

Some recent developments in wave turbulence theory: Part 1

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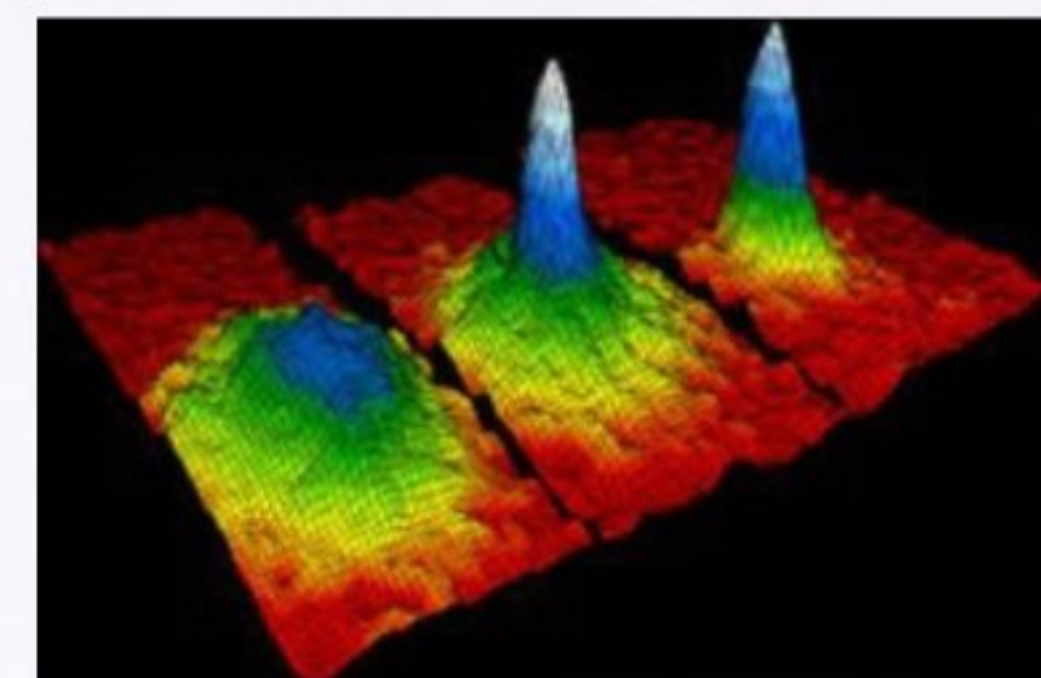
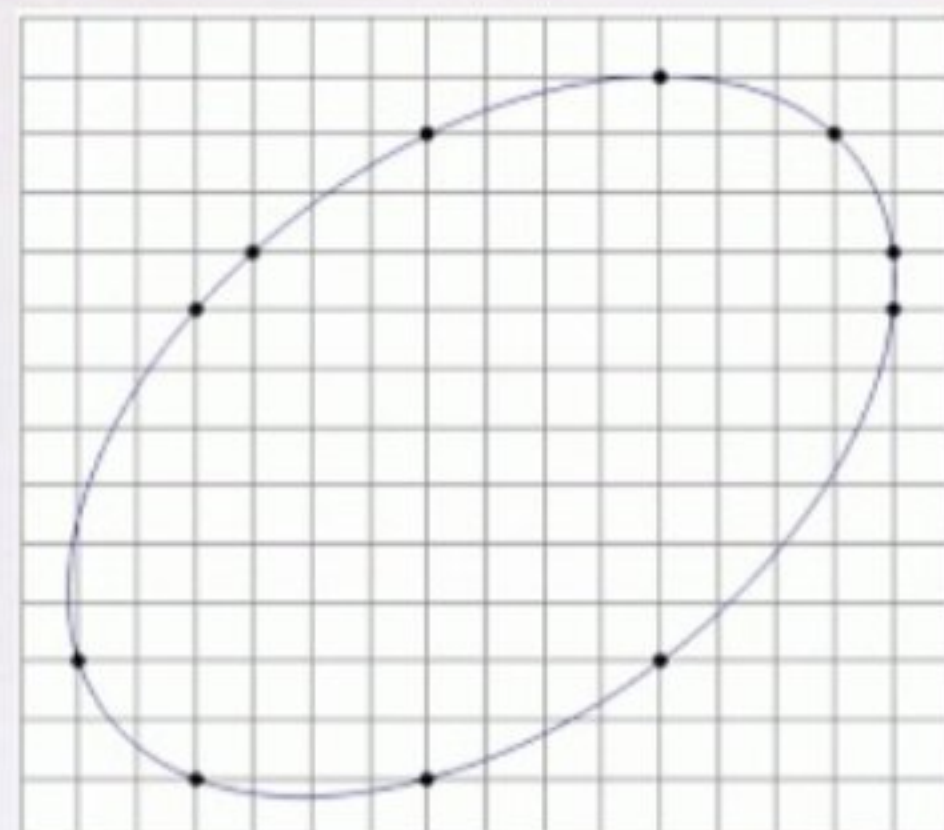
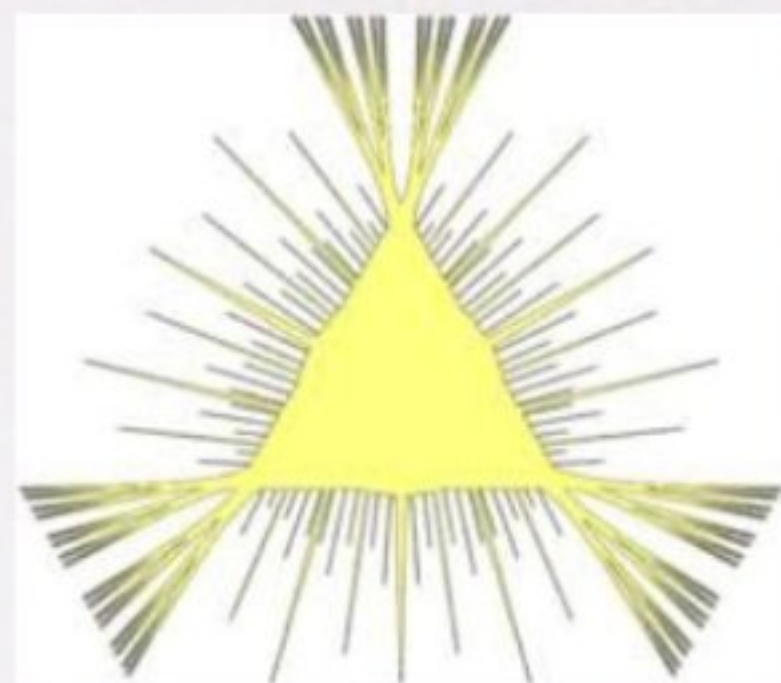
UNC April 5 - April 6



SIMONS
FOUNDATION



What do these pictures have in common?



