 0} e_3.\end{aligned}$$

In the basis of $\Lambda_5$ we have

$$\begin{aligned}\chi_2 &= e_1 \otimes 0 + e_2 \otimes (a - a) + (e_2 + e_3) \otimes a \\ &= e_1 \otimes 0 + e_2 \otimes 0 + (e_2 + e_3) \otimes a\end{aligned}$$

which gives

$$2\phi(\mathbf{L}, \chi_2, \Lambda_5) = 2[0, 0, a]$$

and by symmetry we have

$$2\phi(\mathbf{L}, \chi_2, \Lambda_6) = 2[0, 0, a].$$

So we get the relation

$$[a, a, 0] = 4[0, a, 0] = [0, a, 0]$$

which holds since all elements are trivial.

sect:multi

## 4. Multiplication and co-multiplication

In this section is heavily linked with [2, Section 5], which gives multiplication and co-multiplication maps defined on $\mathcal{M}_n(G)$ and $\mathcal{M}_n^-(G)$. Here we find analogous maps for $\mathcal{B}_n(G) \otimes \mathbb{Q}$, and show these new maps are well defined and compatible with defining relations. Note that in this section all $\mathcal{B}_n(G)$ are tensored with $\mathbb{Q}$, so that it is a $\mathbb{Q}$-vector space.

Now given a short exact sequence of finite abelian groups

$$0 \to G' \to G \to G'' \to 0$$

this induces short exact sequence of character groups

$$0 \to A'' \to A \to A' \to 0$$

since the pullback of $G_2 \to \mathbb{C}^\times$ by $G_1 \to G_2$ gives $A_2 \to A_1$, where $A_i = G_i^\vee = \mathrm{Hom}(G_i, \mathbb{C}^\times)$. Letting

$$n = n' + n'', \quad n', n'' \geq 1,$$

we can define a $\mathbb{Z}$-bilinear 'multiplication' map

$$\nabla_B : \mathcal{B}_{n'}(G') \otimes \mathcal{B}_{n''}(G'') \to \mathcal{B}_{n'+n''}(G)$$

which is defined by

$$[a'_1, \ldots, a'_{n'}] \otimes [a''_1, \ldots, a''_{n''}] \mapsto \sum [a_1, \ldots, a_{n'}, a''_1, \ldots, a''_{n''}]$$

where the sum is over all lifts $a_i \in A$ of $a'_i \in A'$ from the short exact sequence, and the elements $a''_i \in A$ via $A'' \hookrightarrow A$.

The compatibility with defining relations (S) and (B) are obvious for exactly the same reasons as [2, Section 5]. This map descends into a $\mathbb{Z}$-bilinear map via (1.1)

$$\nabla_B^- : \mathcal{B}_{n'}^-(G') \otimes \mathcal{B}_{n''}^-(G'') \to \mathcal{B}_{n'+n''}^-(G)$$

where both $G'$ and $G''$ are nontrivial.

We can also define the corresponding 'co-multiplication' map

$$\Delta_B : \mathcal{B}_{n'+n''}(G) \to \mathcal{B}_{n'}(G') \otimes \mathcal{B}_{n''}^-(G'')$$

by

$$[a_1, ..., a_n] \mapsto \begin{cases} \sum [a_{I'} \bmod A''] \otimes [a_{I''}]^- & a_1, ..., a_n \neq 0 \\ 2\sum [a_{I'} \bmod A''] \otimes [a_{I''}]^- & \text{exactly one of } a_1, ..., a_n \text{ is } 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$[a_{I'} \bmod A''] = [a_{i_1} \bmod A'', \dots, a_{i_{n'}} \bmod A''], \quad I' := \{i_1, \dots, i_{n'}\}$$
$$[a_{I''}]^- = [a_{j_1}, \dots, a_{j_{n''}}]^-, \quad I'' := \{j_1, \dots, j_{n''}\}.$$

The sum is over all subdivisions

$$\{1, \dots, n\} = I' \sqcup I'', \quad \text{such that } \#I' = n', \#I'' = n'',$$

satisfying $a_j \in A''$, for all $j \in I''$ and $a_j$, $j \in I''$, generate $A''$.

**Proposition 9.** *The map $\Delta_B$ extends to a well-defined $\mathbb{Z}$-linear homomorphism.*

*Proof.* The proof is similar to the proof of Proposition 11 in [2], except this time there is more cases to deal with. We just need to check that the relation (B) is compatible. We only care about cases where $a_1, a_2$ are in different sides of the tensor product, since all other cases are trivial as it is the consequence of tensor products. These are

$$(4.1) \quad [a_1, a_2, \dots] \mapsto \lambda\Big(\delta^{gen}_{a_1 \in A''} \cdot [a_2 \bmod A'', ...] \otimes [a_1, ...]^- + \delta^{gen}_{a_2 \in A''} \cdot [a_1 \bmod A'', ...] \otimes [a_2, ...]^-\Big)$$

where

$$\lambda = \begin{cases} 1 & a_1, ..., a_n \neq 0 \\ 2 & \text{exactly one of } a_1, ..., a_n \text{ is } 0 \\ 0 & \text{otherwise} \end{cases}$$

and for $a \in A$,

$$\delta^{gen}_{a \in A''} = \begin{cases} 1 & a \in A'' \text{ and } \mathbb{Z}a + \sum_{j \in J''} \mathbb{Z}a_j = A'' \\ 0 & \text{otherwise} \end{cases}$$

where

$$J' := I' \cap \{3, \dots, n\}, \quad J'' := I'' \cap \{3, \dots, n\}$$

of cardinality $n' - 1$ and $n'' - 1$ respectively.

First suppose $a_1 \neq a_2$, there are three cases:

(1) $a_3, ..., a_n \neq 0$
(2) exactly one of $a_3, ..., a_n$ is 0
(3) more than two of $a_3, ..., a_n$ is 0

In Case 1, if $a_1, a_2 \neq 0$, then this is the same as $\mathcal{M}_n(G)$. So assume WLOG $a_1 = 0, a_2 \neq 0$, the relation is then

$$[0, a_2, \dots] = [0, a_2, \dots] + [-a_2, a_2, \dots]$$

but the last term on the right is mapped to zero since

$$[-a_2, a_2, ...] \mapsto \delta^{gen}_{a_2 \in A''} \cdot [0, ...] \otimes [a_2, ...]^- + \delta^{gen}_{-a_2 \in A''} \cdot [0, ...] \otimes [-a_2, ...]^- = 0.$$

In Case 2, if $a_1, a_2 \neq 0$ then we have basically the same case as $\mathcal{M}_n(G)$ since every term involved contains exactly one zero, except the image of everything is effectively doubled due to $\lambda = 2$. So we can assume WLOG that $a_1 = 0, a_2 \neq 0$, the relation is then

$$[0, a_2, \dots, 0, \dots] = [0, a_2, \dots, 0, \dots] + [-a_2, a_2, \dots, 0, \dots]$$

again the last term is mapped to zero since

$$[-a_2, a_2, ...] \mapsto 2\Big(\delta^{gen}_{a_2 \in A''} \cdot [0, ...] \otimes [a_2, ...]^- + \delta^{gen}_{-a_2 \in A''} \cdot [0, ...] \otimes [-a_2, ...]^-\Big) = 0.$$

In Case 3, everything maps to zero.

So we can assume $a_1 = a_2 = a$. Furthermore if $a = 0$ then everything is mapped to zero, so we can assume further $a \neq 0$. We have the following cases:

(1) $a \in A''$
  (a) $a_3, ..., a_n \neq 0$
  (b) exactly one of $a_3, ..., a_n$ is 0
  (c) two or more of $a_3, ..., a_n$ is 0
(2) $a \notin A''$

In Case 1(a), we have

$$[a, a, ...] \mapsto 2\delta^{gen}_{a\in A''} \cdot [0, ...] \otimes [a, ...]^-$$

and

$$[0, a, ...] \mapsto 2\Big(\delta^{gen}_{0\in A''} \cdot [0, ...] \otimes [0, ...]^- + \delta^{gen}_{a\in A''} \cdot [0, ...] \otimes [a, ...]^-\Big).$$

Clearly they are equal.

In Case 1(b), WLOG let $a_3$ be the zero element, then we have

$$[a, a, ...] \mapsto 4\delta^{gen}_{a\in A''} \cdot [0, ...] \otimes [a, ...]^-.$$

Now if $a_3 = 0$ is in left side of the tensor product, then the term is a torsion element due to [1, Prop 7.1]. If $a_3 = 0$ is in the right side, then the whole term disappears since $[0, ...]^- \in B^-_{n''}$ is equal to 0. This is consistent with

$$[0, a, ...] \mapsto 0.$$

In Case 1(c), everything maps to zero.

In Case 2, we have

$$[a, a, ...] \mapsto 0$$

and

$$[0, a, ...] \mapsto \delta^{gen}_{0\in A''} \cdot [a \bmod A'', ...] \otimes [0, ...]^- = 0$$

again everything maps to 0. □

We will now give a few explicit examples of this map. Consider the short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Z}/30\mathbb{Z} \xrightarrow{g} \mathbb{Z}/15\mathbb{Z} \to 0$$

letting $g_k$ denote the generator of $\mathbb{Z}/k\mathbb{Z}$, the maps $f, g$ are defined on generators by

$$f : g_2 \mapsto g_{30}^{15}$$
$$g : g_{30} \mapsto g_{15}.$$

This is a short exact sequence since

$$\mathrm{im}(f) = \langle g_{30}^{15}\rangle = \ker(g),$$

with $f$ being an injective map and $g$ being an surjective map. Now let $A_k$ denote the character group of $\mathbb{Z}/k\mathbb{Z}$ with generator $\chi_k$, we get the short exact sequence

$$0 \to A_{15} \xrightarrow{g^*} A_{30} \xrightarrow{f^*} A_2 \to 0$$

by the pullbacks $f^*, g^*$ which are defined by

$$\begin{aligned} g^* :& \chi_{15} \mapsto 2\chi_{30} \\ f^* :& \chi_{30} \mapsto \chi_2. \end{aligned}$$

so that

$$\begin{aligned} f^*(\chi_{15})(g) &= \chi_{15}(f(g)) \\ g^*(\chi_{30})(g) &= \chi_{30}(f(g)) \end{aligned}$$

is satisfied.

