Some recent developments in wave turbulence theory: Part 2

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Outline of the talk

- Recalling global well-posedness for cubic periodic 2D NLS
- Transfer of energy.
- Polynomial bounds for the H^s norm of solutions: the rational versus the irrational case.
- Norm growth in the rational case: the C-K-S-T-T result and the Carles-Faou result.
- The irrational case: the old and the new.

On global well-posedness

Consider the initial value problem (IVP)

$$\text{(NLS)}\begin{cases} i\partial_t u + \Delta_{\mathbb{T}^2} u = |u|^2 u \\ u|_{t=0} = u_0 \qquad\qquad x \in \mathbb{T}^2 \end{cases}$$

$$\widehat{\Delta_{\mathbb{T}^2}}(k_1, k_2) = \omega_1^2 k_1^2 + \omega_2^2 k_2^2 =: \lambda_k \qquad (k_1, k_2) \in \mathbb{Z}^2 \qquad \omega_i^2 > 0$$

If $\omega_1^2/\omega_2^2 \in \mathbb{Q}$ rational torus

If $\omega_1^2/\omega_2^2 \in \mathbb{R} \setminus \mathbb{Q}$ irrational torus

$$(\text{NLS})\begin{cases} i\partial_t u + \Delta_{\mathbb{T}^2} u = |u|^2 u \\ u|_{t=0} = u_0 \qquad x \in \mathbb{T}^2 \end{cases}$$

$M = \int_{\mathbb{T}^2} |u|^2 dx$ = mass

$E = \int_{\mathbb{T}^2} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4\right) dx$

= energy (conserved!)

Theorem [Bourgain] If $u_0 \in H^s$, $s \geq 1$, $\exists!$ global solution $u(t,x) \in X_T^{s,b} \subseteq C([0,T], H^s)$ for all $T > 0$ and stable.

Question: What can we say about $\lim_{|t|\to\infty} |u(t,x)|$?

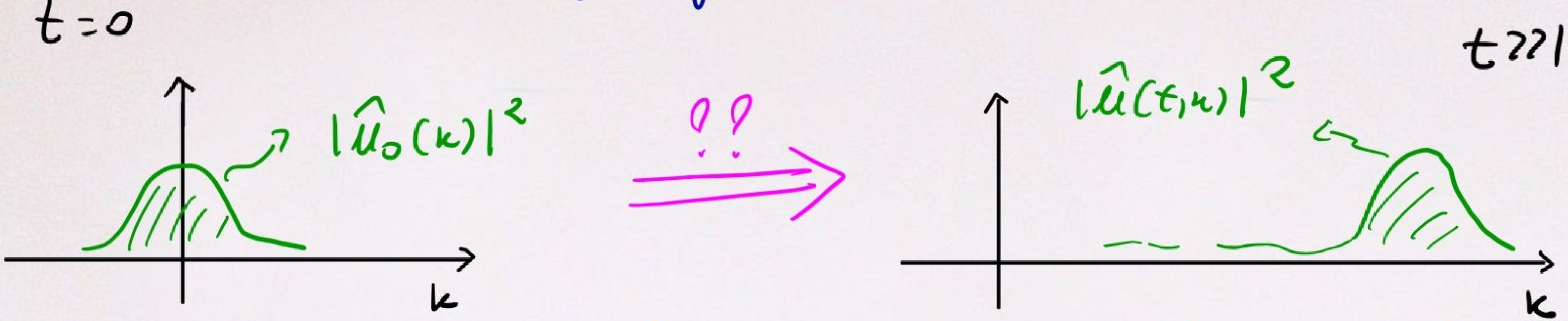
Remark: In $\mathbb{R}^2$ we have scattering: $\exists\, u^+, u^- \in H^s(\mathbb{R}^2)$ s.t.

$$\lim_{t\to\pm\infty} \| u(t) - S(t) u^{\pm} \|_{H^s(\mathbb{R}^2)} = 0 \qquad s \geq 0$$

$$\Rightarrow \quad \| u(t) \|_{H^s} \leq C_s \quad \forall\, t \in \mathbb{R}.$$

(Dodson)

Transfer of energy



This is called transfer of energy or forward cascade.

Remarks: If transfer happens it is a "rigid process".

- Mass conserved $\iff$ area subgraph conserved
  (also conservation of energy)
- the solution $u(t,x)$ takes some time to "see" the boundary and the irrationality of the torus may "mitigate the impact."

Growth of Sobolev Norms

To "see" transfer of energy analyze the growth of

$$\boxed{\sum_k \langle k \rangle^{2s} |\hat{u}(t,k)|^2 \simeq \|u(t)\|^2_{H^s} \qquad s \gg 1 \quad \text{as } t \to \pm\infty}$$

Remarks:

- In $\mathbb{R}^2$ there is no growth thanks to scattering
- In $\mathbb{R}$ and $\mathbb{T}$ there is no growth thanks to integrability. $\Rightarrow$ Infinite many conserved integrals.

Bounds from above

Theorem [Bourgain, Sohinger, Planchon - Tzvetkov - Visciglia]

the solution $u(t,x)$ of NLS is s.t.

$$\|u(t)\|_{H^s} \leq C(1+|t|)^{s-1+\varepsilon} \quad \forall t, \varepsilon > 0$$

note dependence on s!

The proof is based on an improved local estimate:

$$\|u(n)\|_{H^s} \leq 1 \, \|u(n-1)\|_{H^s} + \hat{C} \, \|u(n-1)\|_{H^s}^{1-\sigma}$$

$$\sigma^{-1} = (s-1)+\varepsilon$$

(high-low method, upsidedown I-method, some energy method)

Better for irrational tori

Theorem [Deng - Germain] For NLS with $|u|^{p-1}u$, $\underbrace{3<p<5}_{\text{subcritical}}$ in generic tori $\mathbb{T}^3$ one has

$$\|u(t)\|_{H^s} \le C(1+|t|)^{\frac{2}{\underbrace{5-p+\theta(p)}_{\text{not here for } \mathbb{T}^3 \text{ rational}}}}$$

where $\theta(p) = \min(p-3, 5-p)/182$

Theorem [Deng] If $p=5$ (energy critical) $\exists\, \mathcal{U} \subseteq \mathbb{R}^3_+$ of measure zero s.t. $\forall\, \vec{w} = (T_1, T_2, T_3) \in \mathbb{R}^3_+ \setminus \mathcal{U}$, if $E(u_0)$ is small then for the global solution $u(t,x)$ one has

$$\|u(t)\|_{H^s} \le C \max\left(\|u_0\|_{H^s}, |t|^{300(s-1)}\right) \quad \forall t$$

What is the expected upper bound?