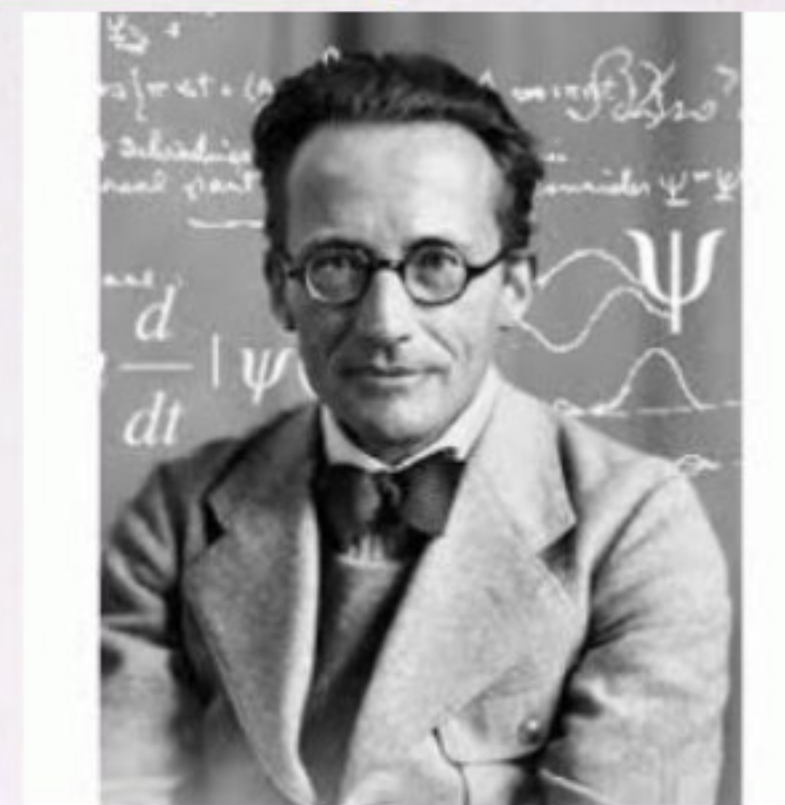
Tools come from many areas

The extraordinary recent progress in the study of waves interactions has involved:

- * Harmonic and Fourier Analysis
- * Analytic Number Theory
- * Math Physics
- * Dynamical Systems
- * Probability
- * Symplectic Geometry

A case study: the nonlinear Schrödinger equation

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u = \lambda |u|^2 u & \lambda = \pm 1 \\ u(0, x) = u_0(x) & x \in \mathbb{T}^d \end{cases}$$



This is the periodic NLS initial value problem.

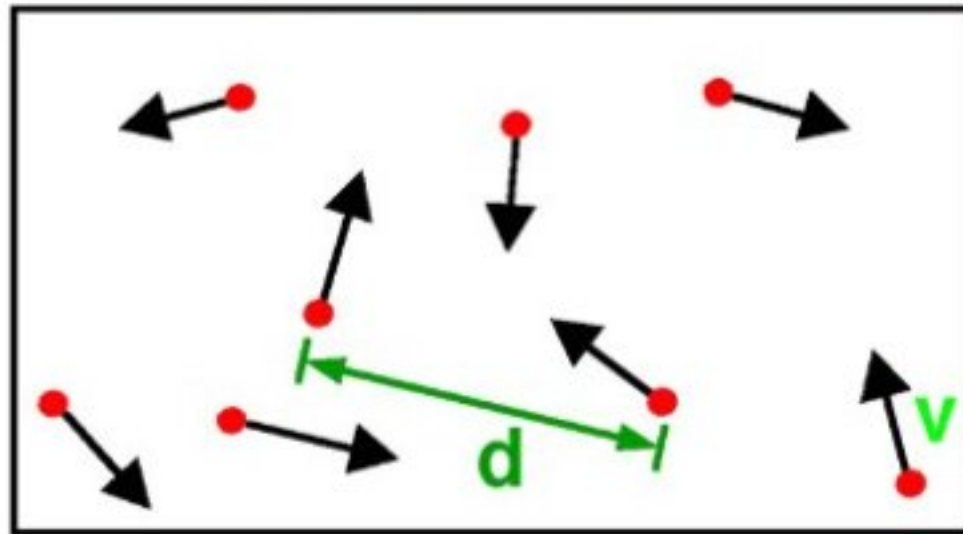
$$\text{Mass} = \int_{\mathbb{T}^d} |u(t, x)|^2 dx$$

$$\text{Hamiltonian} = \int_{\mathbb{T}^d} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{\lambda}{4} \int_{\mathbb{T}^d} |u(t, x)|^4 dx$$

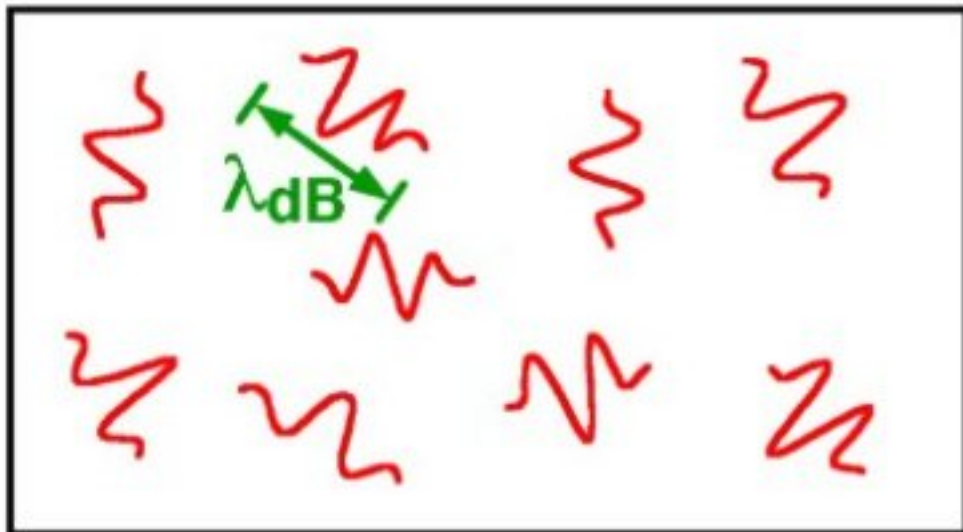
these integrals are conserved!

Where does the NLS come from? An example

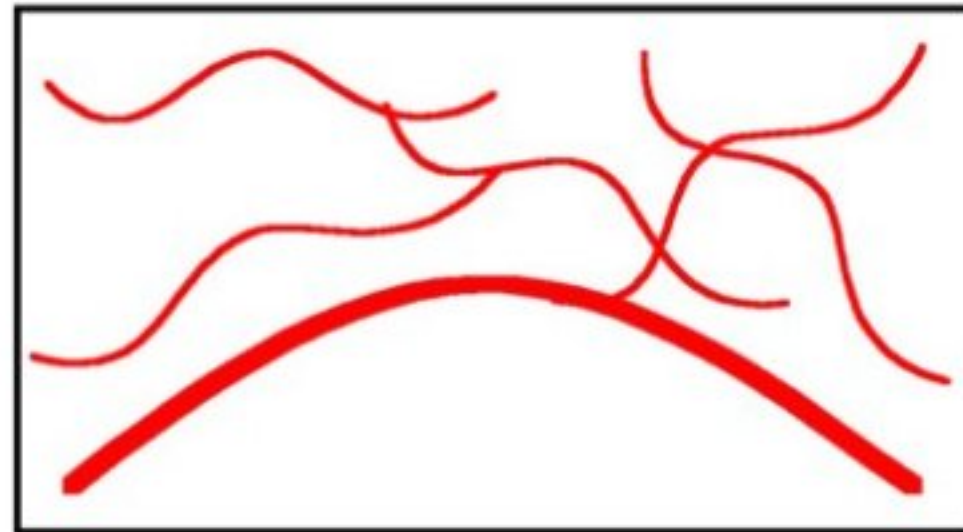
What is Bose-Einstein condensation (BEC)?