So we can now calculate the 'multiplication' and 'co-multiplication' maps induced from the above short exact sequence. Let

$$n' = 1, \quad n'' = 2$$

we will first look at the map

$$\nabla_B : \mathcal{B}_1(\mathbb{Z}/2\mathbb{Z}) \otimes \mathcal{B}_2(\mathbb{Z}/15\mathbb{Z}) \to \mathcal{B}_3(\mathbb{Z}/30\mathbb{Z}).$$

The image of the element

$$[\chi_2] \otimes [3\chi_{15}, 13\chi_{15}]$$

under $\nabla_B$ is

$$\sum_{i=0}^{14} [(2i+1)\chi_{30}, 6\chi_{30}, 26\chi_{30}].$$

Now let's check if $\nabla_B$ is compatible with the following relation

$$\begin{aligned} [\chi_2] \otimes [3\chi_{15}, 13\chi_{15}] &= [\chi_2] \otimes ([20\chi_{15}, 13\chi_{15}] + [3\chi_{15}, 10\chi_{15}]) \\ &= [\chi_2] \otimes [20\chi_{15}, 13\chi_{15}] + [\chi_2] \otimes [3\chi_{15}, 10\chi_{15}], \end{aligned}$$

we have

$$\nabla_B([\chi_2] \otimes [20\chi_{15}, 13\chi_{15}]) = \sum_{i=0}^{14} [(2i+1)\chi_{30}, 10\chi_{30}, 22\chi_{30}]$$

and

$$\nabla_B([\chi_2] \otimes [3\chi_{15}, 10\chi_{15}]) = \sum_{i=0}^{14} [(2i+1)\chi_{30}, 6\chi_{30}, 20\chi_{30}].$$

But since we have for all $i$

$$\begin{aligned} &[(2i+1)\chi_{30}, 6\chi_{30}, 26\chi_{30}] \\ = &[(2i+1)\chi_{30}, 10\chi_{30}, 22\chi_{30}] + [(2i+1)\chi_{30}, 6\chi_{30}, 20\chi_{30}] \end{aligned}$$

the above relation holds under $\nabla_B$. This example illustrates how the compatibility with defining relations (S) and (B) is obvious, as a consequence of tensor products.

Next we will look at the 'co-multiplication' map

$$\Delta_B : \mathcal{B}_3(\mathbb{Z}/30\mathbb{Z}) \to \mathcal{B}_1(\mathbb{Z}/2\mathbb{Z}) \otimes \mathcal{B}_2^-(\mathbb{Z}/15\mathbb{Z}).$$

The image of the element

$$[\chi_{30}, 6\chi_{30}, 26\chi_{30}]$$

under $\Delta_B$ is

$$[\chi_{30} \mod A_{15}] \otimes [3\chi_{15}, 13\chi_{15}]^- = [\chi_2] \otimes [3\chi_{15}, 13\chi_{15}]^-.$$

We can check the case of a relation where $a_1, a_2$ are distributed over different factors

$$[\chi_{30}, 6\chi_{30}, 26\chi_{30}] = [25\chi_{30}, 6\chi_{30}, 26\chi_{30}] + [\chi_{30}, 5\chi_{30}, 26\chi_{30}]$$

we have

$$\Delta_B([25\chi_{30}, 6\chi_{30}, 26\chi_{30}]) = [\chi_2] \otimes [3\chi_{15}, 13\chi_{15}]^-$$

and

$$\Delta_B([\chi_{30}, 5\chi_{30}, 26\chi_{30}]) = 0$$

which means the above relation holds. So far this is exactly the same as the case for $\mathcal{M}_n(G)$, so let's check the relation

$$[2\chi_{30}, 2\chi_{30}, 3\chi_{30}] = [0, 2\chi_{30}, 3\chi_{30}].$$

We have

$$\Delta_B([2\chi_{30}, 2\chi_{30}, 3\chi_{30}]) = [\chi_2] \otimes [\chi_{15}, \chi_{15}]^- = [\chi_2] \otimes [0, \chi_{15}]^- = 0$$

and

$$\Delta_B([0, 2\chi_{30}, 3\chi_{30}]) = [\chi_2] \otimes [0, \chi_{15}]^- = 0$$

so the image of both sides are trivial, which means the relation holds.

Now let's look at a different map in order to show the case where there is a 0 in the symbol

$$\Delta_B : \mathcal{B}_4(\mathbb{Z}/30\mathbb{Z}) \to \mathcal{B}_2(\mathbb{Z}/2\mathbb{Z}) \otimes \mathcal{B}_2^-(\mathbb{Z}/15\mathbb{Z}).$$

i.e. $n' = n'' = 2$. Consider the relation

$$[2\chi_{30}, 2\chi_{30}, \chi_{30}, 0] = [0, 2\chi_{30}, \chi_{30}, 0].$$

We have

$$\begin{aligned}
&\Delta_B([2\chi_{30}, 2\chi_{30}, \chi_{30}, 0]) \\
&= 2([\chi_2, 0] \otimes [\chi_{15}, \chi_{15}]^- + [\chi_2, 0] \otimes [0, \chi_{15}]^- + [\chi_2, 0] \otimes [0, \chi_{15}]^-) \\
&= 2[\chi_2, 0] \otimes [\chi_{15}, \chi_{15}]^- \\
&= 2[\chi_2, 0] \otimes [0, \chi_{15}]^- = 0
\end{aligned}$$

and

$$\Delta_B([0, 2\chi_{30}, \chi_{30}, 0]) = 0$$

so the relation holds. Now for a case where $a_1 \neq a_2$, consider

$$[2\chi_{30}, \chi_{30}, 2\chi_{30}, 0] = [\chi_{30}, \chi_{30}, 2\chi_{30}, 0] + [2\chi_{30}, 29\chi_{30}, 2\chi_{30}, 0]$$

we have

$$\Delta_B([\chi_{30}, \chi_{30}, 2\chi_{30}, 0]) = 2[\chi_2, \chi_2] \otimes [\chi_{15}, 0]^- = 0$$

and

$$\Delta_B([2\chi_{30}, 29\chi_{30}, 2\chi_{30}, 0]) = 2[\chi_2, \chi_2] \otimes [\chi_{15}, \chi_{15}]^- = 0$$

which works too.

## 5. Explanation of proofs of Theorem 12 and 14 in [2]

sect: explain

In this section we give a more detailed explanation of the proofs of Theorem 12 and 14 found in [2, Section 5]. In particular some of the non-trivial steps of the proofs will be proven in more detail, with concrete examples at the end showing the proofs more explicitly.

We have the following multiplication and co-multiplication maps

$$\begin{aligned}
\Delta^- &: \mathcal{M}^-_{n'+n''}(G) \to \mathcal{M}^-_{n'}(G') \otimes \mathcal{M}^-_{n''}(G'') \\
\nabla^- &: \mathcal{M}^-_{n'}(G') \otimes \mathcal{M}^-_{n''}(G'') \to \mathcal{M}^-_{n'+n''}(G)
\end{aligned}$$

given a short exact sequence of finite abelian groups

$$0 \to G' \to G \to G'' \to 0$$

and their corresponding character groups.

We can obtain a chain complex from these maps induced from simplicial complexes, in both directions: let $\mathcal{G}_\bullet$ be a flag of subgroups of length $r$

$$0 = G_{\leq 0} \subsetneq G_{\leq 1} \subsetneq \cdots \subsetneq G_{\leq r} = G.$$

This means that we get the following diagram

$$\mathcal{M}_n^-(G) \rightleftarrows \bigoplus_{\substack{n_1+n_2=n \\ \mathcal{G}_\bullet \text{ of lengths } 2}} \mathcal{M}_{n_1}^-(gr_1(\mathcal{G}_\bullet)) \otimes \mathcal{M}_{n_2}^-(gr_2(\mathcal{G}_\bullet))$$

$$\rightleftarrows \bigoplus_{\substack{n_1+n_2+n_3=n \\ \mathcal{G}_\bullet \text{ of lengths } 3}} \mathcal{M}_{n_1}^-(gr_1(\mathcal{G}_\bullet)) \otimes \mathcal{M}_{n_2}^-(gr_2(\mathcal{G}_\bullet)) \otimes \mathcal{M}_{n_3}^-(gr_3(\mathcal{G}_\bullet)) \rightleftarrows \dots$$

where $gr_i(\mathcal{G}_\bullet) = G_{\leq i}/G_{\leq i-1}$, which is well defined since everything is abelian and hence all subgroups are normal. The left and right arrows are obtained via the following: given

$$\cdots \subsetneq G_i \subsetneq G_{i+1} \subsetneq G_{i+2} \subsetneq \cdots$$

which is part of some flag $\mathcal{G}_\bullet$ of length $s \geq 2$, we get the short exact sequence

$$0 \to G_{i+1}/G_i \to G_{i+2}/G_i \to G_{i+2}/G_{i+1} \to 0$$

which is due to the third isomorphism theorem

$$(G_{i+2}/G_i)/(G_{i+1}/G_i) \cong G_{i+2}/G_{i+1}.$$

Write

$$G_{i+1}/G_i = gr_{i+1}(\mathcal{G}_\bullet), \quad G_{i+2}/G_{i+1} = gr_{i+2}(\mathcal{G}_\bullet), \quad G_{i+2}/G_i = gr_{i+1}(\mathcal{H}_\bullet)$$

where $\mathcal{H}_\bullet = \mathcal{G}_\bullet \setminus \{G_{i+1}\}$ is the flag of length $s-1$ obtained by removing the group $G_{i+1}$, then the short exact sequence will give us the homomorphisms

$$\Delta^- : \mathcal{M}_{n'+n''}^-(gr_{i+1}(\mathcal{H}_\bullet)) \to \mathcal{M}_{n'}^-(gr_{i+1}(\mathcal{G}_\bullet)) \otimes \mathcal{M}_{n''}^-(gr_{i+2}(\mathcal{G}_\bullet))$$
$$\nabla^- : \mathcal{M}_{n'}^-(gr_{i+1}(\mathcal{G}_\bullet)) \otimes \mathcal{M}_{n''}^-(gr_{i+2}(\mathcal{G}_\bullet)) \to \mathcal{M}_{n'+n''}^-(gr_{i+1}(\mathcal{H}_\bullet)).$$

where if $n' + n'' = 1$ then we have the trivial map. This fits into the complex via

$$\cdots \otimes \mathcal{M}_{n'}^-(gr_{i+1}(\mathcal{G}_\bullet)) \otimes \mathcal{M}_{n''}^-(gr_{i+2}(\mathcal{G}_\bullet)) \otimes \cdots \in C^{s-1,-}(G,n), C_{s-1}^-(G,n)$$
$$\cdots \otimes \mathcal{M}_{n'+n''}^-(gr_{i+1}(\mathcal{H}_\bullet)) \otimes \cdots \in C^{s-2,-}(G,n), C_{s-2}^-(G,n),$$

where $C^{s,-}(G,n), C_s^-(G,n)$ is the $s$-th term in the diagram of homomorphism above.