Theorem [Bourgain] Consider the IVP $\begin{cases} i\partial_t u + \Delta u + V(t,x)u = 0 \\ u|_{t=0} = u_0 \quad \text{in } \mathbb{T}^d \end{cases}$

1) If $V(t,x)$ is bounded, periodic also in time rational and smooth, then $\|S_V(t)u_0\|_{H^s} \leq C(|t|+1)^{\varepsilon} \quad \forall \varepsilon > 0$

2) If $V(t,x)$ is bounded, real analytic, quasi periodic in time for $d=1,2$ and small for $d=2$ then.

$$\|S_V(t)u_0\|_{H^s} \leq C_1[\log(1+|t|)]^{C_2} \quad C_1, C_2 > 0$$

Large literature on the topic: Bambusi, Berti, Colliander, Delort, Guardia, Hani, Haus, Maspero, Oh, Procesi, W.M. Wang ....

Bounds from below

Question: Can one exhibit a NLS solution $u(t,x)$ that actually has a growing in time Sobolev norm?

Bourgain: When proving result 2) above he also shows that the log growth is sharp by constructing a growing solution with that rate.

(see also Berti, Maspero ...)

Remark: For the nonlinear problem the situation is not clear (see also Kuksin)

Proving "some" growth in nonlinear cases

Theorem [Colliander - Keel - S - Takaoka - Tao] Consider the IVP

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \quad x \in \mathbb{T}^2 \text{ (rational)} \end{cases}$$

Fix $s > 1$, $0 < \sigma \ll 1$, $K \gg 1$. Then $\exists\, u_0 \in H^s$ s.t. $\|u_0\|_{H^s} \leq \sigma$ and $\exists\, T \gg 1$ s.t. $\|u(T)\|_{H^s} \geq K$.

Also holds for irrational numbers "close" to rational

Giuliani - Guardia

Remark: this is a "weak" result since we do not know what happens after time T.

(See also the work of Guardia - Hani - Haus - Maspero - Procesi)

Idea of the proof

Solve for $u(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t)\, e^{i(t|n|^2 + x \cdot n)}$ (assume $\mathbb{T}^2$ square)

$$\Rightarrow \boxed{i \partial_t a_n = |a_n|^2 a_n - \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \overline{a_{n_2}} a_{n_3}} \quad \text{(FNLS)}$$

$$\Gamma(n) = \left\{ (n_1, n_2, n_3) \;/\; \begin{array}{l} n = n_1 - n_2 + n_3 \\ \omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0 \end{array} \right\}$$

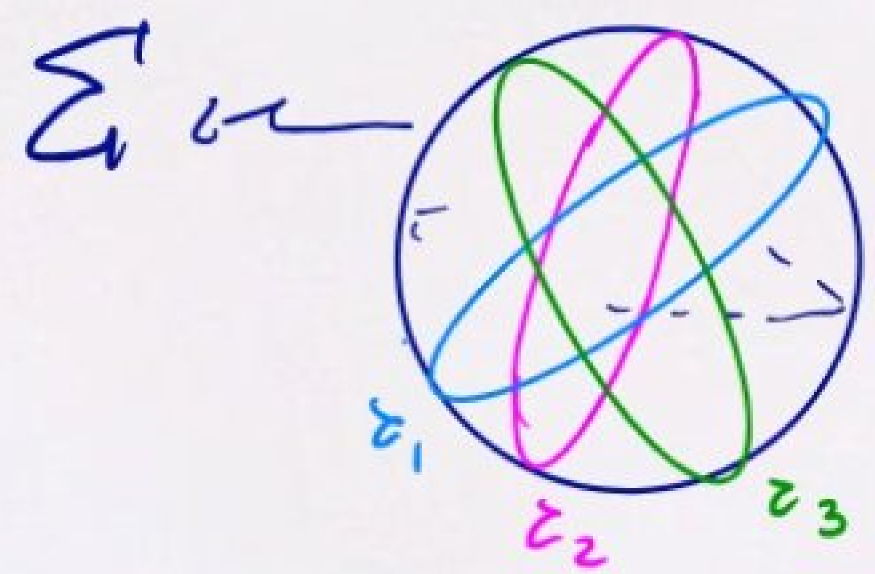
(n, n_1, n_2, n_3) are in resonance

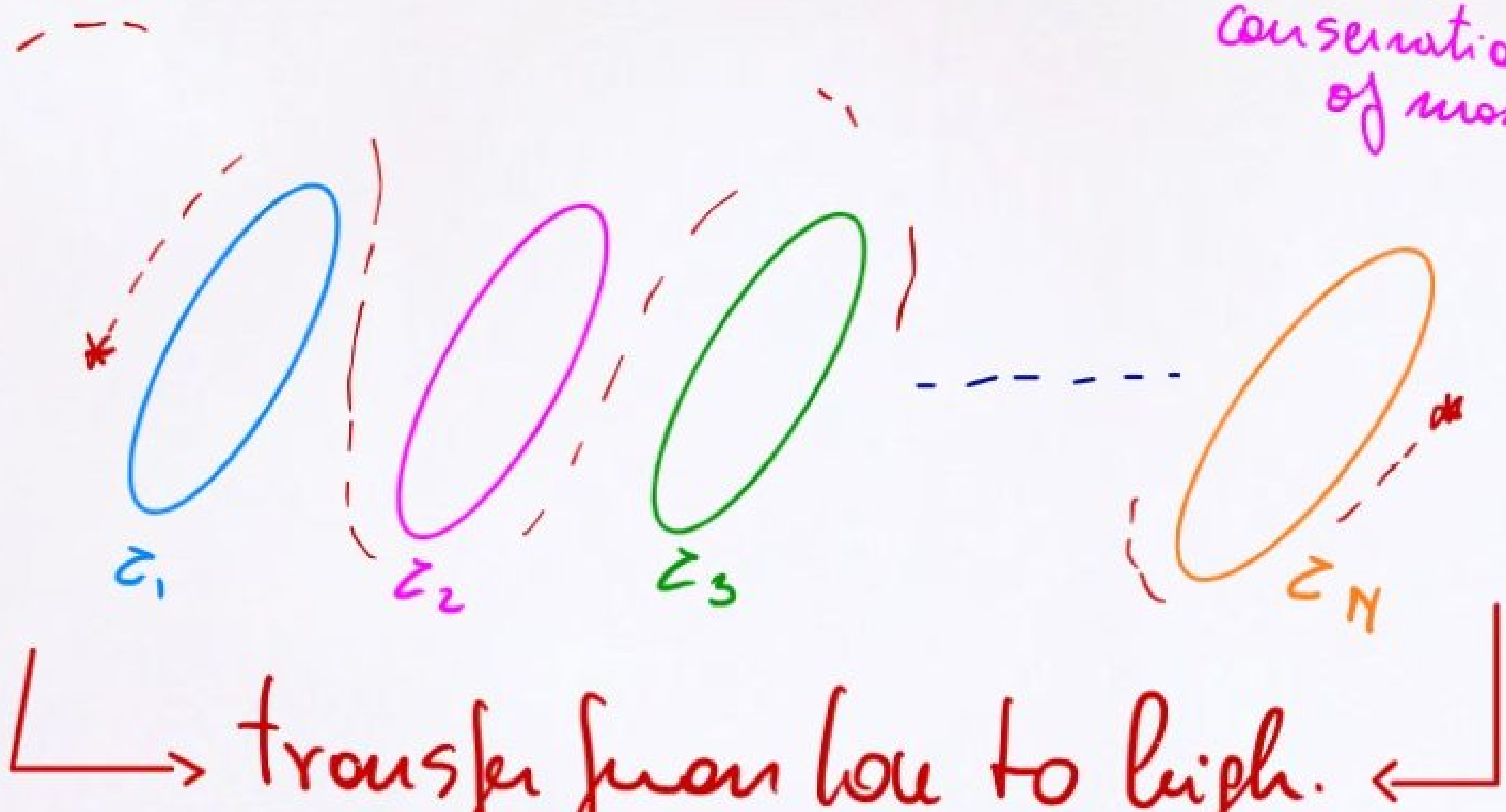
By making further restrictions on the resonant system ...