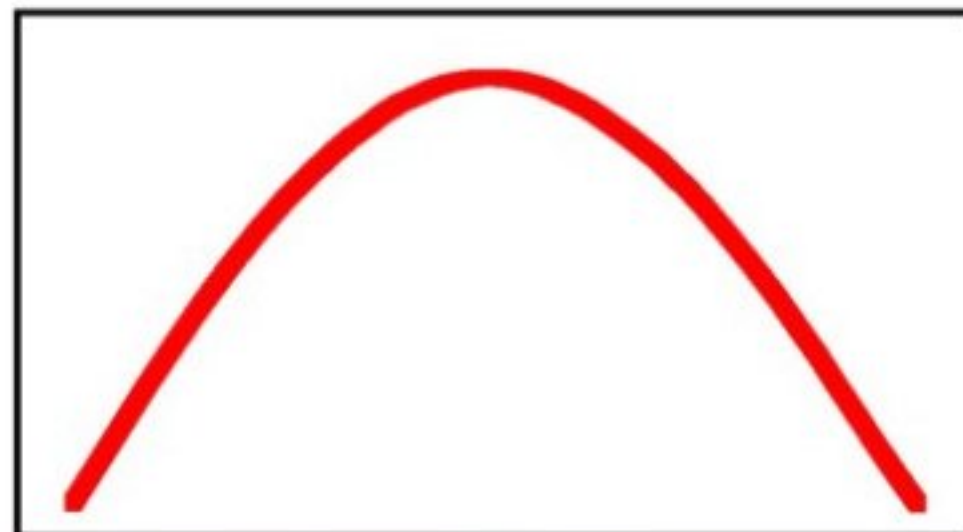
High
Temperature T :
thermal velocity v
density d^{-3}
"Billiard balls"



Low
Temperature T :
De Broglie wavelength
 $\lambda_{dB} = h/mv \propto T^{-1/2}$
"Wave packets"



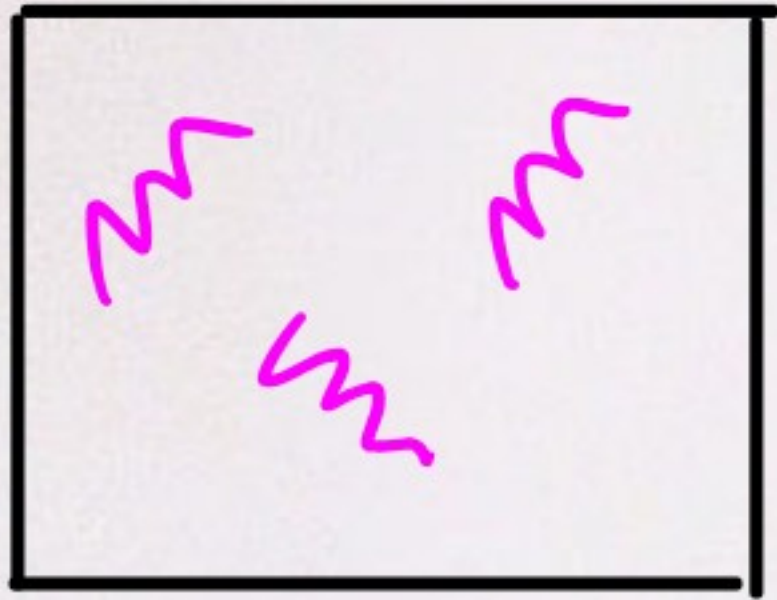
$T = T_{crit}$:
Bose-Einstein
Condensation
 $\lambda_{dB} \approx d$
"Matter wave overlap"



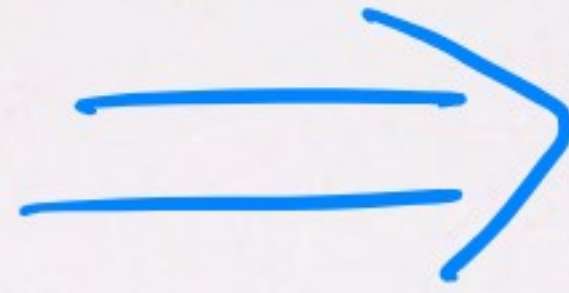
$T = 0$:
Pure Bose
condensate
"Giant matter wave"



Mathematically:



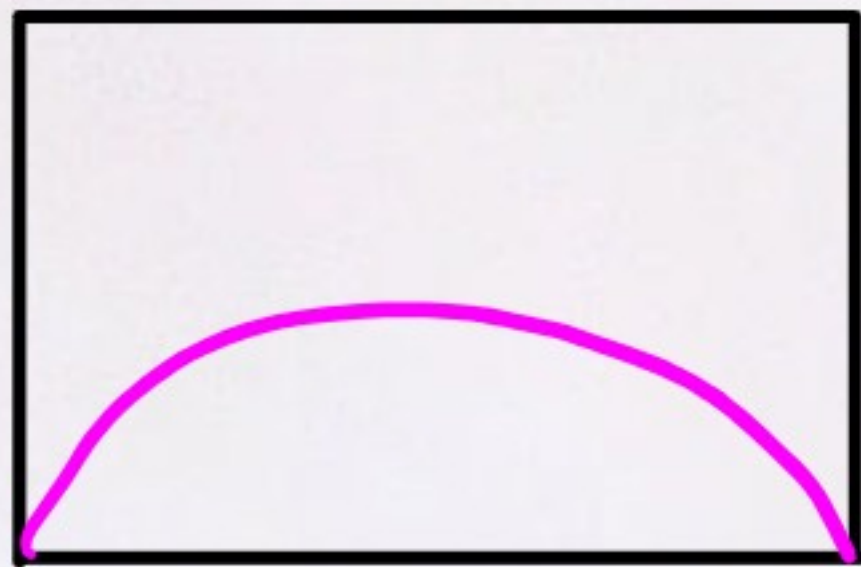
"wave packets"



The BBGKY
hierarchy



take limit
as $N \rightarrow \infty$



"giant matter
wave"



The Gross-Pitaevskii
hierarchy

More Mathematics

If $\underline{x}_K = (x_1, \dots, x_K)$ $x_i \in \mathbb{R}^d, \mathbb{T}^d$

$$\gamma_0^{(n)}(\underline{x}_K, \underline{x}'_K) = \prod_{j=1}^K \mu_0(x_j) \overline{\mu_0(x'_j)}$$

(initial data
of G-P)

Then

$$\gamma^{(n)}(t, \underline{x}_K, \underline{x}'_K) = \prod_{j=1}^K \mu(t, x_j) \overline{\mu(t, x'_j)}$$

(solution to
G-P)

where

$$(NLS) \begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u(0, x) = u_0(x) \end{cases}$$

Spohn; Erdős-Schlein-Yau
Kleinman-Machedon;
Kirkpatrick-Schlein-S.
T. Chen-Pavlovic;
X. Chen-Holmer.