Now we give a simplicial complex structure to the whole diagram. Consider each flag of subgroups $\mathcal{G}_\bullet$ of length $r$ as an oriented $r$-simplex

$$(G_0, G_1, \dots, G_r)$$

we can get the boundary map to oriented $(r-1)$-simplexes, i.e. flags of length $r-1$,

$$(G_0, G_1, \dots, G_r) \mapsto \sum_{i=1}^{r-1} (-1)^i (G_0, \dots, \not{G_i}, \dots, G_r).$$

Using the maps $\nabla^-$ as discussed above, with $\mathcal{H}_\bullet$ being each flag of length $r-1$ on the right, this boundary map generates the left arrows in the diagram, which we denote by the differential $d_{\nabla^-}$. It satisfies the condition $\partial_{r-1}\partial_r = 0$ for $d_{\nabla^-}$ due to the simplicial complex structure, so we obtain a chain complex

$$\mathcal{C}_\bullet^-(G, n).$$

We can also get the corresponding boundary map to oriented $(r+1)$-simplexes, i.e. flags of length $r+1$,

$$(G_0, G_1, \dots, G_r) \mapsto \sum (-1)^i (G_0, \dots, G_i, G_k, G_{i+1}, \dots, G_r).$$

where the sum is over all possible ways $\mathcal{G}_\bullet$ can be extended into a flag of length $r+1$ by adding a subgroup $G_i \subsetneq G_k \subsetneq G_{i+1}$. Again we can use the maps $\Delta^-$ as above, to generate the right arrows in the diagram, which we denote by the differential $d_{\Delta^-}$. It satisfies $\partial^r\partial^{r-1} = 0$ for $d_{\Delta^-}$ since each term

$$(G_0, \dots, G_i, G_k, G_{i+1}, \dots G_j, G_l, G_{j+1}, \dots, G_r)$$

in the image of $\partial^r\partial^{r-1}$ occurs twice, once with sign $(-1)^{i+j}$ and once with sign $(-1)^{i+j+1}$. So we again obtain a chain complex

$$\mathcal{C}^{\bullet,-}(G, n).$$

**Theorem 10.** [2, Theorem 12] *Let $G$ be a finite cyclic group. Then the cohomology of both complexes*

$$\mathcal{C}^{\bullet,-}(G, n), \quad \mathcal{C}_\bullet^-(G, n)$$

*after tensoring by $\mathbb{Q}$, is concentrated in degree 0.*

*Proof.* We will not need to use the assumption that $G$ is a finite cyclic group until the last step of the proof, since all constructions prior does not reply on this assumption.

First we define

$$\mathcal{M}_n^\sim(G)$$

to be $\mathcal{S}_n(G) \otimes \mathbb{Q}$ with the additional condition that $a_j \neq 0$ for all $j$, i.e. the $\mathbb{Q}$-vector space generated by symbols of the form

$$\langle a_1, \dots, a_n \rangle^\sim$$

which satisfy the symmetry condition (S) and the generation condition (G) with $a_j \neq 0$ for all $j$. We get a natural linear map between $\mathbb{Q}$-vector spaces

$$\mathcal{M}_n^{\sim}(G) \to \mathcal{M}_n^{-}(G) \otimes \mathbb{Q},$$

defined on generators by

$$\langle a_1, \dots, a_n \rangle^{\sim} \mapsto \langle a_1, \dots, a_n \rangle^{-}.$$

We can define the co-multiplication

$$\Delta^{\sim} : \mathcal{M}^{\sim}_{n'+n''}(G) \to \mathcal{M}^{\sim}_{n'}(G') \otimes \mathcal{M}^{\sim}_{n''}(G'')$$

defined similarly to $\Delta^{-}$ by

$$\langle a_1, \dots, a_n \rangle^{\sim} \mapsto \sum \langle a_{I'} \mod A'' \rangle^{\sim} \otimes \langle a_{I''} \rangle^{\sim}$$

where $I'$, $I'' \subsetneq I$ are nonempty subsets such that $I' \sqcup I'' = \{1, \dots, n\}$, and $I''$ satisfy $a_i \in A''$ for all $i \in I''$ and $\sum_{i \in I''} \mathbb{Z} a_i = A''$. This map is well defined due to the same reason as $\Delta^{-}$.

We can also define the multiplication

$$\nabla^{\sim} : \mathcal{M}^{\sim}_{n'}(G') \otimes \mathcal{M}^{\sim}_{n''}(G'') \to \mathcal{M}^{\sim}_{n'+n''}(G).$$

defined similarly to $\nabla^{-}$ by

$$\langle a'_1, \dots, a'_{n'} \rangle \otimes \langle a''_1, \dots, a''_{n''} \rangle \mapsto \sum \langle a_1, \dots, a_{n'}, a''_1, \dots, a''_{n''} \rangle$$

where the sum runs over all lifts $a_i \in A$ of $a'_i \in A'$ from the short exact sequence, and the elements $a''_i \in A$ via $A'' \hookrightarrow A$. Again this map is well defined due to the same reason as $\nabla^{-}$.

Using the same construction as above, we get the following diagram of homomorphisms

$$\mathcal{M}_n^{\sim}(G) \rightleftarrows \bigoplus_{\substack{n_1+n_2=n \\ \mathcal{G}_\bullet \text{ of lengths } 2}} \mathcal{M}^{\sim}_{n_1}(gr_1(\mathcal{G}_\bullet)) \otimes \mathcal{M}^{\sim}_{n_2}(gr_2(\mathcal{G}_\bullet))$$

$$\rightleftarrows \bigoplus_{\substack{n_1+n_2+n_3=n \\ \mathcal{G}_\bullet \text{ of lengths } 3}} \mathcal{M}^{\sim}_{n_1}(gr_1(\mathcal{G}_\bullet)) \otimes \mathcal{M}^{\sim}_{n_2}(gr_2(\mathcal{G}_\bullet)) \otimes \mathcal{M}^{\sim}_{n_3}(gr_3(\mathcal{G}_\bullet)) \rightleftarrows \dots$$

which gives us two chain complexes

$$\mathcal{C}^{\bullet,\sim}(G, n), \quad \mathcal{C}^{\sim}_{\bullet}(G, n)$$

with differentials $d_{\Delta^{\sim}}$ and $d_{\nabla^{\sim}}$ respectively. There are natural linear maps

$$g^{\bullet} : \mathcal{C}^{\bullet,\sim}(G, n) \twoheadrightarrow \mathcal{C}^{\bullet,-}(G, n) \otimes \mathbb{Q}, \quad g_{\bullet} : \mathcal{C}^{\sim}_{\bullet}(G, n) \twoheadrightarrow \mathcal{C}^{-}_{\bullet}(G, n) \otimes \mathbb{Q}$$

induced by

$$\langle a_1, \ldots, a_{n_i} \rangle^{\sim} \mapsto \langle a_1, \ldots, a_{n_i} \rangle^{-}$$

which is surjective since nothing in $\mathcal{M}_n^{\sim}(G)$ contains 0, and $\langle a_1, \ldots, a_{n_i} \rangle^{-} = 0$ if at least one $a_j = 0$ for some $j$, since then it is a torsion element of order 2 in $\mathcal{M}_n^{-}(G)$. This also tells us that these maps are compatible with their differentials, so that the following diagrams commutes

$$\begin{array}{ccc}
\mathcal{C}^{s,\sim}(G,n) & \xrightarrow{d_{\Delta^{\sim}}} & \mathcal{C}^{s+1,\sim}(G,n) \\
\Big\downarrow g^{\bullet} & & \Big\downarrow g^{\bullet} \\
\mathcal{C}^{s,-}(G,n) \otimes \mathbb{Q} & \xrightarrow{d_{\Delta^{-}}} & \mathcal{C}^{s+1,-}(G,n) \otimes \mathbb{Q}
\end{array}$$

$$\begin{array}{ccc}
\mathcal{C}_s^{\sim}(G,n) & \xleftarrow{d_{\nabla^{\sim}}} & \mathcal{C}_{s+1}^{\sim}(G,n) \\
\Big\downarrow g_{\bullet} & & \Big\downarrow g^{\bullet} \\
\mathcal{C}_s^{-}(G,n) \otimes \mathbb{Q} & \xleftarrow{d_{\nabla^{-}}} & \mathcal{C}_{s+1}^{-}(G,n) \otimes \mathbb{Q}.
\end{array}$$

**Lemma 11.** *There is a series of implications*

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$$

*where*

(1) $H^{>0}(\mathcal{C}^{\bullet,\sim}(G,n)) = 0,$

(2) *The operator*

$$\mathbf{\Delta}^{\sim} = d_{\Delta^{\sim}} \circ d_{\nabla^{\sim}} + d_{\nabla^{\sim}} \circ d_{\Delta^{\sim}}$$

*in invertible in degree* $> 0,$

(3) *The operator*

$$\mathbf{\Delta}^{-} = d_{\Delta^{-}} \circ d_{\nabla^{-}} + d_{\nabla^{-}} \circ d_{\Delta^{-}}$$

*in invertible in degree* $> 0,$

(4) $H^{>0}(\mathcal{C}^{\bullet,-}(G,n)) = 0, H_{>0}(\mathcal{C}_{\bullet}^{-}(G,n)) = 0.$

*Proof.* (1) $\Leftrightarrow$ (2) : We can verify that the differentials $d_{\nabla^{\sim}}$ and $d_{\Delta^{\sim}}$ are adjoint with respect to inner products defined on each $\mathcal{C}^{j,\sim}(G,n)$, induced from the identity matrix. Letting $(\cdot,\cdot)$ denote the inner products, and

$$\mathbf{\Delta}_j^{\sim} = \partial_j \partial^j + \partial^{j-1} \partial_{j-1}$$

we have the following facts:

- $\mathrm{im}(\partial^{j-1}) \cap \mathrm{im}(\partial_j)$ is trivial: Suppose $z \in \mathrm{im}(\partial^{j-1}) \cap \mathrm{im}(\partial_j)$, then

$$(z,z) = (\partial^{j-1}(y), z) = (y, \partial_{j-1}(z)) = (y, \partial_{j-1}\partial_j(x)) = (y, 0) = 0$$

  which means that $z = 0$.

