Some recent developments in wave turbulence theory: Part 3

Gigliola Staffilani

Massachusetts Institute of Technology

Analysis and PDE Mini-School

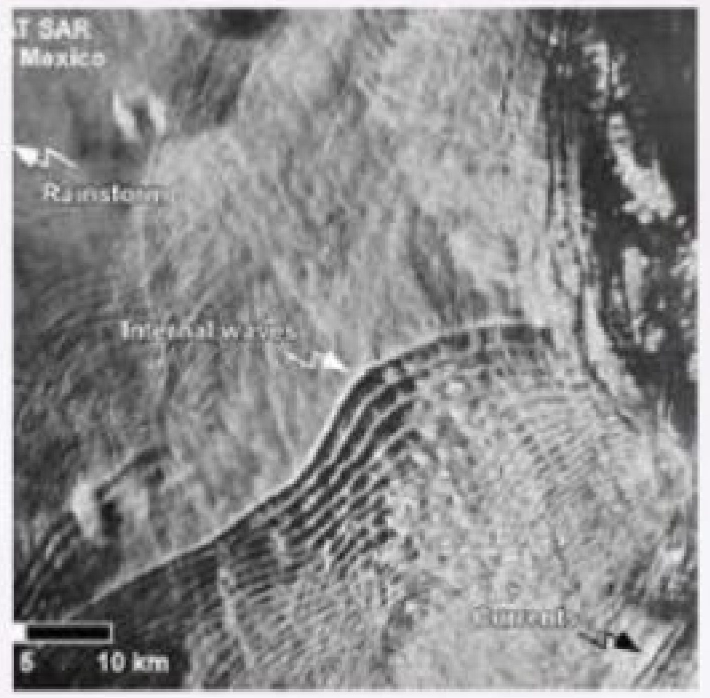
UNC April 5 - April 6

Wave Turbulence Theory

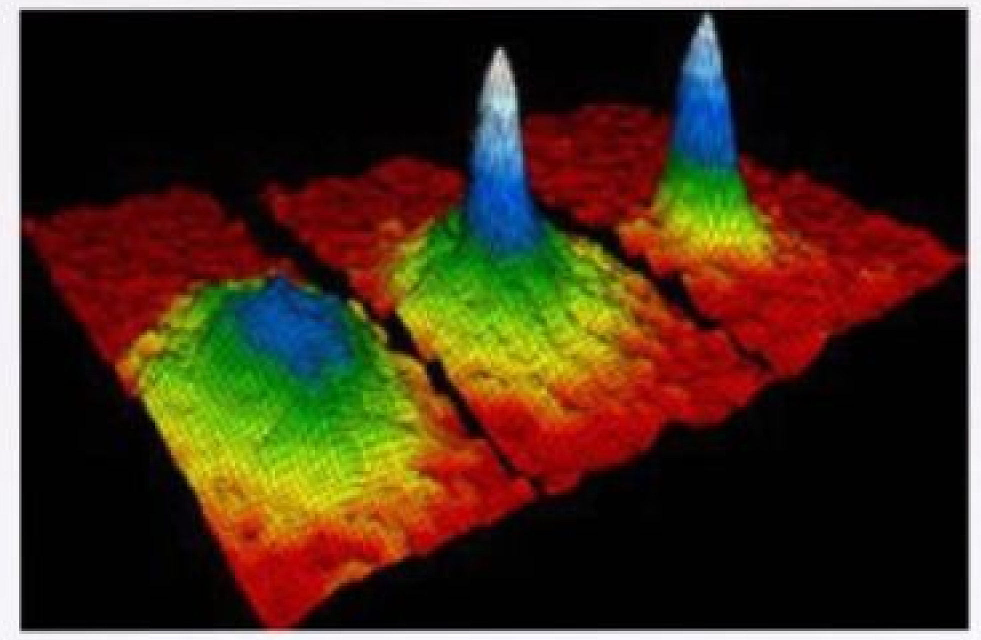
"When in a given physical system a large number of waves are present, the description of each individual wave is neither possible nor relevant. What becomes of physical importance and practical use are the density and the statistics of the interacting waves: this is Wave TurbulenceTheory"

The statistical description of a system of interacting waves is of great importance in physics:

Gravity



Internal waves



Bose Einstein Condensate

The rigorous derivation of this description in different settings is a fundamental problem in mathematics

Energy Spectrum

Given a dispersive equation (NLS, KdV, ZK...) and a periodic solution $u(t,x)$ we define the energy spectrum $\mathcal{E}(t) = \{ |\hat{u}(t,k)|^2 / k \in \mathbb{Z}^d \}$

It would be great to have an effective equation for the spectrum $\mathcal{E}(t) \Longrightarrow$

Wave Kinetic Equation

From the dispersive equation to the Wave Kinetic Equation

Consider the periodic NLS

$$\partial_t u + \Delta u = \varepsilon |u|^2 u$$

$$u|_{t=0} = u_0 \qquad x \in \mathbb{T}_L^d$$

What one wants to study, assuming an initial distribution of u_0, is:

$$\lim_{\varepsilon \to 0} \lim_{L \to \infty} \mathbb{E}\left(|\hat{u}(\varepsilon^{-2}\tau, k)|^2\right) := n_k(\tau)$$

and $\partial_\tau n_k = Q(n_k(\tau))$. $\rightsquigarrow$ Wave Kinetic Equation (WKE)

Can we derive the Wave Kinetic Equation?

Fundamental original work on this by:

Peierls, Hasselman, Benney-Saffman-Newell

Newell, Zakharov, L'vov, Pomeau, Nazarenko, ---

In these works one starts from a certain dispersive equation (NLS, KdV ...) with parameters ε, L, and a background probability, then various types of *formal* approximations and limits are taken $\Longrightarrow$ **WKE** is obtained!

Prediction from physics

Consider the Zakharov-Kuznetsov (ZK) equation

$$\partial_t \phi(x,t) = -\Delta \partial_{x_1} \phi(x,t) + \varepsilon \partial_{x_1}(\phi^2(x,t)) \qquad x \in [-L,L]^d$$

Let $n_k(t) = \mathbb{E}(|\hat{\phi}(k,t)|^2)$. At the kinetic time $t = \tau \varepsilon^{-2}$ $\quad \tau \ll 1$

from:

$$\left\{ \partial_t |\hat{\phi}(k,t)|^2 = \varepsilon^2 Q[|\hat{\phi}(k,t)|^2] + \mathcal{O}(\varepsilon^{2+\delta}) \right\}$$

FORMAL DERIVATION WITH SOME PROBABILITY DISTRIBUTION

$$\Downarrow \qquad \begin{aligned} L &\to \infty \\ \varepsilon &\to 0 \end{aligned}$$

$$\boxed{\partial_\tau n_k(\tau) = Q(n_k(\tau))}$$

wave kinetic equation

When the Collision Operator is:

$$Q(n_{k_1}) = \int dk_2\, dk_3\, |W(k_1, k_2, k_3)|^2 \delta(\omega(k_3) + \omega(k_2) - \omega(k_1))$$

$$\times \delta(k_2 + k_3 - k_1) \left[n_{k_2} n_{k_3} - n_{k_1} n_{k_2} \operatorname{sign}(k_1^1) \operatorname{sign}(k_3^1) - n_{k_1} n_{k_3} \operatorname{sig}(k_1^1) \operatorname{sign}(k_2^1) \right]$$

- $\omega(k) = k^1 |k|^2$, $k = (k^1, \ldots, k^d)$
- $W(k_1, k_2, k_3) = |k_1^1 k_2^1 k_3^1|$

For Schrödinger $\omega(k) = |k|^2$

$\boxed{\text{Note}}$ $\partial_{x_1}(\phi^2(x,t))$ quadratic nonlinearity $\Longrightarrow$ $Q(n_{k_1})$ quadratic collision operator