Example of an integrable system

$$i\partial_t u + \partial_x^2 u = \pm |u|^2 u$$

in \mathbb{R}, \mathbb{T}

is an integrable system

Lax Pairs, Inverse Scattering,
infinitely many conservation
laws:

$$I_S(u) = \int \frac{1}{2} |\partial^S u|^2 dx + \text{l.o.t.}$$

$$S \in \mathbb{N}$$

Gross-Pitaevskii Hierarchy
in \mathbb{R}, \mathbb{T} also admits
infinitely many

conserved quantities

Mikhailovson - Nehmed - Pavlovic-S

Note: We expect even more
structure.

Hamiltonian Structure and Poisson Commuting Energies

Many Body System

BBGKY

$\Downarrow N \rightarrow \infty$

GP hierarchy

\Updownarrow

Schrödinger Equation

What is here?

? \Downarrow

What is here?

\Downarrow ?

Hamiltonian structure
Canonical Poisson structure
If in IR and cubic \Rightarrow geometric integrable structure (Poisson)

Can the GP hierarchy be realized as a Hamiltonian equation of motion

$$\left(\frac{d}{dt} \Gamma\right)(t) = X_{H_{GP}}(\Gamma(t))$$

where $X_{H_{GP}}$ is the unique Hamiltonian vector field defined by H_{GP} w.r.t. a certain Poisson structure?

Can the Poisson structure and the Hamiltonian H_{GP} be derived in a suitable manner from an analogous structure

$$\left(\frac{d}{dt} \Gamma_N\right)(t) = X_{H_{BBGKY,N}}(\Gamma_N(t))$$

for the N -particle system?

Assume now $d = 1$:

Does the cubic GP hierarchy possess an integrable structure in the sense that $\exists \{H_n\}_{n \in \mathbb{N}}$ of Hamiltonians that Poisson commute and contain H_{GP} ?

If $\{H_n\}_{n \in \mathbb{N}}$ exists does each of the H_n generates an Hamiltonian equation of motion related to the known n th-Schrödinger equation?

For all the 4 questions above the answers
are yes, see recent work of:

D. Mandelson, A. Nehmood, N. Pavlovic, M. Rosenzweig, G.S.

Questions 1+2: Our geometric constructions are based
on a "quantized" version of the Poisson structure by
Marsden, Morrison and Weinstein

Questions 3+4: We establish the existence of an infinite
sequence of "energies" that commute w.r.t. the
Poisson structure above. In a sense this is a "quantized"
version of the work of Paleis and many others.

Kell-Poseidon

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u(0, x) = u_0(x) \end{cases}$$

\Rightarrow

$$u(t, x) = S(t)u_0(x) \pm \int_0^t S(t-t') |u|^2 u(t') dt'$$

$S(t)u_0(x)$ = solution to linear Schrödinger:

$$\begin{cases} i\partial_t v + \Delta v = 0 \\ v(0, x) = u_0(x) \end{cases}$$

Solution to Cauchy problem

\Rightarrow

Fixed point of integral equation.

Periodic Strichartz Estimates

We need a good Banach space for a fixed point argument.
The Strichartz Estimates on $S(t)u_0(x)$ help:

$$\|S(t)u_0\|_{L^q_{[0,1]} L^q_{\mathbb{T}^d}} \leq C \|u_0\|_{H^s(\mathbb{T}^d)}$$

Case: $d=2, q=4$

$$S(t)u_0(x) = \sum_{n \in \mathbb{Z}^2} \hat{u}_0(n) e^{it(\alpha_1 n_1^2 + \alpha_2 n_2^2)} e^{in \cdot x}$$

$\alpha_1, \alpha_2 > 0$

- $\alpha_1/\alpha_2 \in \mathbb{Q} \iff \mathbb{T}^2$ rational torus
- $\alpha_1/\alpha_2 \notin \mathbb{Q} \iff \mathbb{T}^2$ irrational torus.

Strichartz Estimates on rational tori

If \mathbb{T}^2 is rational torus then

Bourgain 90's

$$\|S(t)u_0\|_{L^4_{\pi \times \pi^2}} \leq C \|u_0\|_{H^\varepsilon(\mathbb{T}^2)} \quad \varepsilon > 0$$

Ingredients:

a) \mathbb{T}^2 rational $\Rightarrow S(t)u_0(x)$ is also periodic in time.