- $\ker(\partial^j) = (\mathrm{im}(\partial_j))^\perp$: Since $z \in \ker(\partial^j)$ implies for any $y$
$$(z, \partial_j(y)) = (\partial^j(z), y) = (0, y) = 0.$$
- $\ker(\partial_{j-1}) = (\mathrm{im}(\partial^{j-1}))^\perp$: Same argument as above.
- $\ker(\mathbf{\Delta}^\sim) = \ker(\partial_j\partial^j) \cap \ker(\partial^{j-1}\partial_{j-1}) = \ker(\partial^j) \cap \ker(\partial_{j-1})$: First we show
$$\ker(\partial^j) = \ker(\partial_j\partial^j).$$
Obviously $\ker(\partial^j) \subseteq \ker(\partial_j\partial^j)$, so suppose $z \in \ker(\partial_j\partial^j)$, then we have
$$0 = (\partial_j\partial^j(z), z) = (\partial^j(z), \partial^j(z))$$
which implies $\partial^j(z) = 0$ and so $\ker(\partial^j) \supseteq \ker(\partial_j\partial^j)$. We can repeat this argument to also get
$$\ker(\partial_{j-1}) = \ker(\partial^{j-1}\partial_{j-1})$$
as required.
Now we just need to show the first equality, which is equivalent to showing
$$\ker(\mathbf{\Delta}^\sim) \subseteq \ker(\partial_j\partial^j) \cap \ker(\partial^{j-1}\partial_{j-1})$$
since the other inclusion is trivial. Let $z \in \ker(\mathbf{\Delta}^\sim)$, then we have, for any $y$
$$\begin{aligned} 0 = (\mathbf{\Delta}^\sim z, \partial_j\partial^j(y)) &= (\partial_j\partial^j(z) + \partial^{j-1}\partial_{j-1}(z), \partial_j\partial^j(y)) \\ &= (\partial_j\partial^j(z), \partial_j\partial^j(y)) + (\partial^{j-1}\partial_{j-1}(z), \partial_j\partial^j(y)) \\ &= (\partial_j\partial^j(z), \partial_j\partial^j(y)) + (\partial_{j-1}(z), \partial_{j-1}\partial_j\partial^j(y)) \\ &= (\partial_j\partial^j(z), \partial_j\partial^j(y)) + 0 \end{aligned}$$
which implies $z \in \ker(\partial_j\partial^j)$, and so $\ker(\mathbf{\Delta}^\sim) \subseteq \ker(\partial_j\partial^j)$. We can repeat this argument with $\partial^{j-1}\partial_{j-1}(y)$ to also get $\ker(\mathbf{\Delta}^\sim) \subseteq \ker(\partial^{j-1}\partial_{j-1})$.

This means that there is an orthogonal decomposition
$$\mathcal{C}^{j,\sim}(G, n) = \mathrm{im}(\partial^{j-1}) \oplus \ker(\mathbf{\Delta}^\sim) \oplus \mathrm{im}(\partial_j)$$
where
$$\ker(\partial^j) = \mathrm{im}(\partial^{j-1}) \oplus \ker(\mathbf{\Delta}^\sim).$$
So we have
$$H^j(\mathcal{C}^{\bullet,\sim}(G, n)) = \ker(\partial^j)/\mathrm{im}(\partial^{j-1}) \cong \ker(\mathbf{\Delta}^\sim)$$
and therefore these two statements are equivalent.

(2) $\Rightarrow$ (3) : Since $\mathbf{\Delta}^{\sim}$ is invertible, for any $\beta \in \mathcal{C}^{s,\sim}(G,n), \mathcal{C}_s^{\sim}(G,n)$ there exists an $\alpha$ such that

$$\mathbf{\Delta}^{\sim}(\alpha) = \beta$$

i.e.

$$(\mathbf{\Delta}^{\sim})^{-1}(\beta) = \alpha.$$

Since $g$ is compatible with respective differentials, we have

$$\begin{aligned}
g(\beta) &= g \circ \mathbf{\Delta}^{\sim}(\alpha) \\
g(\beta) &= g \circ (d_{\Delta^{\sim}} \circ d_{\nabla^{\sim}} + d_{\nabla^{\sim}} \circ d_{\Delta^{\sim}})(\alpha) \\
g(\beta) &= g \circ d_{\Delta^{\sim}} \circ d_{\nabla^{\sim}}(\alpha) + g \circ d_{\nabla^{\sim}} \circ d_{\Delta^{\sim}}(\alpha) \\
g(\beta) &= d_{\Delta^{-}} \circ g \circ d_{\nabla^{\sim}}(\alpha) + d_{\nabla^{-}} \circ g \circ d_{\Delta^{\sim}}(\alpha) \\
g(\beta) &= d_{\Delta^{-}} \circ d_{\nabla^{-}} \circ g(\alpha) + d_{\nabla^{-}} \circ d_{\Delta^{-}} \circ g(\alpha) \\
g(\beta) &= \mathbf{\Delta}^{-}(g(\alpha))
\end{aligned}$$

which means for all $\beta$

$$\begin{gathered}
g(\alpha) = g \circ (\mathbf{\Delta}^{\sim})^{-1}(\beta) \\
\mathbf{\Delta}^{-}(g(\alpha)) = \mathbf{\Delta}^{-} \circ g \circ (\mathbf{\Delta}^{\sim})^{-1}(\beta) = g(\beta) \\
\mathbf{\Delta}^{-} \circ g \circ (\mathbf{\Delta}^{\sim})^{-1} \circ g^{-1} \circ g(\beta) = g(\beta).
\end{gathered}$$

Using the fact that $g$ is surjective, we must have

$$\mathbf{\Delta}^{-} \circ g \circ (\mathbf{\Delta}^{\sim})^{-1} \circ g^{-1} = \mathrm{Id}$$

and therefore $\mathbf{\Delta}^{-}$ is invertible.

(3) $\Rightarrow$ (4) : Abusing notation, we can consider $\mathbf{\Delta}^{-}$ to be two chain maps

$$\begin{gathered}
\mathbf{\Delta}^{-} : \mathcal{C}^{\bullet,-}(G,n) \to \mathcal{C}^{\bullet,-}(G,n) \\
\mathbf{\Delta}_{-} : \mathcal{C}_{\bullet}^{-}(G,n) \to \mathcal{C}_{\bullet}^{-}(G,n)
\end{gathered}$$

which are both homotopic to zero. Now suppose $z \in Z_j(\mathcal{C}^{\bullet,-}(G,n))$ represents $x \in H_j(\mathcal{C}^{\bullet,-}(G,n))$, then

$$\mathbf{\Delta}_j^{-}(z) - 0_j(z) = \partial^{j-1}\partial_j(z) \in B_j(\mathcal{C}^{\bullet,-}(G,n)).$$

We can repeat this same reasoning with $\mathbf{\Delta}_{-}$ to get that they both induce the same trivial homomorphisms

$$\begin{gathered}
(\mathbf{\Delta}^{-})_* = 0_* : H^j(\mathcal{C}^{\bullet,-}(G,n)) \to H^j(\mathcal{C}^{\bullet,-}(G,n)) \\
(\mathbf{\Delta}_{-})_* = 0_* : H_j(\mathcal{C}_{\bullet}^{-}(G,n)) \to H_j(\mathcal{C}_{\bullet}^{-}(G,n)).
\end{gathered}$$

Now $\mathbf{\Delta}^-$ is invertible in degree $\geq 0$, which implies that $(\mathbf{\Delta}^-)_*$ and $(\mathbf{\Delta}_-)_*$ are also invertible (and trivial), which means that

$$H^{>0}(\mathcal{C}^{\bullet,-}(G,n)) = 0, \quad H_{>0}(\mathcal{C}_{\bullet}^{-}(G,n)) = 0.$$

as required. □

This means that we just need to prove statement (1). We will do this by constructing a homotopy

$$h : C_j^{\sim}(G,n) \to C_{j-1}^{\sim}(G,n)$$

such that

$$\mathbf{\Delta}_h^{\sim} := h \circ d_{\Delta^{\sim}} + d_{\Delta^{\sim}} \circ h$$

is invertible, in degrees $> 0$. This would imply $(\mathbf{\Delta}_h^{\sim})_* = 0_*$ is invertible, as above, and hence

$$H^{>0}(\mathcal{C}^{\bullet,\sim}(G,n)) = 0.$$

If we have a flags of subgroups

$$0 = G_{\leq 0} \subsetneq G_{\leq 1} \subsetneq \cdots \subsetneq G_{\leq r} = G,$$

which give the terms in $C_r^{\sim}(G,n)$, we can obtain a chain of surjective homomorphisms

$$0 = A_{\leq 0} \overset{\neq}{\twoheadleftarrow} A_{\leq 1} \overset{\neq}{\twoheadleftarrow} \ldots \overset{\neq}{\twoheadleftarrow} A_{\leq r} = A$$

by pullbacks of the inclusion maps $G_{\leq i} \hookrightarrow G_{\leq i+1}$, and the surjectivity is from the fact that every element of $A_{\leq i}$ has an induced representation in $A_{\leq i+1}$. This allows us to define the homotopy $h$ as below

$$h := \mathcal{M}_{n_1}^{\sim}(A_{\leq 1}) \otimes \mathcal{M}_{n_2}^{\sim}(\mathrm{Ker}(A_{\leq 2} \twoheadrightarrow A_{\leq 1})) \otimes \cdots \to \mathcal{M}_{n_1+n_2}^{\sim}(A_{\leq 2}) \otimes \cdots$$

where on the first two terms we have

$$\langle a_1, \ldots, a_{n_1} \rangle^{\sim} \otimes \langle b_1, \ldots, b_{n_2} \rangle^{\sim} \mapsto \langle \psi(a_1), \ldots, \psi(a_{n_1}), b_1, \ldots, b_{n_2} \rangle^{\sim},$$

and the identity on the rest. Here

$$\psi : A_{\leq 1} \to A_{\leq 2}$$

is a section of the natural surjective homomorphism, which we will defined below.