$$\Rightarrow \boxed{\begin{cases} i\,\dot b_j = |b_j|^2 b_j - 2 b_{j-1}^2 \bar b_j - 2 b_{j+1}^2 \bar b_j \\ b_1(t) = b_N(t) = 0 \qquad\qquad j = 1, \dots, N \\ b_j(0) = b_j \end{cases} \quad \text{(Toy Model)}}$$

The Toy Model "lives" on $\Sigma_1 = \{ x \in \mathbb{C}^N / |x|^2 = 1 \}$

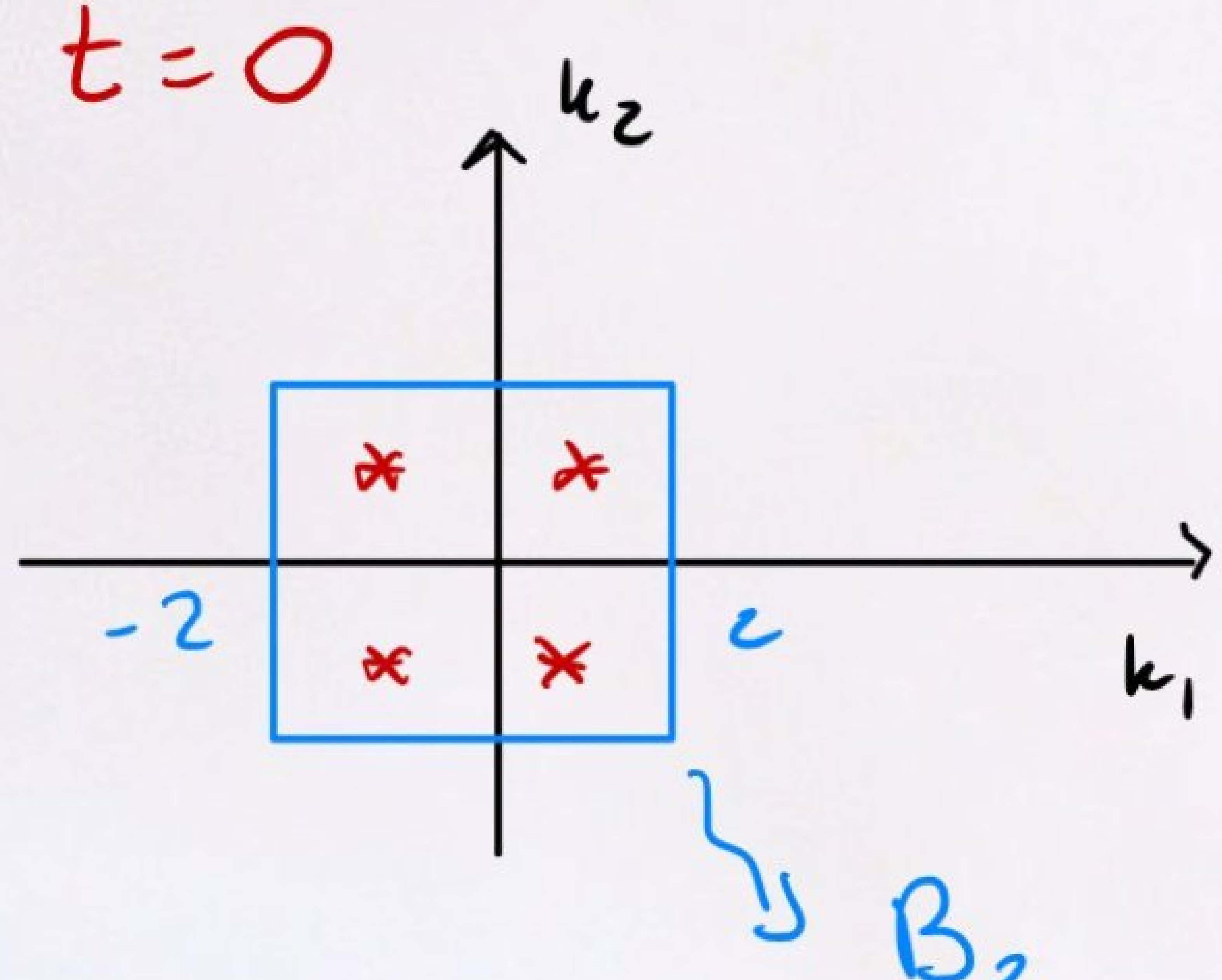
conservation of mass.