To see this take $\alpha_1, \alpha_2 \in \mathbb{N}$

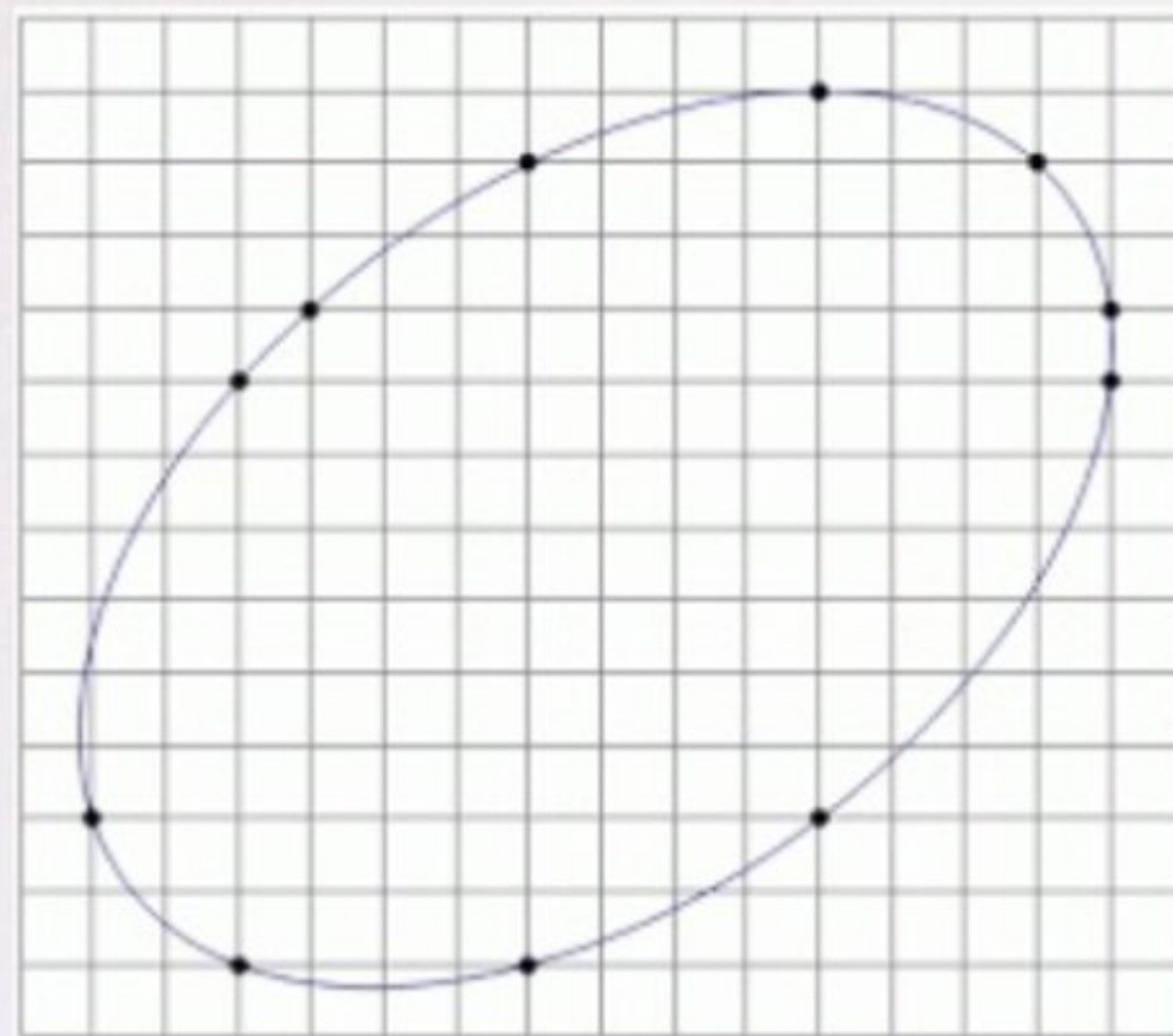
$$S(t)u_0(x) = \sum_{n \in \mathbb{Z}^2} \hat{u}_0(n) e^{i t (\alpha_1 n_1^2 + \alpha_2 n_2^2) + i n \cdot x}$$

\searrow time periodicity

b) If π^2 is rational one can count the set

$$\{n \in \mathbb{Z}^2 / \alpha_1 n_1^2 + \alpha_2 n_2^2 = R^2\} = \Sigma'$$

$\alpha_1, \alpha_2, R \in \mathbb{N}$



In fact

$$|\Sigma| \approx \exp c \frac{\log R}{\log \log R} \ll R^\varepsilon$$

(Gauss lemma)

Analytic Number Theory \Rightarrow Harmonic Analysis

Strichartz Estimates on any Torus

$$\|S(t)u_0\|_{L^4_{[0,1]} L^4_{\mathbb{T}^2}} \leq C \|u_0\|_{H^{\varepsilon}} \quad \varepsilon > 0$$

Bourgain - Demeter '14

Surprisingly ANT was not part of the proof. It is in fact consequence of the

l^2 Decoupling Theorem

this theorem had been a major conjecture in HA. It is related to the Fourier Restriction theorem and the Kakeya problem.

Improvements and consequences

- ✧ Longer time Strichartz estimates were proved on irrational tori by:

Y. Deng - P. Germain - L. Guth.

- ✧ Sharp Decoupling for curves \Rightarrow Vinogradov Mean Value Theorem

J. Bourgain - C. Demeter - L. Guth

Harmonic Analysis \Rightarrow Analytic Number Theory

Global well-posedness and properties

Now using Strichartz estimates and a fixed point argument one can claim that the Cauchy problem:

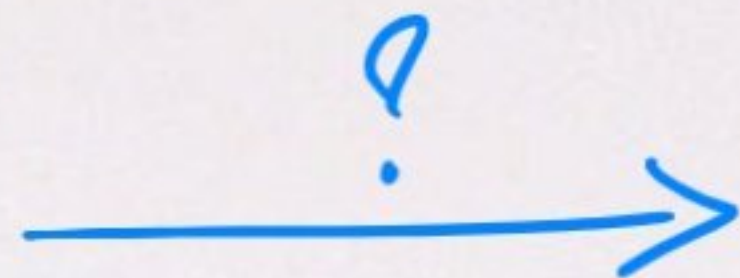
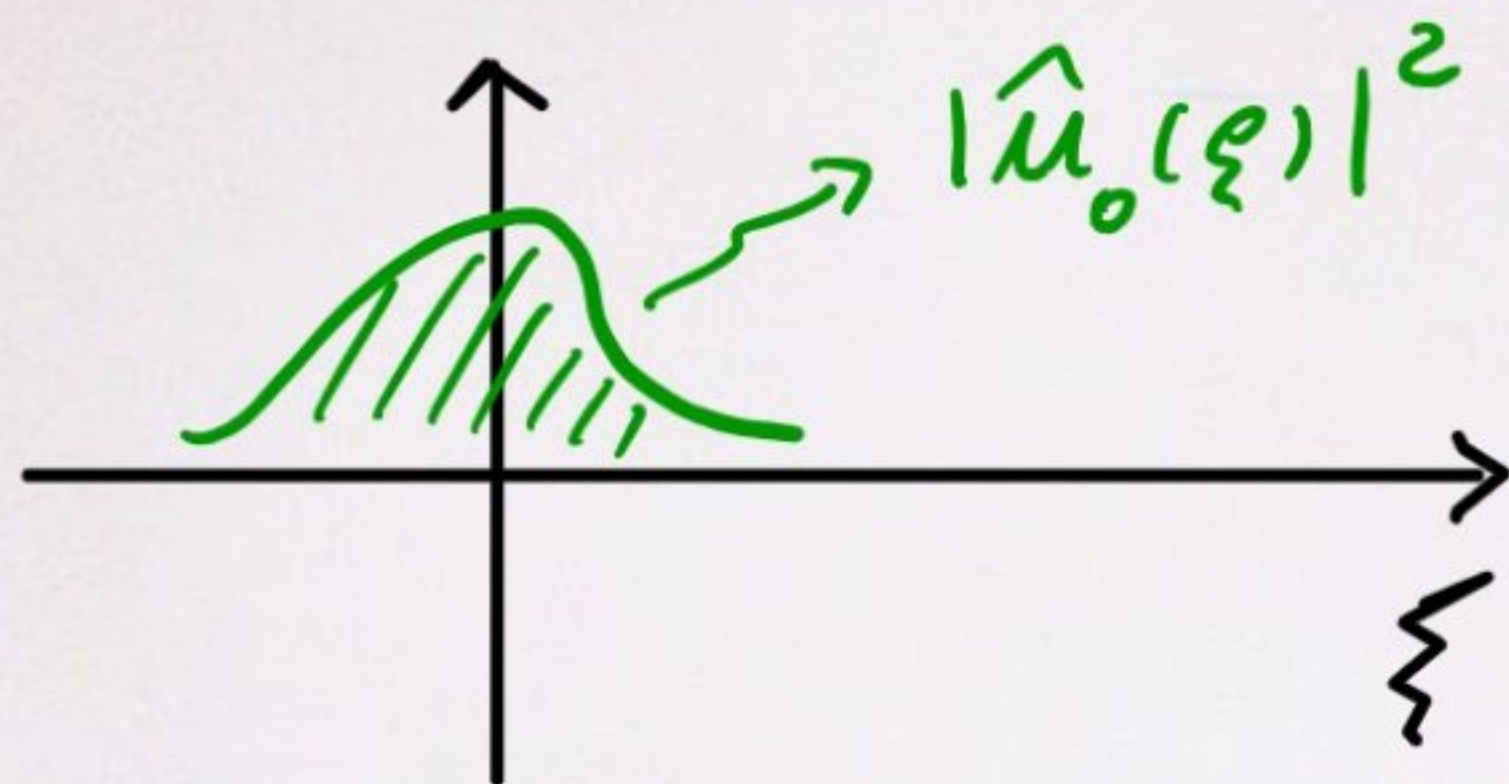
$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^2 u \\ u|_{t=0} = u_0(x) \quad x \in \mathbb{T}^2 \end{cases} \quad \lambda = \pm 1$$