Now we will use the assumption that $G$ is cyclic, which also implies all $A_{\geq n}$ are also cyclic. This means we have

$$G = \mathbb{Z}/N\mathbb{Z} = \prod_i \mathbb{Z}/p_i^{k_i}\mathbb{Z},$$

and we can identify as follows

$$\mathbb{Z}/p_i^{k_i}\mathbb{Z} = \{0, \ldots, p_i - 1\}^{k_i},$$

by considering each element as a string of digits in the base $p_i$. Then we can have a natural map

$$\psi : A_{\leq 1} \to A_{\leq 2}$$

by adding zeroes to the corresponding strings of digits, for all $p_i$, such that the size of the cyclic group fits.

The generation condition (G) is satisfied on the RHS because $a_1, \dots, a_{n_1}$ generate $A_{\leq 1}$ and $b_1, \dots, b_{n_2}$ generate $\mathrm{Ker}(A_{\leq 2} \twoheadrightarrow A_{\leq 1})$, and the map $\psi$ is essentially lifts from $A_{\leq 1}$ to $A_{\leq 2}$ and

$$A_{\leq 1} \cong A_{\leq 2}/\mathrm{Ker}(A_{\leq 2} \twoheadrightarrow A_{\leq 1})$$

due to the isomorphism theorem.

The differential $d_{\Delta^\sim}$ is given by removing digits in this presentation, since the co-multiplication map is defined by

$$\langle a_1, \dots, a_n \rangle^\sim \mapsto \sum \langle a_{I'} \mod A'' \rangle^\sim \otimes \langle a_{I''} \rangle^\sim ;$$

the $\langle a_{I''} \rangle^\sim$ part is obviously obtained by removing digits, and subgroups of $\mathbb{Z}/p_i^{k_i}\mathbb{Z}$ are of the form

$$\mathbb{Z}/p_i^{l_i}\mathbb{Z} = \{0, \dots, p_i - 1\}^{l_i}$$

for some $l_i \leq k_i$, so we can represent $\langle a_{I'} \mod A'' \rangle^\sim$ by removing $l_i$ digits for each $p_i$.

Now consider the operator

$$\mathbf{\Delta}_h^\sim - \mathrm{Id}$$

acting on $C^{j,\sim}(G, n)$, for $j \geq 1$. Since the digit 0 is invariant under $\mathbf{\Delta}_h^\sim - \mathrm{Id}$, and a non-trivial part is invariant under $\mathbf{\Delta}_h^\sim$ from the definition of $h$ and $d_{\Delta^\sim}$, this implies that the number of zeros in the string of digits is strictly increased by the operator. Therefore it is nilpotent. So suppose $(\mathbf{\Delta}_h^\sim - \mathrm{Id})^m = 0$ for some $m$, then

$$\begin{aligned}
\mathrm{Id} &= \mathrm{Id}^m - (\mathrm{Id} - \mathbf{\Delta}_h^\sim)^m \\
&= \Big(\mathrm{Id} - (\mathrm{Id} - \mathbf{\Delta}_h^\sim)\Big)\Big(\mathrm{Id} + (\mathrm{Id} - \mathbf{\Delta}_h^\sim) \\
&\qquad\qquad + (\mathrm{Id} - \mathbf{\Delta}_h^\sim)^2 + \cdots + (\mathrm{Id} - \mathbf{\Delta}_h^\sim)^{m-1}\Big) \\
&= \Big(\mathbf{\Delta}_h^\sim\Big)\Big(\mathrm{Id} + (\mathrm{Id} - \mathbf{\Delta}_h^\sim) + (\mathrm{Id} - \mathbf{\Delta}_h^\sim)^2 + \cdots + (\mathrm{Id} - \mathbf{\Delta}_h^\sim)^{m-1}\Big)
\end{aligned}$$

Which means we can conclude that $\mathbf{\Delta}_h^\sim$ is invertible in degrees $\geq 1$. $\square$

Now we will go through each step of the proof with a specific example. Consider the case of

$$G = \mathbb{Z}/30\mathbb{Z} = \langle g \mid g^{30} = 1 \rangle, \qquad n = 3$$

which has the following subgroups

$$\langle g^{15} \rangle = C_2, \quad \langle g^{10} \rangle = C_3, \quad \langle g^{6} \rangle = C_5,$$
$$\langle g^{5} \rangle = C_6, \quad \langle g^{3} \rangle = C_{10}, \quad \langle g^{2} \rangle = C_{15}.$$

Using the $C_n$ notation from now on, this gives the following flags of subgroups of length 2

$$0 \subsetneq C_2 \subsetneq G, \quad 0 \subsetneq C_3 \subsetneq G, \quad 0 \subsetneq C_5 \subsetneq G$$
$$0 \subsetneq C_6 \subsetneq G, \quad 0 \subsetneq C_{10} \subsetneq G, \quad 0 \subsetneq C_{15} \subsetneq G$$

and the following flags of subgroups of length 3

$$0 \subsetneq C_2 \subsetneq C_6 \subsetneq G, \quad 0 \subsetneq C_2 \subsetneq C_{10} \subsetneq G$$
$$0 \subsetneq C_3 \subsetneq C_6 \subsetneq G, \quad 0 \subsetneq C_3 \subsetneq C_{15} \subsetneq G$$
$$0 \subsetneq C_5 \subsetneq C_{10} \subsetneq G, \quad 0 \subsetneq C_5 \subsetneq C_{15} \subsetneq G.$$

So we have the chain complexes

$$\begin{gathered}
\mathcal{M}_3^-(\mathbb{Z}/30\mathbb{Z}) \rightleftarrows \mathcal{M}_1^-(C_2) \otimes \mathcal{M}_2^-(C_{15}) \oplus \mathcal{M}_2^-(C_2) \otimes \mathcal{M}_1^-(C_{15}) \\
\oplus \mathcal{M}_1^-(C_3) \otimes \mathcal{M}_2^-(C_{10}) \oplus \mathcal{M}_2^-(C_3) \otimes \mathcal{M}_1^-(C_{10}) \oplus \mathcal{M}_1^-(C_5) \otimes \mathcal{M}_2^-(C_6) \\
\oplus \mathcal{M}_2^-(C_5) \otimes \mathcal{M}_1^-(C_6) \oplus \mathcal{M}_1^-(C_6) \otimes \mathcal{M}_2^-(C_5) \oplus \mathcal{M}_2^-(C_6) \otimes \mathcal{M}_1^-(C_5) \\
\oplus \mathcal{M}_1^-(C_{10}) \otimes \mathcal{M}_2^-(C_3) \oplus \mathcal{M}_2^-(C_{10}) \otimes \mathcal{M}_1^-(C_3) \oplus \mathcal{M}_1^-(C_{15}) \otimes \mathcal{M}_2^-(C_2) \\
\oplus \mathcal{M}_2^-(C_{15}) \otimes \mathcal{M}_1^-(C_2) \\
\rightleftarrows \mathcal{M}_1^-(C_2) \otimes \mathcal{M}_1^-(C_3) \otimes \mathcal{M}_1^-(C_5) \oplus \mathcal{M}_1^-(C_2) \otimes \mathcal{M}_1^-(C_5) \otimes \mathcal{M}_1^-(C_3) \\
\oplus \mathcal{M}_1^-(C_3) \otimes \mathcal{M}_1^-(C_2) \otimes \mathcal{M}_1^-(C_5) \oplus \mathcal{M}_1^-(C_3) \otimes \mathcal{M}_1^-(C_3) \otimes \mathcal{M}_1^-(C_2) \\
\oplus \mathcal{M}_1^-(C_5) \otimes \mathcal{M}_1^-(C_2) \otimes \mathcal{M}_1^-(C_3) \oplus \mathcal{M}_1^-(C_5) \otimes \mathcal{M}_1^-(C_3) \otimes \mathcal{M}_1^-(C_2)
\end{gathered}$$

with corresponding differentials $d_{\nabla^-}$ and $d_{\Delta^-}$ respectively. Note that we used the notation $C_{(30/k)}$ to denote the cyclic group obtained by $G/C_k$. We now verify the condition $\partial^1\partial^0 = 0$ for the element

$$\langle \chi_{30}, 24\chi_{30}, 14\chi_{30} \rangle^- \in \mathcal{M}_3^-(\mathbb{Z}/30\mathbb{Z})$$

where $\chi_{30}$ is the generator of the character group of $\mathbb{Z}/30\mathbb{Z}$ as in sections before. We will use similar notations for subgroups and quotients of

$\mathbb{Z}/30\mathbb{Z}$ below. The image of this element under $\partial^0$ is

$$(\langle\chi_2\rangle^- \otimes \langle 12\chi_{15}, 7\chi_{15}\rangle^-) \oplus \overbrace{(\langle 0, \chi_2\rangle^- \otimes \langle 7\chi_{15}\rangle^-)}^{=0} \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0$$
$$\oplus(\langle\chi_6, 2\chi_6\rangle^- \otimes \langle 4\chi_5\rangle^-) \oplus 0 \oplus 0 \oplus 0 \oplus 0$$

and the image of the above element under $\partial^1$ is

$$((-1)^1\langle\chi_2\rangle^- \otimes \langle\chi_3\rangle^- \otimes \langle 4\chi_5\rangle^-) \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0$$
$$+((-1)^0\langle\chi_2\rangle^- \otimes \langle\chi_3\rangle^- \otimes \langle 4\chi_5\rangle^-) \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0$$
$$= 0$$

which is what is expected. Here we can see that the flag of subgroup associated with the non-trivial element

$$0 \subsetneq C_2 \subsetneq C_6 \subsetneq G$$

occurs twice, once from each flag

$$0 \subsetneq C_2 \subsetneq G, \quad 0 \subsetneq C_6 \subsetneq G$$

with opposite signs, which cancels out.