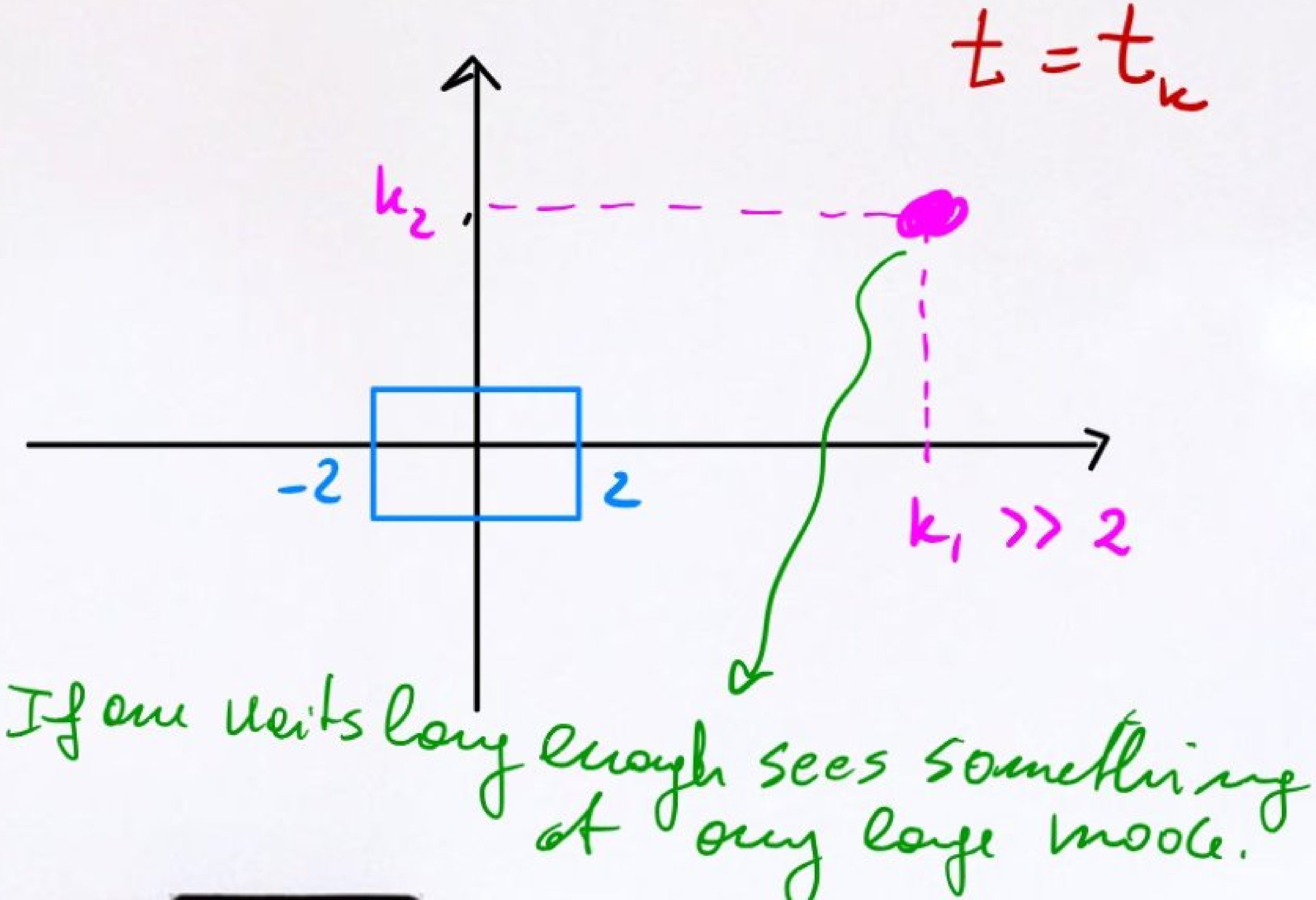




Theorem [Carles - Faou] Consider the cubic (focusing/defocusing) NLS in $\mathbb{T}^2$ rational. Fix $s>1$. $\exists$ a solution $u(t,x)$ s.t. $\operatorname{supp} \hat{u}_0 \subseteq B_2$, $\|u_0\|_{H^s} < \varepsilon$ and $\forall k \in \mathbb{Z}^2$ $\exists\, t_k$ s.t.

$$|\hat{u}(t_k, k)| > \varepsilon^{1+\sigma} \quad \text{for } \sigma > 0$$





Other important results

- Hani - Pausader - Tzvetkov - Visciglia:
the cubic, defocusing NLS on $\underline{\mathbb{R}\times\mathbb{T}^d}$ (rational) for $d=2,3,4$
at $t=\pm\infty$ presents a dynamics dictated by the Toy Model

$\Longrightarrow$

$$\boxed{\|u(t_n)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} \geq \exp\left(C(\log\log t_n)^{\frac{1}{2}}\right) \quad \text{for a sequence } t_n\to\infty}$$

- Gérard - Grellier: they prove a very precise
asymptotic growth for the Szëgo system.

What can we say when $\mathbb{T}^d$ is irrational?

Theorem [S-Wilson] Consider $\mathbb{T}^2$ irrational and fix $s > 1$ and $M > 0$. Suppose $u_0 \in C^\infty(\mathbb{T}^2)$ and $\operatorname{supp} \hat{u}_0 \subseteq B_M$. $\exists\, \varepsilon > 0$ s.t. if $\|u_0\|_{H^s} < \varepsilon$ the unique solution to the cubic NLS $u \in C([0, \varepsilon^{-2}], H^s)$ and $|\hat{u}(t,k)| \leq \varepsilon^3$ for all $t \in [0, \varepsilon^{-2}]$ and $k \notin B_M$.

Remarks:

1) Comparing this result with that of Colliander - Faou one can see that their dynamics cannot happen in an irrational torus.

2) This theorem is for small data. Also it does not say that growth cannot happen on irrational tori, but that one needs possibly a different mechanism to construct it.

3) In a sense this result is in agreement with the better polynomial bounds for the Sobolev norms of solutions of NLS equations obtained on irrational tori by Germain and Deng.

Idea of the proof:

Step 1: We use an infinite dimensional Birkoff Normal Form Reduction. Here the description of the 4-waves resonant set is key.

Step 2: We analyze the dynamics of the reduced system (resonant part) and we prove that for this system the solution is completely concentrated in the ball B_{2M}.

Step 3: We go back to the original problem with the change of variables dictated by the BNFR and we show that outside B_{2M} all modes stay "very" small.

The 4-waves resonant set for $\mathbb{T}^2$ irrational

Recall that $\widehat{\Delta_{\mathbb{T}^2}}(k_1,k_2) = \omega_1^2 k_1^2 + \omega_2^2 k_2^2 := \lambda_k$

$$\mathcal{R} = \left\{ (k_1,k_2,k_3,k_4) \;/\; \begin{array}{l} k_1 - k_2 + k_3 - k_4 = 0 \\ \lambda_{k_1} - \lambda_{k_2} + \lambda_{k_3} - \lambda_{k_4} = 0 \end{array} \right\} \quad \text{resonant set}$$

$\Downarrow$ (thanks to the irrationality!)

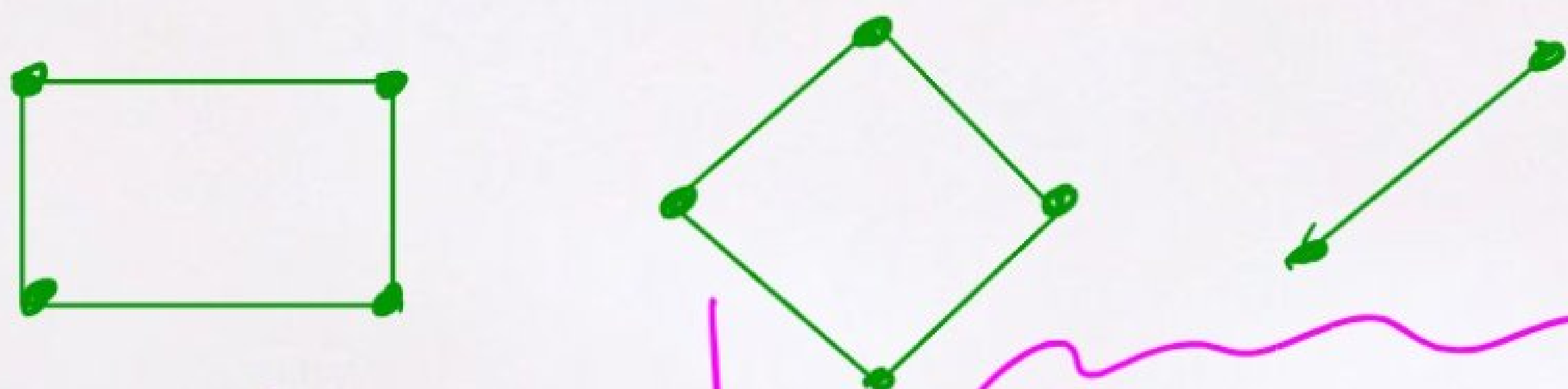
$$\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$$