The formal derivation

We take Fourier transform and we normalize:

$$a_k(t) := \hat{\psi}(t,k) / \sqrt{|k|}$$

Assume that a_k are Random Phase Amplitude (RPA) fields. We want to write

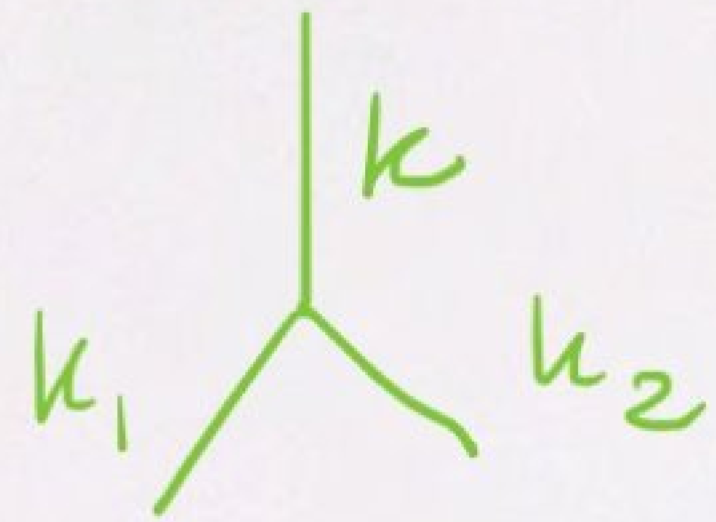
$$a_k(t) = a_k^{(0)}(t) + \varepsilon a_k^{(1)}(t) + \varepsilon^2 a_k^{(2)}(t) + \cdots$$

We derive $a_k^{(i)}$ $i = 0, 1, 2$ from the equation

$$\boxed{\dot{a}_k = i\omega(k) a_k + i\varepsilon \sum_{k=k_1+k_2} \operatorname{sig}(k^1)\, a_{k_1} a_{k_2}}$$

$$\boxed{a_k^{(0)}} = a_k(0) \quad \text{(initial datum)}$$

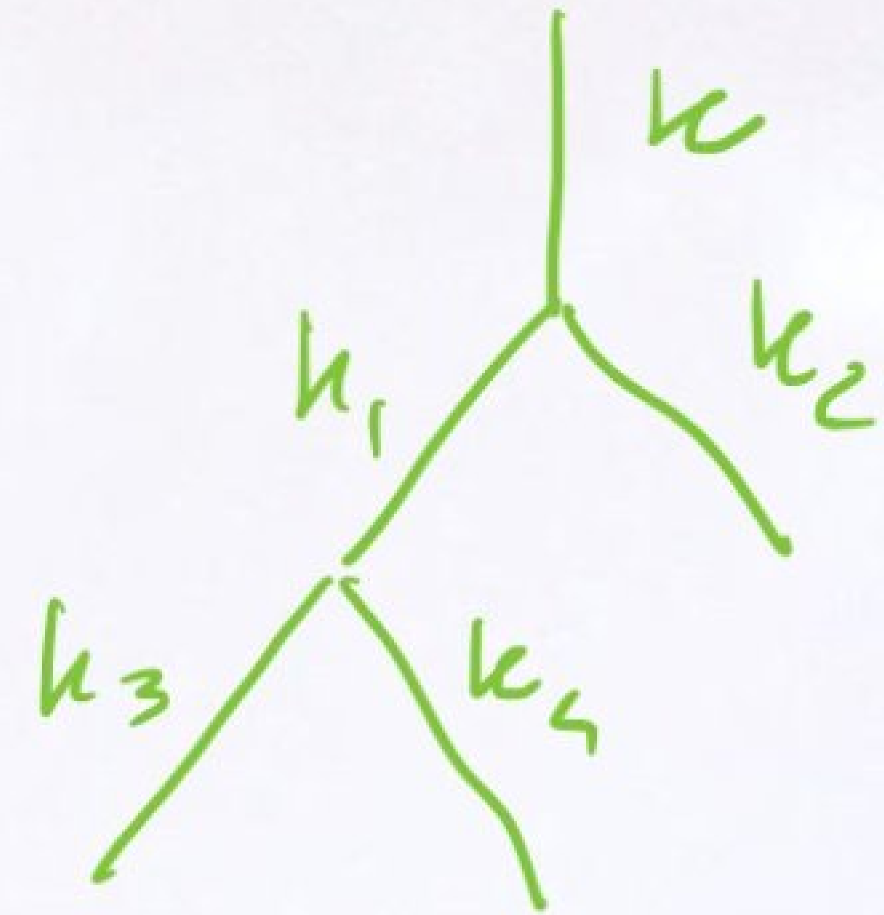
$$\boxed{a_k^{(1)}(t)} = -i\,\mathrm{sig}(k') \sum_{k=k_1+k_2} V_{k_1 k_2 k}\, a_{k_1}^{(0)} a_{k_2}^{(0)} \int_0^t e^{i\omega_{12}^k s}\, ds$$