The $\mathbb{Q}$-vector space

$$\mathcal{M}_3^\sim(\mathbb{Z}/30\mathbb{Z})$$

has similar complexes as above

$$\mathcal{C}^{\bullet,\sim}(\mathbb{Z}/30\mathbb{Z}, 3), \quad \mathcal{C}_\bullet^\sim(\mathbb{Z}/30\mathbb{Z}, 3)$$

with differentials compatible with the surjective natural linear maps $g^\bullet, g_\bullet$. Consider the homotopy constructed above

$$h : C_1^\sim(\mathbb{Z}/30\mathbb{Z}, 3) \to C_0^\sim(\mathbb{Z}/30\mathbb{Z}, 3)$$

and

$$\mathbf{\Delta}_h^\sim := h \circ d_{\Delta^\sim} + d_{\Delta^\sim} \circ h.$$

We will look at the image of

$$\alpha := (\langle\chi_2\rangle^\sim \otimes \langle 12\chi_{15}, 7\chi_{15}\rangle^\sim) \oplus 0 \oplus 0 \oplus \cdots \in C^{1,\sim}(\mathbb{Z}/30\mathbb{Z}, 3)$$

under both of these maps. The element has the flag of subgroups

$$0 \subsetneq C_2 \subsetneq G$$

and induces the surjective homomorphisms

$$0 \overset{f}{\twoheadleftarrow} A_2 \overset{g}{\twoheadleftarrow} A$$

where $A_2$ is the character group of $C_2$. The maps $f$ and $g$ can by explicitly given as

$$f := \chi_2 \mapsto 0$$
$$g := \chi_{30} \mapsto \chi_2$$

so we have

$$\mathrm{Ker}(A \twoheadrightarrow A_2) = \langle \chi_{30}^2 \rangle \cong \mathbb{Z}/15\mathbb{Z}$$

which we will denote by $A_{15}$ generated by $\chi_{15}$. This means the map $h$ is only relevant on

$$h : \mathcal{M}_1^{\sim}(A_2) \otimes \mathcal{M}_2^{\sim}(A_{15}) \oplus 0 \oplus \cdots \to \mathcal{M}_3^{\sim}(A).$$

Now we need find the map

$$\psi : A_2 \to A$$

explicitly. To do this we just need to find the image of the element $\chi_2$, converting this into strings of digits we get

$$\chi_2 = \{1\} \in \mathbb{Z}/2\mathbb{Z}.$$

Now we can write

$$\mathbb{Z}/30\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$$

so by adding zeroes we get

$$\psi(\chi_2) = \{100\} = 15\chi_{30} \in \mathbb{Z}/30\mathbb{Z}$$

so we have

$$\psi : \chi_2 \mapsto 15\chi_{30}.$$

Therefore we have

$$h(\alpha) = \langle 15\chi_{30}, 24\chi_{30}, 14\chi_{30} \rangle^{\sim}$$

and

$$
\begin{aligned}
\mathbf{\Delta}_h^{\sim}(\alpha) &= h \circ d_{\Delta^{\sim}}(\alpha) + d_{\Delta^{\sim}} \circ h(\alpha) \\
&= h((-\langle \chi_2 \rangle^{\sim} \otimes \langle \chi_3 \rangle^{\sim} \otimes \langle 4\chi_5 \rangle^{\sim}) \oplus 0 \oplus \cdots) \\
&\quad + d_{\Delta^{\sim}}(\langle 15\chi_{30}, 24\chi_{30}, 14\chi_{30} \rangle^{\sim}) \\
&= \cdots \oplus 0 \oplus (-\langle 3\chi_6, 2\chi_6 \rangle^{\sim} \otimes \langle 4\chi_5 \rangle^{\sim}) \oplus 0 \oplus \cdots \\
&\quad + (\langle \chi_2 \rangle^{\sim} \otimes \langle 12\chi_{15}, 7\chi_{15} \rangle^{\sim}) \oplus 0 \oplus (\langle 2\chi_3 \rangle^{\sim} \otimes \langle 5\chi_{10}, 8\chi_{10} \rangle^{\sim}) \\
&\quad \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus (\langle 3\chi_6, 2\chi_6 \rangle^{\sim} \otimes \langle 4\chi_5 \rangle^{\sim}) \oplus 0 \oplus 0 \oplus 0 \\
&\quad \oplus (\langle 9\chi_{15}, 14\chi_{15} \rangle^{\sim} \otimes \langle \chi_2 \rangle^{\sim}) \\
&= (\langle \chi_2 \rangle^{\sim} \otimes \langle 12\chi_{15}, 7\chi_{15} \rangle^{\sim}) \oplus 0 \oplus (\langle 2\chi_3 \rangle^{\sim} \otimes \langle 5\chi_{10}, 8\chi_{10} \rangle^{\sim}) \\
&\quad \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \\
&\quad \oplus (\langle 9\chi_{15}, 14\chi_{15} \rangle^{\sim} \otimes \langle \chi_2 \rangle^{\sim})
\end{aligned}
$$

which gives

$$
\begin{aligned}
(\mathbf{\Delta}_h^{\sim} - \mathrm{Id})(\alpha) = 0 \oplus 0 \oplus (\langle 2\chi_3 \rangle^{\sim} \otimes \langle 5\chi_{10}, 8\chi_{10} \rangle^{\sim}) \\
\oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \\
\oplus (\langle 9\chi_{15}, 14\chi_{15} \rangle^{\sim} \otimes \langle \chi_2 \rangle^{\sim}).
\end{aligned}
$$

Now the string of digits for $\alpha$ is

$$
(\langle \{1\} \rangle^{\sim} \otimes \langle \{02\}, \{12\} \rangle^{\sim}) \oplus 0 \oplus 0 \oplus \ldots
$$

and for $(\mathbf{\Delta}_h^{\sim} - \mathrm{Id})(\alpha)$ is

$$
\begin{aligned}
0 \oplus 0 \oplus (\langle \{2\} \rangle^{\sim} \otimes \langle \{10\}, \{03\} \rangle^{\sim}) \\
\oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \\
\oplus (\langle \{04\}, \{24\} \rangle^{\sim} \otimes \langle \{1\} \rangle^{\sim}).
\end{aligned}
$$

so we can see that the number of zeroes in the presentation has been strictly increased from 1 to 3, which implies that $\mathbf{\Delta}_h^{\sim} - \mathrm{Id}$ is nilpotent. Therefore $\mathbf{\Delta}_h^{\sim}$ is invertible as expected.

Now we will verify that the differentials $d_{\nabla^{\sim}}$ and $d_{\Delta^{\sim}}$ are adjoint with respect to inner products induced from the identity matrix on the elements $\alpha$ and

$$
\beta := 2(\langle \chi_2 \rangle^{\sim} \otimes \langle \chi_3 \rangle^{\sim} \otimes \langle 4\chi_5 \rangle^{\sim}) \oplus 0 \oplus \cdots \in C^{2,\sim}(\mathbb{Z}/30\mathbb{Z}, 3).
$$

We have

$$
\begin{aligned}
(\beta, d_{\Delta^{\sim}}(\alpha)) &= (\beta, (-\langle \chi_2 \rangle^{\sim} \otimes \langle \chi_3 \rangle^{\sim} \otimes \langle 4\chi_5 \rangle^{\sim}) \oplus 0 \oplus \cdots) \\
&= -2
\end{aligned}
$$

and

$$\begin{aligned}(d_{\nabla^\sim}(\beta),\alpha) &= (2(\sum_{i=1,3,5}\langle i\chi_6, 2\chi_6\rangle^\sim \otimes \langle 4\chi_5\rangle^\sim)\oplus 0\oplus\cdots \\ &\qquad \oplus 2((-1)^1\sum_{i=1,4,7,10,13}\langle \chi_2\rangle^\sim\otimes\langle i\chi_{15}, 12\chi_{15}\rangle^\sim)\oplus 0\oplus\cdots,\alpha) \\ &= -2\end{aligned}$$

which is as expected.

Now we will look at a new diagram of homomorphisms

$$\begin{aligned}\mathcal{M}_n(G) &\to \bigoplus_{\substack{n_1+n_2=n\\ \mathcal{G}_\bullet \text{ of lengths } 2}} \mathcal{M}_{n_1}(gr_1(\mathcal{G}_\bullet))\otimes\mathcal{M}^-_{n_2}(gr_2(\mathcal{G}_\bullet)) \\ &\to \bigoplus_{\substack{n_1+n_2+n_3=n\\ \mathcal{G}_\bullet \text{ of lengths } 3}} \mathcal{M}_{n_1}(gr_1(\mathcal{G}_\bullet))\otimes\mathcal{M}^-_{n_2}(gr_2(\mathcal{G}_\bullet))\otimes\mathcal{M}^-_{n_3}(gr_3(\mathcal{G}_\bullet))\to\dots\end{aligned}$$

where $\mathcal{G}_\bullet$ is a flag of subgroups as follows

$$0 = G_{\le 0}\subseteq G_{\le 1}\subsetneq\cdots\subsetneq G_{\le r} = G,\quad r\ge 1$$

where every inclusion is strict except the $G_{\le 0}\subseteq G_{\le 1}$ part; and the leftmost part of the tensor product is not the quotient $\mathcal{M}^-_n(G)$ but the full group $\mathcal{M}_n(G)$.

There is both maps $\Delta$ and $\Delta^-$ in the differential. This is a complex with the same construction as above but we only use the map $\Delta$ when we map the leftmost terms

$$\mathcal{M}_{n_1}(gr_1(\mathcal{H}_\bullet))\otimes\cdots\to\mathcal{M}_{n_1}(gr_1(\mathcal{G}_\bullet))\otimes\mathcal{M}^-_{n_2}(gr_2(\mathcal{G}_\bullet))\otimes\dots$$

where $\mathcal{H}_\bullet = \mathcal{G}_\bullet\setminus\{G_1\}$ is the flag obtained by removing the group $G_1$. This complex is denoted by

$$\mathcal{C}^\bullet(G,n).$$

**Theorem 12.** [2, Theorem 14] *Let $G$ be a finite cyclic group. Then the cohomology of the complex*

$$\mathcal{C}^\bullet(G,n)$$

*after tensoring by $\mathbb{Q}$, is concentrated in degree 0.*

*Proof.* The proof is similar to the one given above. Here we show that for finite cyclic groups, the projection

$$\mu^-:\mathcal{M}_n(G)\to\mathcal{M}^-_n(G)$$

has a section

$$\nu : \mathcal{M}_n^-(G) \to \mathcal{M}_n(G)$$

defined on generators by

$$\langle a_1, \dots, a_n \rangle^- \mapsto \sum_{\varepsilon_1, \dots, \varepsilon_n} \varepsilon_1 \varepsilon_2 \dots \varepsilon_n \langle \varepsilon_1 a_1, \dots, \varepsilon_n a_n \rangle$$

where $\varepsilon_i \in \{+1, -1\}$, and the sum is over all possibilities.