$$\mathcal{R}_i := \left\{ (k_1,k_2,k_3,k_4) \;/\; \begin{array}{l} k_1^i + k_3^i = k_2^i + k_4^i \\ (k_1^i)^2 + (k_3^i)^2 = (k_2^i)^2 + (k_4^i)^2 \end{array} \right\}$$

There is a decoupling into two 1D resonant sets!

Remarks: If (k_1, k_2, k_3, k_4) are in resonance, either for a rational or irrational torus, then (k_1, k_2, k_3, k_4) are vertices of rectangles.

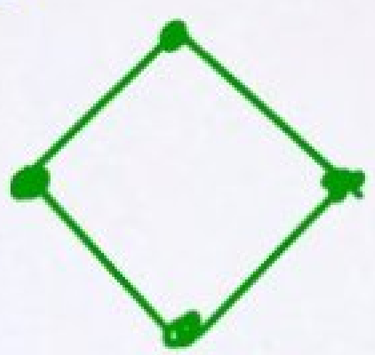
- If $\mathbb{T}^2$ is rational the rectangles could be



degenerate

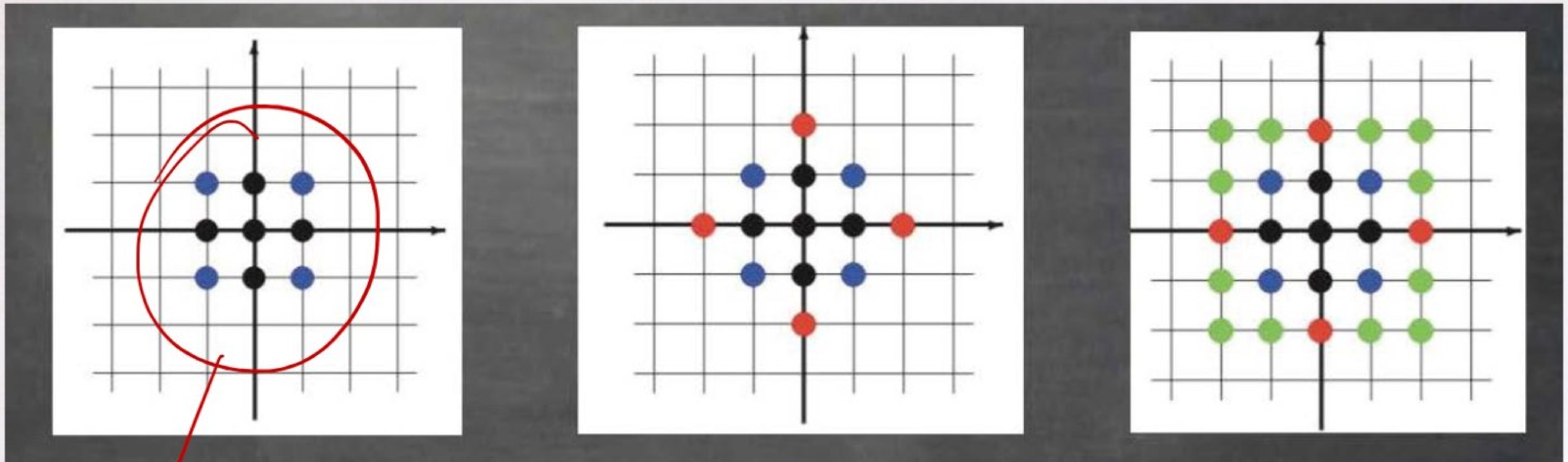
these are both needed in C-K-S-T-T and C-F.

- If $\mathbb{T}^2$ is irrational the diamond configuration



cannot occur.

Set for the dynamics of C-K-S-T-T and C-F examples



the initial data is supported in the 4 blue dots.

the diamond configuration of the 4 red dots is not allowed in the irrational case

Work with A. Hrabski, Y. Pan, B. Wilson

In these results we remove the smallness assumption and developed numerical experiments. In all the theorems below we assume $\omega_1^2/\omega_2^2 = \alpha$ irrational and algebraic. (think $\sqrt{2}$).

"Theorem": Consider a "quasiresonant" cubic NLS in $\mathbb{T}^2_\alpha$, α algebric. Then if the initial data is compact supported in the frequency space, the H^s norms of the solution are uniformely bounded.

The quasi resonant set

Definition: Fix $\Lambda, \varepsilon > 0$, we define the quasi-resonant set

$$\boxed{\Omega(\Lambda,\varepsilon) = \left\{ (k_1, k_2, k_3, k_4) \;/\; k_1 + k_3 = k_2 + k_4 \text{ and } \left|\lambda_{k_1} - \lambda_{k_2} + \lambda_{k_3} - \lambda_{k_4}\right| \leq \frac{\Lambda}{(|k_1|^2 + |k_2|^2 + |k_3|^2 + |k_4|^2)^{1+\varepsilon}} \right\}}$$

where λ_k = k-th eigenvalue of $\Delta_{\mathbb{T}^2_\alpha}$

Theorem Consider the IVP

$$(NLS)^* \begin{cases} i\partial_t v + \Delta v = C(|v|^2 v)^* \\ v|_{t=0} = u_0 \end{cases}$$

this is the non-resonant part of $|v|^2 v$

$\operatorname{supp} \hat{u}_0 \subseteq B_R$ Then the $(NLS)^*$ conserves the mass (i.e. $\|v(t)\|_{L^2} = \|u_0\|_{L^2}$) is globally well-posed in $L^2(\mathbb{T}^2_\alpha)$. Moreover $\exists\, M > 0$ s.t.

$$\|F^{-1}(\chi_{B_M^c} \hat{v}(t))\|_{H^s} = 0 \quad \forall t, \ \forall s \geq 0$$

Note: Here M depends on (Λ, ε), on R and on the irrationality of α.

Remarks on theorem

1) In the C-K-S-T-T work the dynamics of the Toy model was happening in within that of the (NLS)*. Theorem 2 confirms more in details that in the irrational case there is no growth from the resonant system.