$$\omega_{12}^k = \omega(k_1) + \omega(k_2) - \omega(k)$$

$$\boxed{a_k^{(2)}(t)} = -2 \sum_{\substack{k=k_1+k_2 \\ k_1=k_3+k_4}} \mathrm{sgn}(k^1 k_1^1)\, V_{k_1 k_2 k}\, V_{k_3 k_4 k_1} \cdot$$

$$a_{k_2}^{(0)} a_{k_3}^{(0)} a_{k_4}^{(0)} \int_0^t \int_0^s e^{i(\omega_{34}^1 \sigma + \omega_{12}^k s)}\, d\sigma\, ds$$



(Ladder)

Finally one writes

$$n_k(t) = \mathbb{E}(|a_k(t)|^2) \approx \langle (a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)})(a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)}) \rangle$$

(RPA) and keeping only ε^2

$$\Longrightarrow$$

$L \to \infty$ then $\varepsilon \to 0$ one gets the WKE

$$\boxed{\partial_t n_k(t) = Q(n_k(t))}$$

Mathematical Literature : Rigorous Derivation

- Erdos- Yau, Erdos- Salmhofer- Yau :

  Random $\boxed{\text{linear}}$ Schrödinger on a lattice setting

  → linear Boltzmann (kinetic time) → heat equation (diffusion time $t = \varepsilon^{-2-\varepsilon}$)

- F. Hernandez : Proved the same result above with completely different method.

- Lukkarinen - Spohn : Random $\boxed{\text{cubic}}$ NLS at equilibrium and on a lattice setting

  → (linearized) wave kinetic equation at kinetic time.

Recent Results : Out of equilibrium

Random Initial Data :

- Buckmaster - Germain - Hani - Shatah : NLS in continuum case
  $\longrightarrow$ below kinetic time (linear kinetic equation)

- Collot - Germain , Deng - Hani : NLS in continuum case
  $\longrightarrow$ strictly below kinetic time (linear kinetic equation)

- Deng - Hani : NLS in continuum case
  $\longrightarrow$ at kinetic time (nonlinear kinetic equation)

$i\partial_t \phi + \Delta\phi = \varepsilon|\phi|^2\phi$, on periodic torus $[0,L]^d$ $d \geq 3$, L, ε linked.

- Lukkarinen - Vuoksenmaa : NLS in lattice case
  $\longrightarrow$ at kinetic time $d \geq 4$.

- Ma : ZK equation with dissipation $\longrightarrow$ WKE before kinetic time.

Our result: the setting

We consider the stochastic ZK equation

$$\begin{cases} d\phi(x,t) = -\Delta\partial_x \phi(x_1,t)\,dt + \varepsilon\partial_{x_1}(\phi^2(x,t))\,dt + \underbrace{\varepsilon^{\theta}\partial_{x_1}\phi \odot dW(t)}_{\text{Stochastic term}} \\ \phi(x,0) = \phi_0(x) \longrightarrow \text{randomly distributed} \end{cases}$$

$\varepsilon \ll 1, \quad 0 < \theta \ll 1$

The equation is considered on a lattice

$$\Lambda = \{0, 1, \dots 2D\}^d$$

$d \geq 2$ (dimension)

D in $\mathbb{N}$.

Passing to frequency space

We write $k = (k^1, \dots, k^d) \in \Lambda_* = \left\{ -\frac{D}{2D+1}, \dots, 0, \dots, \frac{D}{2D+1} \right\}^d$

$$\omega_k = \omega(k) = \sin(2\pi k^1)\left[\sin^2(2\pi k^1) + \dots + \sin^2(2\pi k^d)\right]$$

dispersion relation

$(k^1 |k|^2)$

(in continuum)

$\overline{\omega_k} = \sin(2\pi k^1)$

$$W(x,t) = \sum_{k \neq 0} \frac{W_k(t)}{\overline{\omega}(k)} e^{i 2\pi k \cdot x}$$ [Stochastic term]

$\{W_k(t)\}$ = sequence of independent real Wiener processes on (Ω, F, P).

