THE UNIVERSITY
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at CHAPEL HILL

Analysis and PDE Seminar

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PH 385

On the long-time behavior of scale-invariant solutions to the 2d Euler equation

Ayman R. Said (Duke)

Abstract. We give a complete description of the long-time behavior of scale-invariant solutions the 2d Euler equation. We show that all scale-invariant solutions relax in infinite time to rigidly rotating or steady states, which are fully classified and shown to be piece-wise constant profiles with countably many jumps. Consequently, all non-constant scale-invariant solutions that are smooth on $\mathbb{S}^1$ become singular in infinite time. On $\mathbb{R}^2$, this corresponds to generic infinite time spiral and cusp formation. In the process, we also show that for scale invariant solutions, the measure (on $\mathbb{S}^1$) of particles moving away from the origin and toward spatial infinity is a strictly increasing function of time.

The scale invariant solutions solve a relatively simple equation on $\mathbb{S}^1$:

$$\begin{cases} \partial_t g + 2G\partial_\theta g = 0, \\ -(4 + \partial_{\theta\theta} G) = g. \end{cases}$$

Now the main result states that for $g_0 \in L^\infty(\mathbb{S}^1)$, there exist two constants $a_{\pm\infty} \in \mathbb{R}$ and two piecewise constant profiles $g_{\pm\infty} \in \mathrm{BV}(\mathbb{S}^1)$ so that

$$\|g(\cdot, t) - g_{\pm\infty}(\cdot - a_{\pm\infty})\|_{L^p} \to 0$$

as $t \to \pm\infty$ for any $1 \le p < \infty$. Moreover, the asymptotic profiles $g_{\pm\infty}(\cdot - a_{\pm\infty})$ are traveling wave solutions with speeds $a_{\pm\infty}$. When $g_0 \in C(\mathbb{S}^1)$, the asymptotic profiles $g_{\pm\infty}$ are identically constant and equal to $\frac{1}{2\pi}\int_{\mathbb{S}^1} g_0$.

This is joint work with Tarek M. Elgindi, Ryan Murray.