2) Note that global well posedness in L^2 for the full periodic cubic NLS is a major open problem. This is due to the loss of ε-derivative in the Strichartz estimates.

Corollary: Assume u_0 is such that

$$\operatorname{supp} \hat{u}_0 \subseteq B_R,$$

if v is solution to $(NLS)^*$ s.t. $v(0) = u_0$, then

$$\|v(t)\|_{H^s} \leq C \qquad \forall t \in \mathbb{R} \text{ and } \forall s \gg 1.$$

Proof: $\exists M$ s.t. $\displaystyle\sum_{|k|>M} \langle k\rangle^{2s} |\hat{v}(t,k)|^2 = 0$, then

$$\sum_{k\in\mathbb{Z}^2} \langle k\rangle^{2s} |\hat{v}(t,k)|^2 = \sum_{|k|\leq M} \cdots + \underbrace{\sum_{|k|\geq M} \cdots}_{=0} \leq M^{2s} \sum_k |\hat{v}(t,k)|^2$$

$$= M^{2s} \|u_0\|_{L^2}^2$$

Ingredients for the proof

- Roth's theorem : This theorem allows us to say that

quasi-resonant

finite

$$\# \left\{ (k_1, k_2, k_3, k_4) \Big/ \begin{array}{c} k_1 + k_3 = k_2 + k_4 \\ 0 < |\lambda_{k_1} + \lambda_{k_3} - \lambda_{k_2} - \lambda_{k_4}| < \frac{\Delta}{(|k_1|^2 + |k_2|^2 + |k_4|^2 + |k_3|^2)^{1+\varepsilon}} \end{array} \right\} \leq C_{\Delta, \alpha, \varepsilon}$$

non resonant.

- The decoupling of the resonant set $R = R_1 \cup R_2$ into two 1D resonant sets: Recall that in 1D the cubic NLS is globally well-posed in L^2 and it is also integrable.

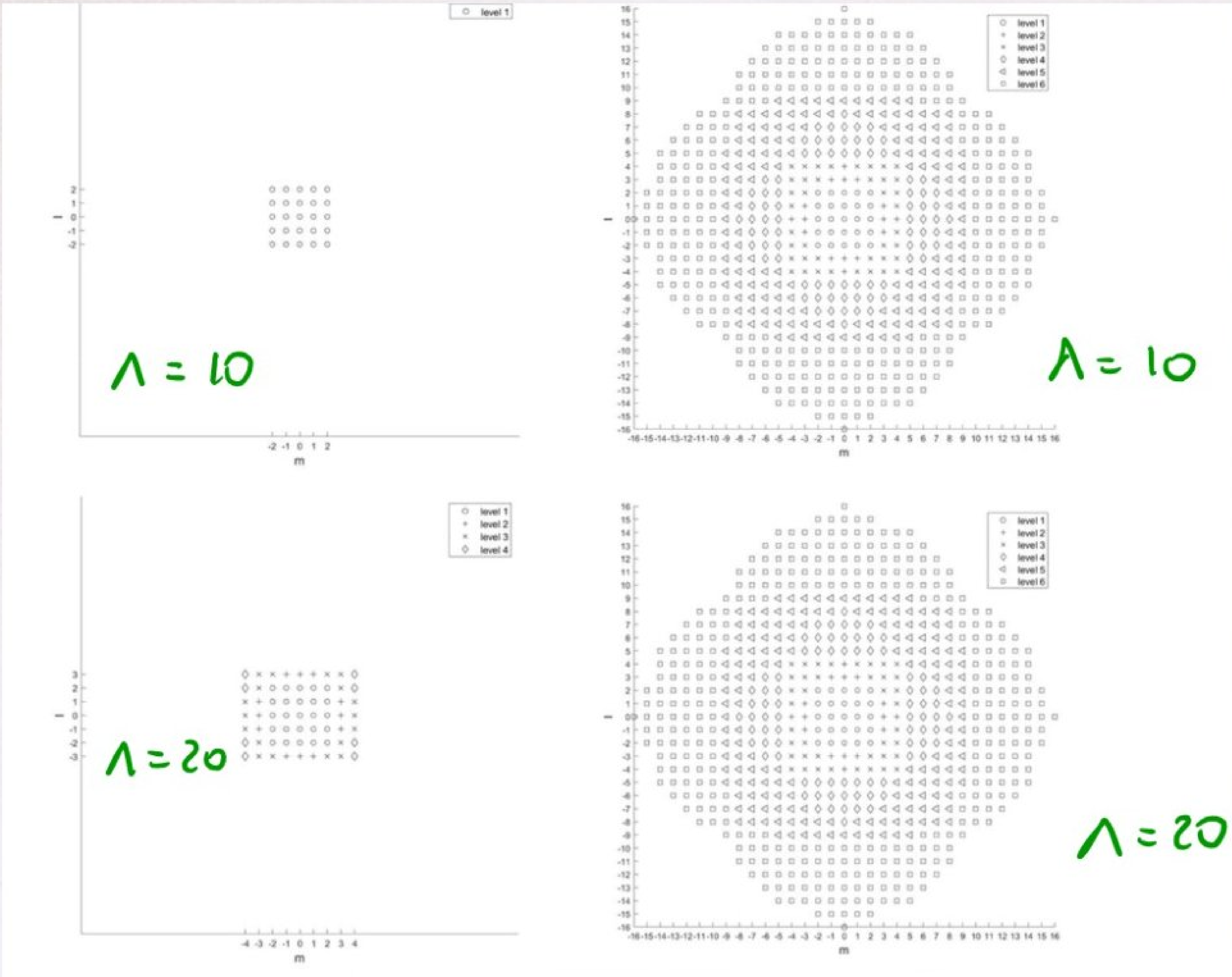
Main Proposition: let $\hat{v}(t,k) =: z_k(t)$, v solution to $(NLS)^*$.

If:
$$N^s_M(z) := \sum_{\substack{k=(m,\ell) \\ |m|>M}} \left[(1+|m|^2)^s + (1+|\ell|^2)^s\right] |z_k|^2 + \sum_{\substack{k=(m,\ell) \\ |\ell|>M}} \left[(1+|m|^2)^s + (1+|\ell|^2)^s\right] |z_k|^2$$

Then $\exists M>0$ s.t. $\frac{d}{dt} N^s_M(z) = 0$ for all t, for all $s \geq 0$

Remark: the proof of this proposition is based on the fact that the resonant set splits into two 1D resonant sets, and outside B_M there are only resonant frequencies for $(NLS)^*$.

Some numerical results



$\alpha = \sqrt{2}$

$\alpha = 1$

levels of (Λ, ε)-quasi-resonant sets for $\varepsilon = 0.1$ that can be excited when starting at $[-2,2]\times[-2,2]$. left column is for $\alpha = \sqrt{2}$ (irrational torus) and right column is for $\alpha = 1$ (rational torus).