$W_{-k}(t) = -W_k(t) \quad \forall\, k \in \Lambda^* = \Lambda_* \setminus \{0\}$.

Set $a_k = \dfrac{\hat{\psi}(k)}{\sqrt{|\bar{\omega}(k)|}}$ and rewrite the equation

$$\boxed{\begin{aligned} da_k &= i\omega(k)\, a_k\, dt + i\varepsilon^{\theta} a_k \circ dW_k + \\ &\quad i\varepsilon \sum_{k=k_1+k_2} \mathrm{sig}(k^1) \sqrt{|\bar{\omega}(k)\, \bar{\omega}(k_1)\, \bar{\omega}(k_2)|}\; a_{k_1} a_{k_2}\, dt \end{aligned}}$$

Definition [two points correlation function] → density function

$$f(a(t)) = \int |a(t)|^2 \, d\rho(t) := \langle a\, \bar{a} \rangle$$

Statement of main result

Consider the two-point correlation function

$$f(k,t) = \langle a(t,k)\, \bar{a}(t,k) \rangle = \int d\rho \, |a_t(k)|^2$$

Main Theorem [S. Tran] Assume dimension $d \geq 2$. Under suitable (but general) assumptions on the initial density function $\rho(0, b_1, b_2)$, if $t = \varepsilon^{-2}\tau = O(\varepsilon^{-2})$, $\tau \leq \tau_0$

"$$\lim_{\varepsilon \to 0, D \to \infty} f(k, \varepsilon^{-2}\tau) = f^{\infty}(k,\tau)$$" and

$$\boxed{\frac{\partial}{\partial \tau} f^{\infty}(k,\tau) = Q(f^{\infty})(k,\tau)}$$

3-wave kinetic equation

Some Preliminary Remarks

Dispersion Relation: $\omega(k) = \sin(2\pi k^1)\left[\sin^2(2\pi k^1) \cdots \sin^2(2\pi k^d)\right]$

- If $k^1 = 0$ then $\omega(k) = 0$. the hyperplane $k^1 = 0$ is a serious obstruction for 3-waves interactions since lots of them are allowed to happen here!

This does not happen for NLS ($\omega(k) = |k|^2$)

- the noise we add in the equation, that preserves energy and vanishes as $\varepsilon \to 0$, has the role of "kicking" the frequencies trapped in the hyperplane away from it.

The density function ρ: (Inspired by work of Faou)

Consider the Hamiltonian $H(a,\bar{a}) = \underbrace{H_1}_{\text{kinetic}} + \varepsilon \underbrace{H_2}_{\text{potential}}$

We introduce the real processes $B_{1,k}$, $B_{2,k}$ s.t. $a_k = B_{1,k} + i B_{2,k}$

$$\left.\begin{aligned} dB_{1,k} &= -\omega_k B_{2,k}\, dt - \varepsilon^{\theta} B_{2,k} \circ dW_k - \varepsilon \frac{\partial H_2(B_1, B_2)}{\partial B_{2,k}} \\ dB_{2,k} &= -\omega_k B_{1,k}\, dt - \varepsilon^{\theta} B_{1,k}\, dW_k + \varepsilon \frac{\partial H_2(\cdot)}{\partial B_{1,k}} \end{aligned}\right\} (*)$$

Question: If we start with an initial density $\rho(0, b_1, b_2)$ how does it evolve if the relative process evolves as in $(*)$?

Answer: the density function $\rho(t)$ solves the equation

$$\boxed{\begin{aligned} d\rho = &\sum_{k\in\Lambda^*} \omega_k \left(b_{2,k}\partial_{b_{1,k}} - b_{1,k}\partial_{b_{2,k}}\right)\rho + \\ &\varepsilon^{2\theta} \sum_k \left(-b_{2,k}\partial_{b_{1,k}} + b_{1,k}\partial_{b_{2,k}}\right)^2 \rho \\ &+ \varepsilon\left(\sum_k h_a(k)\partial_{b_{1,k}}\rho + \sum_k h_b(k)\partial_{b_{2,k}}\rho\right) \end{aligned}}$$

important

Liouville equation

Where $h_a(k)$, $h_b(k)$ are functions of $b_{1,k}$, $b_{2,k}$.