For $n = 1$, we have

$$\langle a_1 \rangle^- \mapsto \sum_{\varepsilon_1} \varepsilon_1 \langle \varepsilon_1 a_1 \rangle = \langle a_1 \rangle + -\langle -a_1 \rangle$$

which is trivially compatible. Now we just need to check the case $n = 2$, where for

$$a, b \in \mathbb{Z}/N\mathbb{Z}, \quad \gcd(a, b, N) = 1,$$

the equation above gives

$$\langle a, b \rangle^- \mapsto \langle a, b \rangle + \langle -a, -b \rangle - \langle -a, b \rangle - \langle a, -b \rangle.$$

We just need to verify that both sides of the relation

$$\langle a, b \rangle^- = \langle a, b - a \rangle^- + \langle a - b, b \rangle^-$$

is mapped to the same thing. This means we need show the following equation holds

$$\begin{aligned} &\langle a, b \rangle + \langle -a, -b \rangle - \langle -a, b \rangle - \langle a, -b \rangle \\ &\quad \overset{?}{=} \langle a, b - a \rangle + \langle -a, a - b \rangle - \langle -a, b - a \rangle - \langle a, a - b \rangle \\ &\qquad + \langle a - b, b \rangle + \langle b - a, -b \rangle - \langle b - a, b \rangle - \langle a - b, -b \rangle. \end{aligned}$$

The first terms and the second terms on each line are

$$\langle a, b \rangle = \langle a, b - a \rangle + \langle a - b, b \rangle$$
$$\langle -a, -b \rangle = \langle -a, a - b \rangle + \langle b - a, -b \rangle$$

which are relations in $\mathcal{M}_2(\mathbb{Z}/N\mathbb{Z})$. So it is sufficient to check

$$\begin{aligned} &- \langle -a, b \rangle - \langle a, -b \rangle \\ &\quad \overset{?}{=} -\langle -a, b - a \rangle - \langle a, a - b \rangle - \langle b - a, b \rangle - \langle a - b, -b \rangle. \end{aligned}$$

We can replace $a$ with $-a$ to make the equation more symmetrical, this then gives

$$\begin{aligned} &\langle a, b \rangle + \langle -a, -b \rangle \\ &\quad \overset{?}{=} \langle a, b + a \rangle + \langle -a, -a - b \rangle + \langle b + a, b \rangle + \langle -a - b, -b \rangle. \end{aligned}$$

Plug in the relations

$\langle a, b+a\rangle = \langle a, b\rangle + \langle -b, b+a\rangle, \quad \langle -a, -b-a\rangle = \langle -a, -b\rangle + \langle b, -b-a\rangle$

we get

$$\begin{aligned} &\langle a, b\rangle + \langle -a, -b\rangle \\ &\stackrel{?}{=} \langle a, b\rangle + \langle -b, b+a\rangle + \langle -a, -a-b\rangle + \langle b+a, b\rangle + \langle -a, -b\rangle + \langle b, -b-a\rangle \end{aligned}$$

which is equivalent to

$$0 \stackrel{?}{=} \langle -b, b+a\rangle + \langle -a, -a-b\rangle + \langle b+a, b\rangle + \langle b, -b-a\rangle.$$

We will use

$$\boldsymbol{\delta}(a+b, b)$$

to denote the four terms on the right hand side. Here we can replace $a+b$ in the notation by $a$ to simplify the above to

$$\boldsymbol{\delta}(a, b) \stackrel{?}{=} 0 \in \mathcal{M}_2(\mathbb{Z}/N\mathbb{Z})$$

which is what we will need to show. Now we have

$$\boldsymbol{\delta}(a+b, b) = \boldsymbol{\delta}(a+b, a), \quad \boldsymbol{\delta}(a, b) = \boldsymbol{\delta}(-a, b) = \boldsymbol{\delta}(b, a),$$

we can verify the first relation by using the same relations as above

$$\begin{aligned} &\boldsymbol{\delta}(a+b, b) \\ &= \langle a+b, b\rangle + \langle -(a+b), b\rangle + \langle a+b, -b\rangle + \langle -(a+b), -b\rangle \\ &= \langle a+b, b\rangle + (\langle -a, -b-a\rangle - \langle -a, -b\rangle) \\ &\qquad + (\langle a, b+a\rangle - \langle a, b\rangle) + \langle -(a+b), -b\rangle \\ &= (\langle a, b\rangle + \langle -a, b+a\rangle) + (\langle -a, -b-a\rangle - \langle -a, -b\rangle) \\ &\qquad + (\langle a, b+a\rangle - \langle a, b\rangle) + (\langle -a, -b\rangle + \langle a, -b-a\rangle) \\ &= \langle a+b, a\rangle + \langle -(a+b), a\rangle + \langle a+b, -a\rangle + \langle -(a+b), -a\rangle \\ &= \boldsymbol{\delta}(a+b, a) \end{aligned}$$

and the latter relation is from the fact that they are all the same equation

$$\boldsymbol{\delta}(a, b) = \boldsymbol{\delta}(-a, b) = \boldsymbol{\delta}(b, a) = \langle a, b\rangle + \langle -a, b\rangle + \langle a, -b\rangle + \langle -a, -b\rangle.$$

By replacing $a+b$ with $a$ in the first relation above we get

$$\boldsymbol{\delta}(a, b) = \boldsymbol{\delta}(a, a-b) = \boldsymbol{\delta}(a-b, a)$$

and so $\boldsymbol{\delta}$ is invariant under the matrices

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which generate $\mathrm{SL}_2(\mathbb{Z})$, and thus $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, so that $\boldsymbol{\delta}(a,b)$ is constant.

Now by applying the defining relation to each term in the sum $\boldsymbol{S}$, defined below, we obtain

$$\begin{aligned}
\boldsymbol{S} &:= \sum_{a,b} \boldsymbol{\delta}(a,b) \\
&= \sum_{a,b} \Big( \langle a,b\rangle + \langle -a,b\rangle + \langle a,-b\rangle + \langle -a,-b\rangle \Big) \\
&= \sum_{a,b} \langle a-b,b\rangle + \sum_{a,b} \langle a,b-a\rangle + \sum_{a,b} \langle -a-b,b\rangle + \sum_{a,b} \langle -a,b+a\rangle \\
&\qquad + \sum_{a,b} \langle -a+b,-b\rangle + \sum_{a,b} \langle -a,-b+a\rangle + \sum_{a,b} \langle a+b,-b\rangle + \sum_{a,b} \langle a,-b-a\rangle \\
&= 8 \sum_{a,b} \langle a,b\rangle \\
&= 2\Big( \sum_{a,b} \langle a,b\rangle + \sum_{a,b} \langle -a,b\rangle + \sum_{a,b} \langle a,-b\rangle + \sum_{a,b} \langle -a,-b\rangle \Big) \\
&= 2 \sum_{a,b} \boldsymbol{\delta}(a,b) \\
&= 2\boldsymbol{S}
\end{aligned}$$

which gives $\boldsymbol{S} = 0$ and thus $\boldsymbol{\delta}(a,b) = 0$. So the section

$$\nu : \mathcal{M}_n^-(G) \to \mathcal{M}_n(G)$$

is compatible with the defining relations.

To prove the statement of the theorem we just need to show that the map

$$\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \to \bigoplus_{N=N'N''} \mathcal{M}_{n'}(\mathbb{Z}/N'\mathbb{Z}) \otimes \mathcal{M}_{n''}^-(\mathbb{Z}/N''\mathbb{Z}), n = n' + n''$$

is surjective, where the sum is over all exact sequences

$$0 \to \mathbb{Z}/N''\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/N'\mathbb{Z} \to 0, \quad N = N'N'', N \geq 2,$$

of finite cyclic groups. Since the image is contained inside the kernel of the next map in the complex, this map being surjective would mean that the image is equal to the kernel in this part of the differential. This, along with the previous results on the complexes of $\mathcal{M}_n^-(G)$ implies the vanishing of the cohomology in degrees $> 0$.

This can be shown by finding the inverse

$$\widetilde{\nabla}\colon \mathcal{M}_{n'}(\mathbb{Z}/N'\mathbb{Z}) \otimes \mathcal{M}^-_{n''}(\mathbb{Z}/N''\mathbb{Z}) \to \mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}), \quad n = n' + n''$$

which on generators is given by

$$\langle a'_1, \ldots, a'_{n'} \rangle \otimes \langle b_1, \ldots, b_{n''} \rangle^- \mapsto \sum_{\substack{\text{all lifts} \\ \varepsilon_1, \cdots, \varepsilon_{n''}}} (\varepsilon_1 \ldots \varepsilon_{n''}) \langle a_1, \ldots, a_{n'}, \varepsilon_1 b_1, \ldots, \varepsilon_{n''} b_{n''} \rangle$$

where the sum is over all lifts $a_i \in \mathbb{Z}/N\mathbb{Z}$ of $a'_i \in \mathbb{Z}/N'\mathbb{Z}$ and all possibilities for $\varepsilon_j \in \{+1, -1\}$ similar to definition the of $\nu$ above. This is compatible with defining equations since we are using the sections $\nu$ to get

$$\mathcal{M}_{n'}(\mathbb{Z}/N'\mathbb{Z}) \otimes \mathcal{M}^-_{n''}(\mathbb{Z}/N''\mathbb{Z}) \xrightarrow{\mathrm{Id}\otimes\nu} \mathcal{M}_{n'}(\mathbb{Z}/N'\mathbb{Z}) \otimes \mathcal{M}_{n''}(\mathbb{Z}/N''\mathbb{Z}) \xrightarrow{\nabla} \mathcal{M}_n(\mathbb{Z}/N\mathbb{Z})$$

in other words we have $\nabla \circ (\mathrm{Id} \otimes \nu) = \widetilde{\nabla}$ and both $\nabla$ and $\nu$ are compatible. ☐