is locally well-posed in $H^s(\mathbb{T}^2)$, $s > 0$. If $\lambda = 1$ (defocusing) then energy conservation \Rightarrow global well-posedness for $s \geq 1$.

Question: Can we learn more about the behaviour of the solution $u(t, x)$ as $t \rightarrow \infty$?

Transfer of energy

$t = 0$



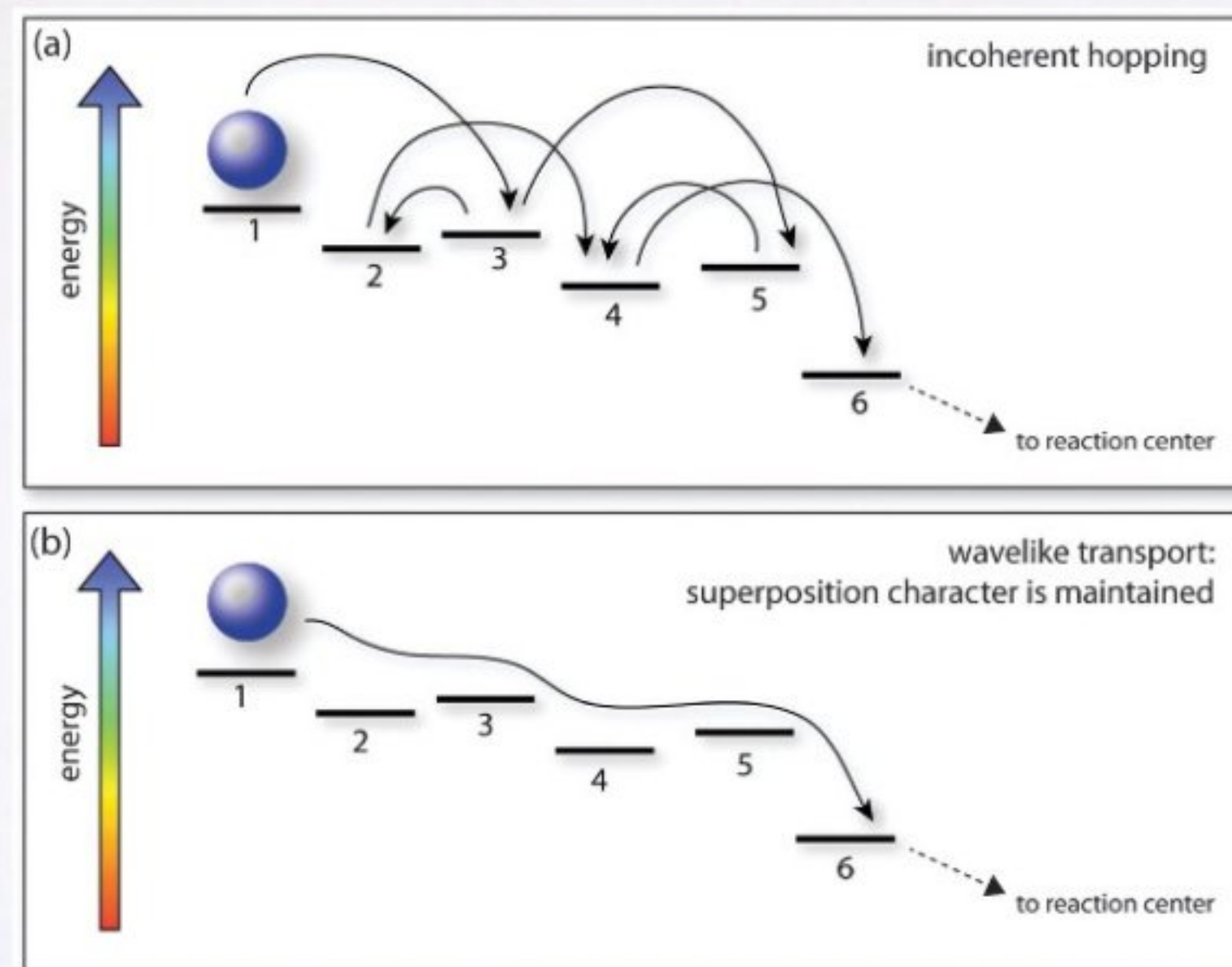
$t > 0$



Question: 1) Does the support of $|\hat{u}(t, \xi)|^2$ move to higher frequencies?

(Weak turbulence, forward cascade)

2) If such a "migration" happens, is it done in a incoherent hopping way or more like a wavelike transport?



Lecture # 2

Two different approaches

Approach # 1 : We look at $\sum_k |\hat{u}(\epsilon, k)|^2 \langle k \rangle^{2s} =: \|u(\epsilon)\|_{H^s}^2$

and we study $\lim_{t \rightarrow \infty} \|u(t)\|_{H^s}^2$.

PDE Approach : Bourgain, Kenig, S., Solinger, Deng-Germain,
Colliander-Keel-S.-Takaoka-Tao, Colles-Four, S.-Wilson,
Hani-Pausader-Tzvetkov-Visciglia ...

Computational Approach : Colliander-Sulem, Four, Y. Pan ...

Dynamical System Approach : Hase-Procesi, Kenoshin-Guadalupe,
Berti-Mespero, Giuliani-Guadalupe ...

Lecture #3

Approach #2: This is based on finding an effective dynamics for the quantity $\|\hat{u}(t, \xi)\|^2 =: n(t, \xi)$.

One approximates the equation, where the nonlinearity is weak ($\lambda \rightarrow 0$), (this is done in various ways)

and then "takes limits" to get to the Wave Kinetic Equation



Wave Turbulence Theory.