Note: In our work the analysis of this equation is fundamental.

On the equation for the density function:

The noise effects only the angular part!

If we write $b_{1k} + i\, b_{2k} = \sqrt{2 c_{1,k}}\, e^{i c_{2,k}}$ then

$$\boxed{\partial_t \rho = \sum_k 4 c_{1,k} \omega_k \partial_{c_{2,k}} \rho + \varepsilon^{2\theta} \sum_k 16 c_{1,k}^2 \partial_{c_{2,k}}^2 \rho + \varepsilon \sum_k h^a(k) \partial_{c_{1,k}} \rho + \varepsilon \sum_k h^b(k) \partial_{c_{2,k}} \rho}$$

Liouville Equation in polar coordinates

Note: there is dissipation w.r.t. the angular variable, so it will behave well.

Ingredients of the proof

- the properties from the Liouville equation
- Duhamel expansion and Feynmann Diagrams/Graphs
- Study of the graphs and the difficulties coming from the singularities of $\omega(k)$.
- Leading Graphs and Non Leading Graphs
- Crossing estimates and the counterexample of Lukkarinen.
- Resonance Broadening

Properties of the density ρ

A key property that we use about the density function ρ are the moments bounds.

- In the pioneering work of Lukkarinen - Spohn these bounds come from imposing the assumption on the "ℓ^1-clustering estimate" at equilibrium.

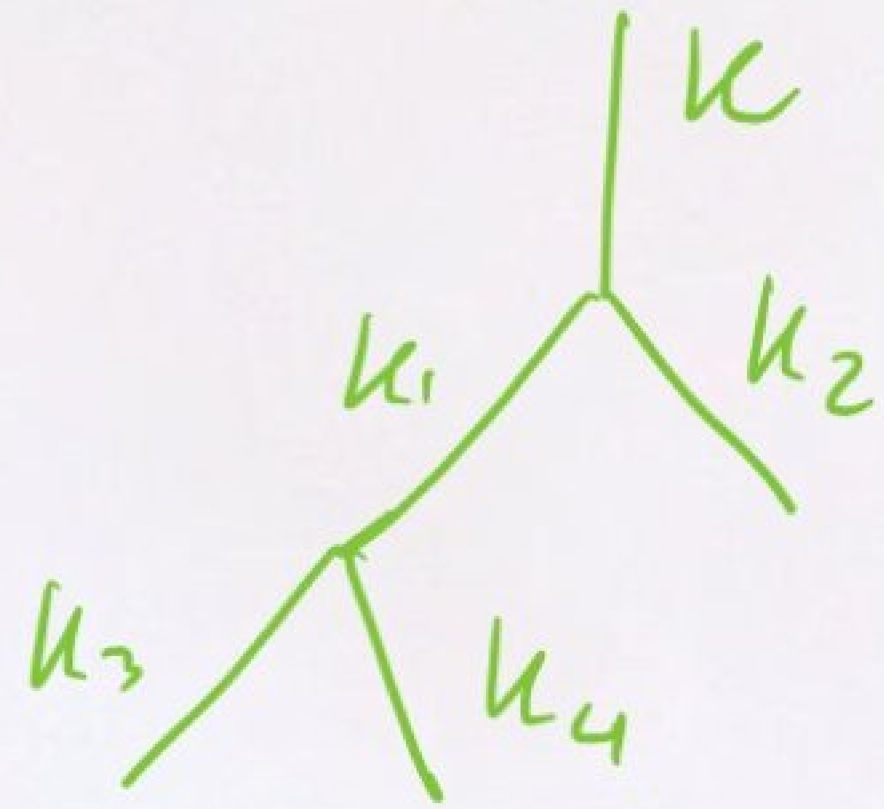
- this assumption has been proved only for the zero boundary conditions and at equilibrium (Abdesselam)

- In our case these estimates are not an assumption but they are derived using the Liouville equation of the density function.

Feynman Diagrams

An important tool for the analysis is the iteration via Feynman graphs using the Duhamel Expansion:

In formal derivation ignoring all ε^n, $n > 2$:



Full derivation with all ε^n, $n \geq 0$

Lots and lots of diagrams ...