The barriers

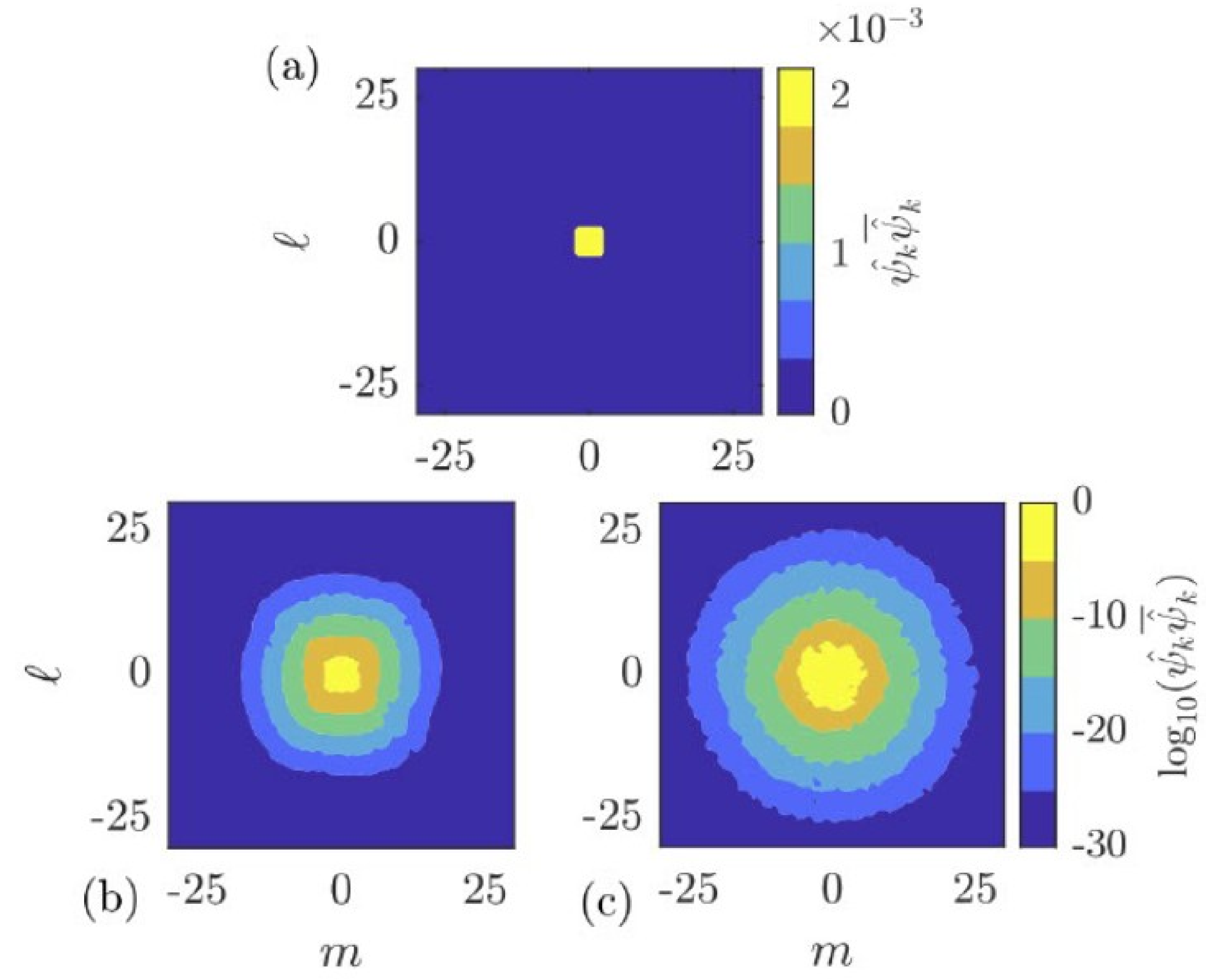


FIGURE 2. The 2D energy spectra of (a) the initial condition, (b) the irrational torus at $t = 20T_f$ and (c) the rational torus at $t = 20T_f$. Note that (b) and (c) share the color bar. The zero mode has amplitude 0, but is colored for simplicity.