Fundamental original work on this by:

Peierls, Hasselmann, Zakharov, Newell, L'vov,
Pomeau, Nazarenko, - - -

Approach #1: Growth of Sobolev Norms

Fact 1: Complete integrability may prevent the growth of Sobolev norms (1D cubic NLS, KdV)

Fact 2: Scattering prevents the growth of Sobolev norms:
(Defocusing Cubic NLS in \mathbb{R}^2 . If $u(t, x)$ is solution in $H^s(\mathbb{R}^2)$ then $\exists u^+ \in H^s(\mathbb{R}^2)$ s.t.

$$s \geq 0 \quad \boxed{\|S(t)u^+ - u\|_{H^s} \xrightarrow{t \rightarrow +\infty} 0} \quad (\text{Dodson '16})$$

As a consequence for $t \gg 1$

$$\|u(t)\|_{H^s} \leq \|S(t)u^+ - u\|_{H^s} + \|S(t)u^+\|_{H^s} \leq \varepsilon + \|u^+\|_{H^s}.$$

$\nearrow S(t)$ is unitary!

A bound from above.

Assume $u(t, x)$ is the global smooth solution to

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \quad x \in \mathbb{T}^2, \quad S \gg 1 \end{cases}$$

Fact 1 $\ast \quad \|u(t)\|_{H^S} \leq C |t|^{S-1+\varepsilon} \quad |t| \geq 1$

for any torus (Bourgain, Solinger, Planchon-Visaghe)

Better results available for irrational tori.

(Deng-Germain-Guth, Hrabowski, Pan, S., Wilson)

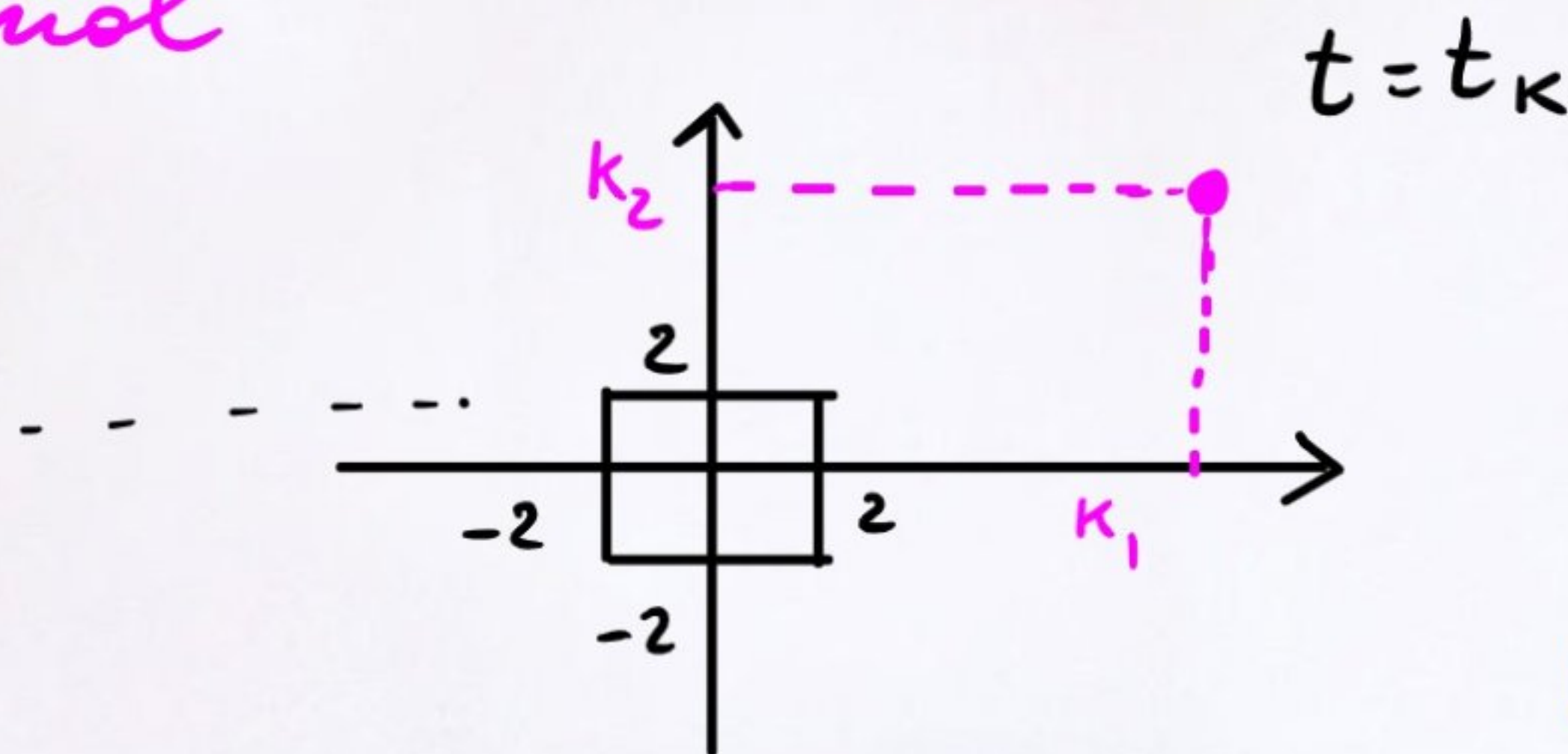
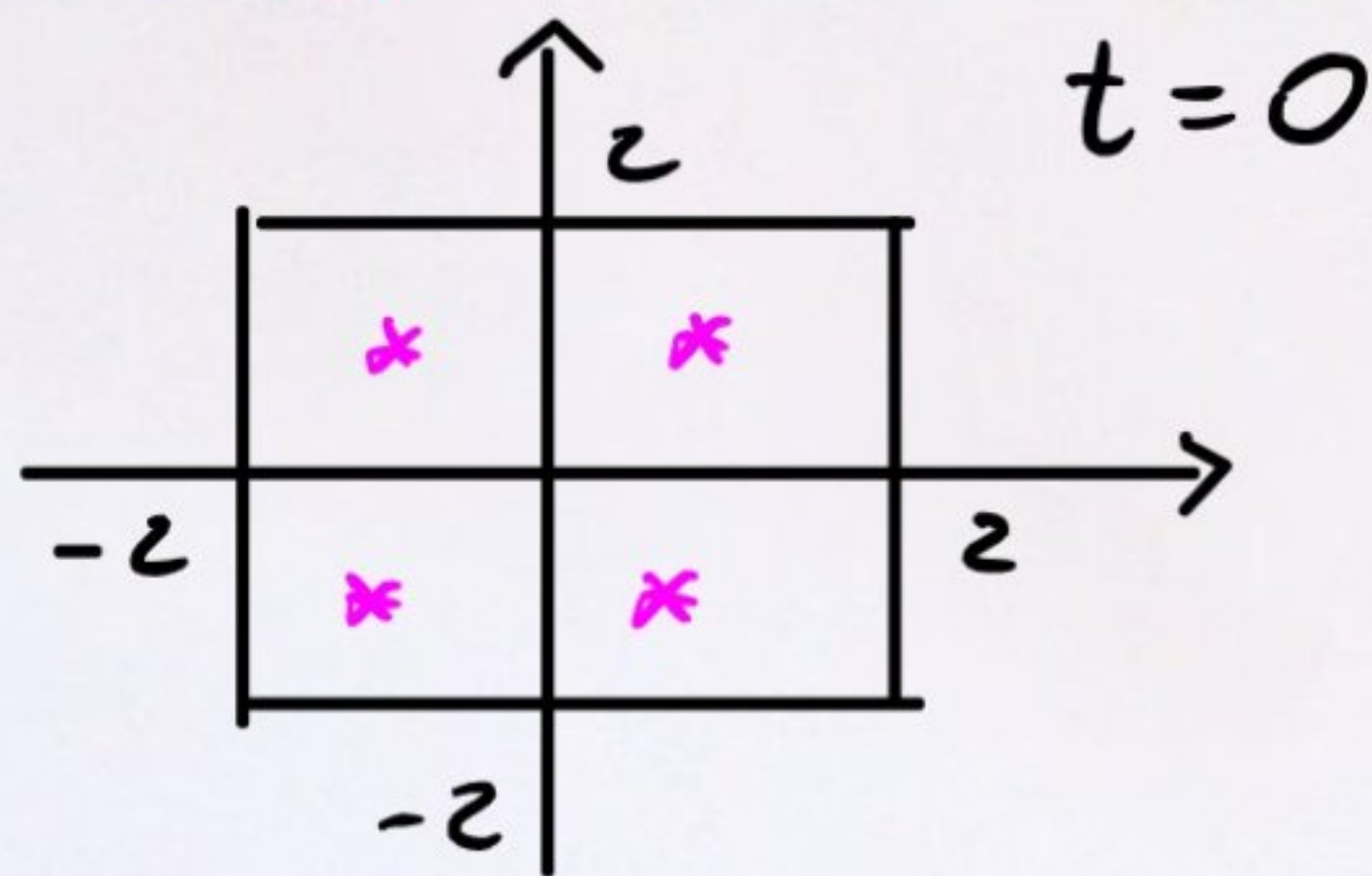
Are there solutions that grow?