Duhamel expansions and graphs

We employ a framework that is built on top of the techniques developed in work of Erdos-Salmhofer-Yau and Lukkarinen-Spohn.

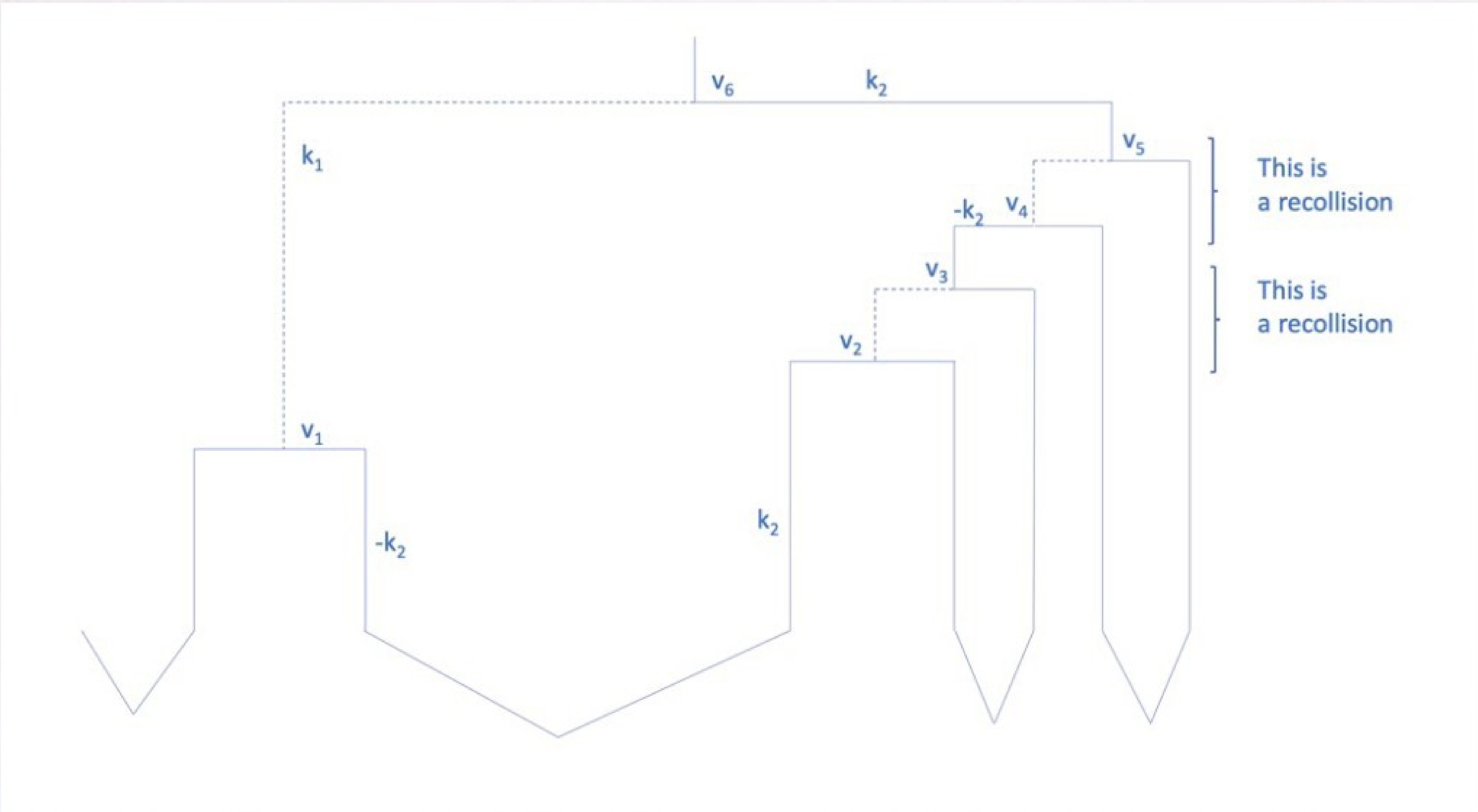
- Duhamel's expansion and Feynmann's diagrams with a stopping rule. We expand up to m layers, $m=m(\varepsilon)$ and $m(\varepsilon) \xrightarrow[\varepsilon \to 0]{} \infty$.
- We separate the graphs into leading graph and non leading graphs:

$\mathcal{G}_L$: leading graphs $\rightsquigarrow$

$\mathcal{G}_{NL}$: Non - leading $\overset{\theta,\ \text{large}}{\rightsquigarrow}$

$\boxed{\text{these are ladder Graphs} \Rightarrow \text{the kinetic equation}}$

$\boxed{\text{"}\lim_{\varepsilon \to 0} \mathcal{G}_{NL} = 0\text{"}}$



Example of a ladder graph

- Due to the singularity of the dispersion function $\omega(k)$ we need to define Ghost Manifolds. they only are in NL graphs.

NL graphs + GM

here we use the noise for convergence since no oscillations present.

NL graphs without GM

no need for noise here.

<u>Note</u> the noise does not appear in the limiting WKE because it acts only on angles and the WKE equation governs the amplitudes.

Challenges in the implementation

Bad news #1: The counterexample of Lukkarinen in estimating the non-leading graphs.

Bad news #2: The bilinear nonlinearity

Bad news #3: the limits in the leading graphs need to be taken in a weaker sense

Remarks: these challenges do not surface for the continuum NLS equation.

Bad News 1): The counterexample

"An analytic dispersion relation $\omega(k)$ suppresses crossings if and only if it is not a constant on any affine hyperplane" [Lukkarinen]

A counterexample, in which the crossing estimate fails to hold true, has also been introduced which covers the ZK equation on the lattice case:

$$\omega(k) = \sin(2\pi k^1)\left[\sin^2(2\pi k^1) + \cdots + \sin(2\pi k^d)\right]$$

Bad News 2) : the bilinear nonlinearity

- In the NLS case the cubic nonlinearity allows for the use of an L^3 norm that decays in time.

- In the ZK case with the same argument one would need to use an L^2 norm, which is conserved and no decay comes from it.

Fixing : We note that the noise cannot fix this since it does not appear in all of these graphs. So the only option is to understand better the graphs at hand and use weaker norms.

Bad news 3): The convergence of the leading graphs

Unlike the Schrödinger dispersion relation, the lattice ZK dispersion relation not only creates major issues in the crossing estimates, but also it prevents the convergence of the leading diagrams.

<u>What is</u> the <u>problem</u>? Making sense of a certain measure.

It is common practice to assume that

$$\int_{\mathbb{T}^{2d}} dk_2\, dk_3\, \delta(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))\, F(k_2+k_3, k_2, k_3)$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{T}^{2d}} dk_2\, dk_3\, e^{-is(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))} F(k_2+k_3, k_2, k_3) \quad (*)$$

for any test function $F \in \mathcal{C}^\infty(\mathbb{T}^{2d})$ and for any dispersion relation $\omega(k)$.

Problem: For singular dispersion relations the quantity $\delta(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))$ cannot be defined as positive measure, hence the equality above is not mathematically rigorous.

Resonance Broadening

- When ω is the lattice ZK dispersion relation, the delta function $\delta(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))$ cannot be defined as a positive measure, yielding the divergence of the leading graphs that contain osillatory integrals as in $(*)$

- there are a few common resonance broadening strategies (very common in physics). We pick the following

$$\delta_\ell\left(\omega(k_3)+\omega(k_2)-\omega(k_1)\right)=\frac{1}{2\ell}\int_{-\ell}^{\ell} d\xi \int_{\mathbb{R}} ds\, e^{-is(\omega(k_3)+\omega(k_2)-\omega(k_1))-i2\pi s\xi}$$

$\ell > 0$

and we use δ_ℓ, $\ell > 0$ instead of δ.

Looking Forward

- How is Energy Cascade linked to WKE?
- What happens after the Kinetic time $t = \varepsilon^{-2}$?
- there is a predicted quantistic wave kinetic equation (see recent result of well posed of this by Rosenzweig - S.). How can we derive it?
- Wave kinetic equation for other dispersive systems.
- Study the inhomogeneous 3-wave kinetic equation (this was recently solved for ZK by Hannani - Rosenzweig - S - Tran.)


Thank you