Now we will give an example for the above proof. Consider the same group as above

$$G = \mathbb{Z}/30\mathbb{Z}, \quad n = 3$$

by similar construction we get the complex

$$\begin{gathered} \mathcal{M}_3(\mathbb{Z}/30\mathbb{Z}) \rightleftarrows \mathcal{M}_1(C_2) \otimes \mathcal{M}^-_2(C_{15}) \oplus \mathcal{M}_2(C_2) \otimes \mathcal{M}^-_1(C_{15}) \\ \oplus \mathcal{M}_1(C_3) \otimes \mathcal{M}^-_2(C_{10}) \oplus \mathcal{M}_2(C_3) \otimes \mathcal{M}^-_1(C_{10}) \oplus \mathcal{M}_1(C_5) \otimes \mathcal{M}^-_2(C_6) \\ \oplus \mathcal{M}_2(C_5) \otimes \mathcal{M}^-_1(C_6) \oplus \mathcal{M}_1(C_6) \otimes \mathcal{M}^-_2(C_5) \oplus \mathcal{M}_2(C_6) \otimes \mathcal{M}^-_1(C_5) \\ \oplus \mathcal{M}_1(C_{10}) \otimes \mathcal{M}^-_2(C_3) \oplus \mathcal{M}_2(C_{10}) \otimes \mathcal{M}^-_1(C_3) \oplus \mathcal{M}_1(C_{15}) \otimes \mathcal{M}^-_2(C_2) \\ \oplus \mathcal{M}_2(C_{15}) \otimes \mathcal{M}^-_1(C_2) \\ \rightleftarrows \mathcal{M}_1(C_2) \otimes \mathcal{M}^-_1(C_3) \otimes \mathcal{M}^-_1(C_5) \oplus \mathcal{M}_1(C_2) \otimes \mathcal{M}^-_1(C_5) \otimes \mathcal{M}^-_1(C_3) \\ \oplus \mathcal{M}_1(C_3) \otimes \mathcal{M}^-_1(C_2) \otimes \mathcal{M}^-_1(C_5) \oplus \mathcal{M}_1(C_3) \otimes \mathcal{M}^-_1(C_3) \otimes \mathcal{M}^-_1(C_2) \\ \oplus \mathcal{M}_1(C_5) \otimes \mathcal{M}^-_1(C_2) \otimes \mathcal{M}^-_1(C_3) \oplus \mathcal{M}_1(C_5) \otimes \mathcal{M}^-_1(C_3) \otimes \mathcal{M}^-_1(C_2). \end{gathered}$$

Consider the element

$$\langle \chi_{30}, 24\chi_{30}, 14\chi_{30} \rangle \in \mathcal{M}_3(\mathbb{Z}/30\mathbb{Z})$$

the image under $\partial^0$ is

$$(\langle\chi_2\rangle \otimes \langle 12\chi_{15}, 7\chi_{15}\rangle^-) \oplus (\langle 0, \chi_2\rangle \otimes \langle 7\chi_{15}\rangle^-) \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0$$
$$\oplus(\langle\chi_6, 2\chi_6\rangle \otimes \langle 4\chi_5\rangle^-) \oplus 0 \oplus 0 \oplus 0 \oplus 0$$

and the image of the above element under $\partial^1$ is

$$((-1)^1\langle\chi_2\rangle \otimes \langle\chi_3\rangle^- \otimes \langle 4\chi_5\rangle^-) \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0$$
$$+((-1)^0\langle\chi_2\rangle \otimes \langle\chi_3\rangle^- \otimes \langle 4\chi_5\rangle^-) \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 = 0$$

and so the condition $\partial^1\partial^0 = 0$ is satisfied similar to above.

Now we will look at the section

$$\nu : \mathcal{M}_3^-(\mathbb{Z}/30\mathbb{Z}) \to \mathcal{M}_3(\mathbb{Z}/30\mathbb{Z}).$$

Consider the relation

$$\langle\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle^- = \langle -23\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle^- + \langle\chi_{30}, 23\chi_{30}, 14\chi_{30}\rangle^-$$

we have

$$\begin{aligned}
&\nu(\langle\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle^-)\\
&= \langle\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle - \langle -\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle\\
&\quad - \langle\chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle - \langle\chi_{30}, 24\chi_{30}, -14\chi_{30}\rangle\\
&\quad + \langle\chi_{30}, -24\chi_{30}, -14\chi_{30}\rangle + \langle -\chi_{30}, 24\chi_{30}, -14\chi_{30}\rangle\\
&\quad + \langle -\chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle - \langle -\chi_{30}, -24\chi_{30}, -14\chi_{30}\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\nu(\langle -23\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle^-)\\
&= \langle -23\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle - \langle 23\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle\\
&\quad - \langle -23\chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle - \langle -23\chi_{30}, 24\chi_{30}, -14\chi_{30}\rangle\\
&\quad + \langle -23\chi_{30}, -24\chi_{30}, -14\chi_{30}\rangle + \langle 23\chi_{30}, 24\chi_{30}, -14\chi_{30}\rangle\\
&\quad + \langle 23\chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle - \langle 23\chi_{30}, -24\chi_{30}, -14\chi_{30}\rangle\\
&\nu(\langle\chi_{30}, 23\chi_{30}, 14\chi_{30}\rangle^-)\\
&= \langle\chi_{30}, 23\chi_{30}, 14\chi_{30}\rangle - \langle -\chi_{30}, 23\chi_{30}, 14\chi_{30}\rangle\\
&\quad - \langle\chi_{30}, -23\chi_{30}, 14\chi_{30}\rangle - \langle\chi_{30}, 23\chi_{30}, -14\chi_{30}\rangle\\
&\quad + \langle\chi_{30}, -23\chi_{30}, -14\chi_{30}\rangle + \langle -\chi_{30}, 23\chi_{30}, -14\chi_{30}\rangle\\
&\quad + \langle -\chi_{30}, -23\chi_{30}, 14\chi_{30}\rangle - \langle -\chi_{30}, -23\chi_{30}, -14\chi_{30}\rangle.
\end{aligned}$$

If the first two elements in the symbol have the same sign, then the relation trivially holds. So the relation reduces to

$$\begin{aligned}
&-\langle -\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle - \langle \chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle \\
&+\langle \chi_{30}, -24\chi_{30}, -14\chi_{30}\rangle + \langle -\chi_{30}, 24\chi_{30}, -14\chi_{30}\rangle \\
&\stackrel{?}{=} -\langle 23\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle - \langle -23\chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle \\
&\quad + \langle -23\chi_{30}, -24\chi_{30}, -14\chi_{30}\rangle + \langle 23\chi_{30}, 24\chi_{30}, -14\chi_{30}\rangle \\
&\quad - \langle -\chi_{30}, 23\chi_{30}, 14\chi_{30}\rangle - \langle \chi_{30}, -23\chi_{30}, 14\chi_{30}\rangle \\
&\quad + \langle \chi_{30}, -23\chi_{30}, -14\chi_{30}\rangle + \langle -\chi_{30}, 23\chi_{30}, -14\chi_{30}\rangle
\end{aligned}$$

we just need to show

$$\begin{aligned}
&-\langle -\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle - \langle \chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle \\
&\stackrel{?}{=} -\langle 23\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle - \langle -23\chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle \\
&\quad - \langle -\chi_{30}, 23\chi_{30}, 14\chi_{30}\rangle - \langle \chi_{30}, -23\chi_{30}, 14\chi_{30}\rangle
\end{aligned}$$

since all the terms with $-14\chi_{30}$ just have the opposite sign. Using the relations

$$\begin{aligned}
\langle -\chi_{30}, 23\chi_{30}, 14\chi_{30}\rangle &= \langle -24\chi_{30}, 23\chi_{30}, 14\chi_{30}\rangle + \langle -\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle \\
\langle \chi_{30}, -23\chi_{30}, 14\chi_{30}\rangle &= \langle 24\chi_{30}, -23\chi_{30}, 14\chi_{30}\rangle + \langle \chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle
\end{aligned}$$

it is further reduced to

$$\begin{aligned}
\boldsymbol{\delta}(23,24) := &\langle 23\chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle + \langle -23\chi_{30}, -24\chi_{30}, 14\chi_{30}\rangle \\
&\langle -24\chi_{30}, 23\chi_{30}, 14\chi_{30}\rangle + \langle 24\chi_{30}, -23\chi_{30}, 14\chi_{30}\rangle \\
\stackrel{?}{=} &\ 0
\end{aligned}$$

which holds by the proof given above. So the section $\nu$ is compatible with the defining relation $(M)$.

Now we will look at the map

$$\widetilde{\Delta}\colon \mathcal{M}_3(\mathbb{Z}/30\mathbb{Z}) \to \bigoplus_{30=N'N''} \mathcal{M}_{n'}(\mathbb{Z}/N'\mathbb{Z}) \otimes \mathcal{M}^-_{n''}(\mathbb{Z}/N''\mathbb{Z}), \quad 3 = n' + n''$$

and its inverse. The image of the element

$$\langle \chi_{30}, 24\chi_{30}, 14\chi_{30}\rangle \in \mathcal{M}_3(\mathbb{Z}/30\mathbb{Z})$$

under this map is

$$\begin{aligned}
&(\langle \chi_2\rangle \otimes \langle 12\chi_{15}, 7\chi_{15}\rangle^-) \oplus (\langle 0, \chi_2\rangle \otimes \langle 7\chi_{15}\rangle^-) \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \\
&\quad \oplus (\langle \chi_6, 2\chi_6\rangle \otimes \langle 4\chi_5\rangle^-) \oplus 0 \oplus 0 \oplus 0 \oplus 0
\end{aligned}$$

and the image of this element under $\widetilde{\nabla} = \nabla \circ (\mathrm{Id} \otimes \nu)$ is

$$\sum_{\substack{\mathrm{i}\,=\,0,\ldots,14 \\ \varepsilon_1,\varepsilon_2\in\{+1,-1\}}} \varepsilon_1\varepsilon_2\langle(2i+1)\chi_{30}, \varepsilon_1 24\chi_{30}, \varepsilon_2 14\chi_{30}\rangle$$
$$+ \sum_{\substack{\mathrm{i,j}\,=\,0,\ldots,14 \\ \varepsilon_1\in\{+1,-1\}}} \varepsilon_1\langle(2i+1)\chi_{30}, (2j)\chi_{30}, \varepsilon_1 14\chi_{30}\rangle$$
$$+ \sum_{\substack{\mathrm{i,j}\,=\,0,\ldots,4 \\ \varepsilon_1\in\{+1,-1\}}} \varepsilon_1\langle(6i+1)\chi_{30}, \varepsilon_1 24\chi_{30}, (6j+2)\chi_{30}\rangle.$$

## References

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