Fact 2: Fix $s > 1$, $0 < \delta \ll 1$, $K \gg 1$, then for the cubic, defocusing NLS in \mathbb{T}^2 rational, \exists an initial state $u_0 \in H^s$ and a time $T \gg 1$ s.t.

$$\|u_0\|_{H^s} < \delta \text{ and } \|u(T)\|_{H^s} > K$$

(Colliander-Keel-S-Takaoke-Tao)

Fact 3: For \mathbb{T}^2 rational



arbitrarily large modes exists.

(Carlen-Faou)

Some Remarks

- * We do not know what happens after time T .
- * In the work of *Coles-Foon* the procedure is different but the same set Δ of frequencies is used.

Question: What happens when π^2 is irrational?

Answer: With *B. Wilson* we prove that something different happens: Both constructions presented above cannot work to show transfer of energy!

Recently *Giuliani-Guondie* proved that if the force is irrational but "close to rational" one can adjust the argument.

Approach #2: The wave kinetic equation

Let $u(t, x)$ sol. to a dispersive eq. with weak nonlinearity.

Set $\hat{u}(t, k) = a_k(t)$.

- Start with $\{a_k(0)\}$ as independent random variables

- Define $n(k, t) = \mathbb{E} |a_k(t)|^2$

We need to use a probabilistic setting!

- "Take limits" (Large box limit, weak-nonlinearity limit, both limits together --)

- Obtain an effective equation $n(k, t)$ often referred to as the wave kinetic equation

The 4-wave kinetic equation

Cubic NLS equation



Wave kinetic equation

$$\partial_t n(t, k) = \mathcal{K}(n(t, k), n(t, k), n(t, k))$$

$$\begin{aligned} \mathcal{K}(\phi_1, \phi_2, \phi_3)(k) := & \int \{ \phi_1(k_1) \phi_2(k_2) \phi_3(k_3) - \phi_1(k) \phi_2(k_2) \phi_3(k_3) \\ & + \phi_1(k_1) \phi_2(k) \phi_3(k_3) - \phi_1(k_1) \phi_2(k_2) \phi_3(k) \} \delta(k_1 - k_2 + k_3 - k) \\ & \delta(|k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2) dk_1 dk_2 dk_3 \end{aligned}$$

Recent Results

Recently a series of great papers have appeared that, under certain conditions, give a rigorous justification to the "take the limit" process almost to the correct time scale.

- Lukkerinen - Sphar } at statistical equilibrium
 - Faou
 - Buchmester - Germain - Hani - Sketch
 - Deng - Hani (at the kinetic time)
 - Collet - Germain
- } out of statistical equilibrium.

The ZK Equation

Consider the ZK equation

$$\partial_t \phi + \partial_{x_1} \Delta \phi + \lambda \partial_{x_1} \phi^2 = 0$$

$$x = (x_1, \dots, x_d)$$

$$\prod_{D=1}^d$$

An informal derivation of the wave kinetic equation was derived by Nazarenko assuming that

$a_k(t) := \hat{\phi}(t, k)$ are RPA fields (Random Phase & amplitude)

$$\begin{matrix} l \rightarrow 0 \\ \Delta \rightarrow \infty \end{matrix} \Downarrow$$

$$f(z, k) \cong \mathbb{E}(|a_k(z)|^2)$$

$$\frac{d}{dt} f(z, k) = Q(f)(z, k)$$

WKE

A Stochastic z_k Equation

with Binh Tran

$$(z_k) \begin{cases} d\phi = -\Delta \phi dt + \lambda \phi^2 dt + c \lambda^\theta \phi \odot dW(t) \\ \phi(x, 0) = \phi_0(x) \end{cases}$$

$1 > \theta > 0$ convolution / Stratonovich product

Actually the name (z_k) is defined in hypercubic lattice of size D .

Remark: This problem was inspired by recent work of Fara.

The presence of the noise is important and necessary.

Dispersive relation: $\omega(k) = k^2 |k|^2$ too singular!

Statement of main result

Let $\hat{\phi}(k, t) = a(k, t)$.

Consider the two-point correlation function

$$f(k, t) = \langle a(k, t), \bar{a}(k, t) \rangle = \int dg |a(k, t)|^2$$

Main Theorem [S. - Tran] Assume we are in \mathbb{T}^d , $d \geq 2$.

Under very general assumptions on the initial

density function f_0 , if $t = \lambda^{-2} \tau = \mathcal{O}(\lambda^{-2})$, $c \leq \tau$

$$\lim_{\lambda \rightarrow 0, \tau \rightarrow \infty} f(k, \lambda^{-2} \tau) = f^\infty(k, \tau) \quad \text{and}$$

Kinetic time scaling.

Collision operator

$$\frac{\partial}{\partial \tau} f^\infty(k, \tau) = Q(f^\infty)(k, \tau)$$

3-wave kinetic equation predicted (see Nazarenko)

Kolmogorov summarizes very well.....

"Mathematics is vast. One person is unable to study all its branches. In this sense specialization is inevitable. But at the same time mathematics is a united science. More and more links appear between its areas, sometimes in a most unexpected way. Some areas serve as tools for other areas. Therefore an isolation of mathematicians in too narrow borders should be destructive for our science."



Thank you!