Growth of the Sobolev Norm

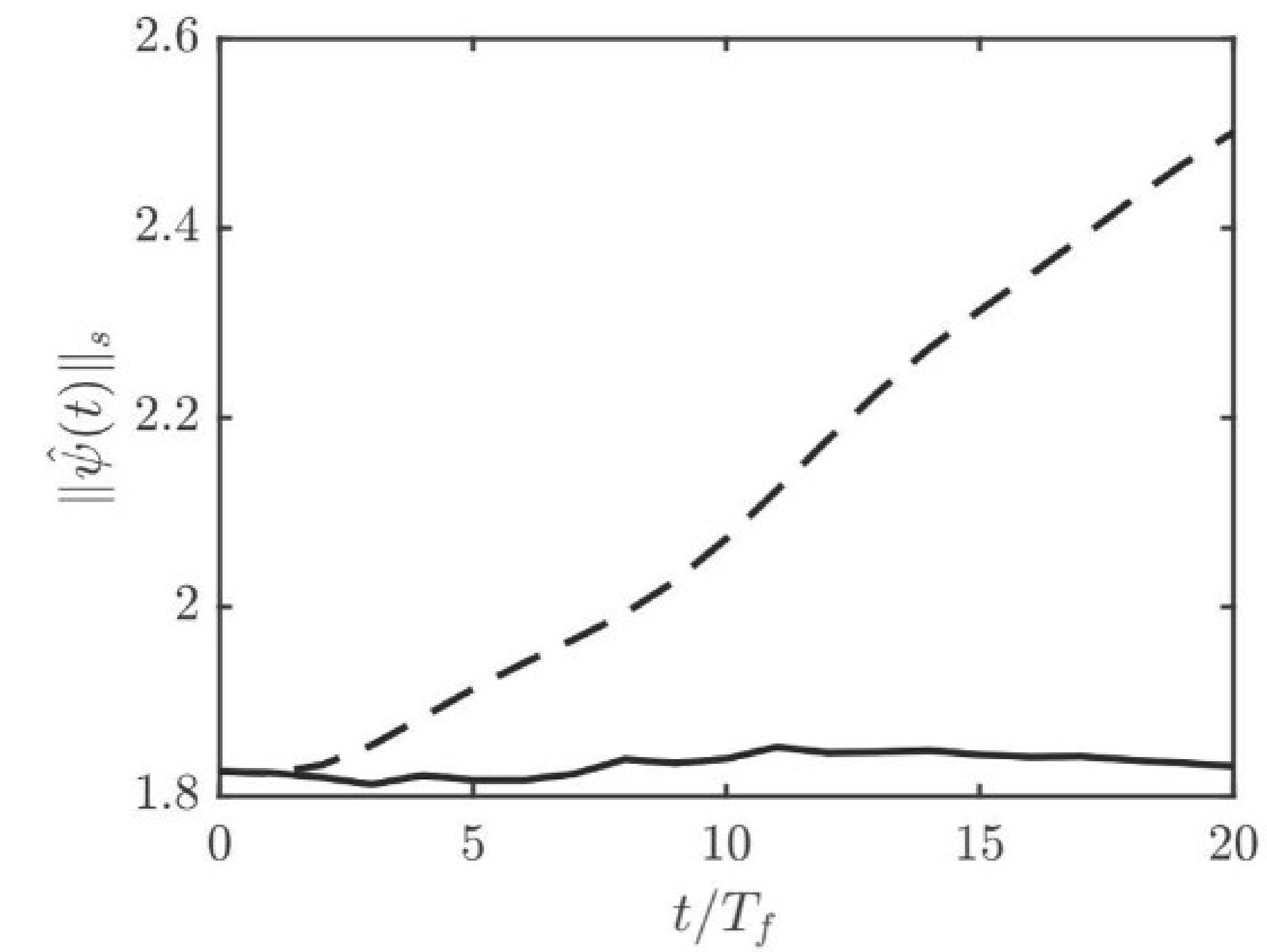


FIGURE 3. The growth of $\|\hat{\psi}(t)\|_s$ for $\omega^2 = \sqrt{2}$ (——) and $\omega^2 = 1$ (– – –)

Growth depending on initial size

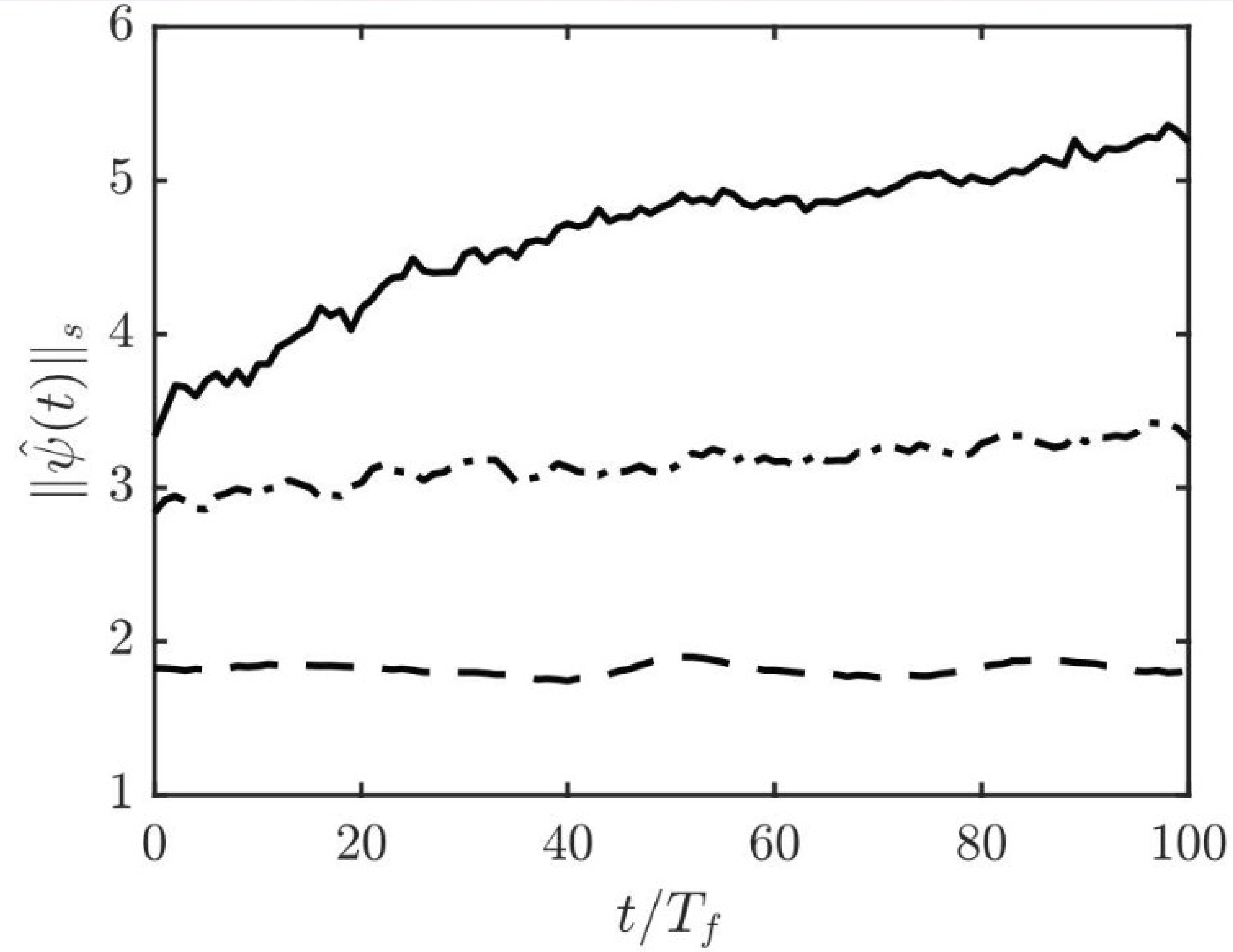


FIGURE 4. The long-time growth of $\|\hat{\psi}(t)\|_s$ on the irrational torus for $R = (3.3343, 2.8390, 1.8263)$.

Work with Nicolas Camps

Consider
$$\begin{cases} i\partial_t u + \partial_x^2 + \operatorname{div}(A\nabla_y)\, u = |u|^2 u \\ u|_{t=0} = u_0 \end{cases} \qquad (x,y) \in \mathbb{R} \times \mathbb{T}^d \quad d \geq 1$$

Assume $\exists\, \tau_* > 0$ and $c > 0$ s.t. $\forall\, (a,b) \in (\mathbb{Z}^d \setminus \{0\})^2$

$$|a^T A b| \geq \frac{c}{\|a\|_{\ell^2}^{\tau_*} \|b\|_{\ell^2}^{\tau_*}}$$

Note: For $\tau_* > \dfrac{d(d+1)}{2}$ the set of matrices A as above is of full Lebesgue measure.

Theorem [Camps-S] For all $d \geq 1$, $s > d$ $\exists\, \varepsilon_*(s)$ s.t. $\forall\, \varepsilon < \varepsilon_*$ if $\|u_0\|_{H^s} \leq \varepsilon$ then the solution $u(t,x)$ exists globally in time and

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s_{xy}} \lesssim \varepsilon, \qquad \sup_{t \in \mathbb{R}} (1+|t|)^{\frac{1}{2}} \|u(t)\|_{L^\infty_{xy}} \leq \varepsilon$$

$\rightsquigarrow$ No growth of Sobolev norms!


Thanks for your attention!