

Fermi gases and dilute neutron matter with low-momentum interactions

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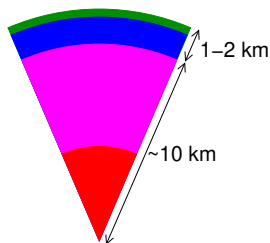
Outline

- ▶ Neutron stars, dilute neutron matter, and ultracold atoms
- ▶ Usual regularization procedure for a contact interaction
- ▶ Low-momentum interactions with zero range
- ▶ Hartree-Fock-Bogoliubov with perturbative corrections
- ▶ Results for cold atoms
- ▶ Results for neutron matter
- ▶ Conclusions and outlook

More details: M.U. and S. Ramanan, Phys. Rev. A **103**, 063306 (2021).

Motivation: neutron stars

- ▶ Typical density in the center of a neutron star: several times nuclear saturation density $\rho_0 = 2.7 \times 10^{14} \text{ g/cm}^3$, $n_0 = 0.16 \text{ fm}^{-3}$
- ▶ Typical temperatures $T \approx 10^6 - 10^9 \text{ K} \approx 0.1 - 100 \text{ keV}$
 $T \ll E_F \rightarrow T = 0$ formalism sufficient for many purposes
- ▶ Inner structure of a neutron star:



outer crust: Coulomb lattice of neutron rich nuclei in a degenerate electron gas

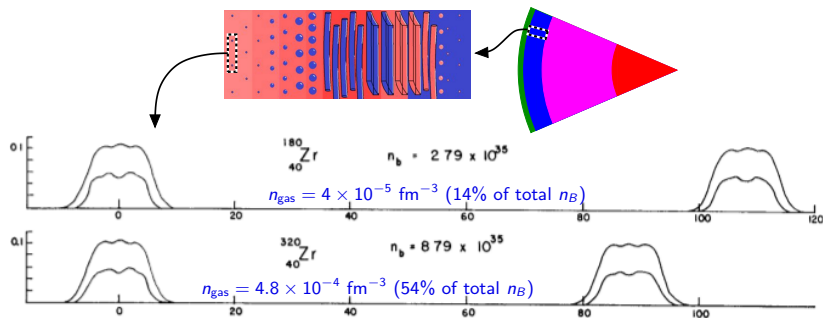
inner crust: unbound neutrons form a superfluid neutron gas between the nuclei (clusters) which is responsible for pulsar glitches

outer core: homogeneous matter (n, p, e^-)

inner core: densities up to a few times ρ_0 , new degrees of freedom: hyperons? quark matter?

What is “dilute” neutron matter?

- Upper layers of the inner crust (close to neutron-drip density $\sim 2.5 \times 10^{-4} \text{ fm}^{-3}$)



[Negele and Vautherin, NPA 207 (1973); similar results by Baldo et al., PRC 76 (2007)]

- In spite of its “low” density (still $\rho \gtrsim 10^{11} \text{ g/cm}^3$), the neutron gas is relevant because it occupies a much larger volume than the clusters
- Deeper in the crust: n_{gas} increases up to $\sim n_0/2 = 0.08 \text{ fm}^{-3}$

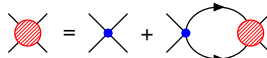
Comparison with ultracold trapped Fermi gases

	neutron gas	trapped Fermi gas (e.g. ${}^6\text{Li}$)
n	$4 \times 10^{-5} \dots 0.08 \text{ fm}^{-3}$	$\sim 1 \text{ } \mu\text{m}^{-3}$
$k_F = (3\pi^2 n)^{1/3}$	$0.1 \dots 1.3 \text{ fm}^{-1}$	$\sim 1 \text{ } \mu\text{m}^{-1}$
$E_F = k_F^2/2m$	$0.2 \dots 35 \text{ MeV}$	$\sim 1 \text{ } \mu\text{K} \sim 10^{-10} \text{ eV}$
scattering length a	-18 fm	adjustable (Feshbach resonance)
effective range r_{eff}	2.5 fm	$\sim 1 \text{ nm}$
$1/k_F a$	$-0.5 \dots -0.07$	unitary limit: 0 BCS-BEC crossover: $-1 \dots 1$
$k_F r_{\text{eff}}$	$0.25 \dots 3$	10^{-3}

- ▶ r_{eff} can be neglected in cold atoms but not in neutron matter
- ▶ the neutron gas is close to the crossover regime but not in the unitary limit

Standard regularization procedure for a contact interaction

- ▶ Scattering length for coupling constant $g < 0$ and cutoff Λ ($\epsilon_k = \frac{k^2}{2m}$)

$$\frac{4\pi a}{m} = g + g \int^{\Lambda} \frac{d^3 k}{(2\pi)^3} \frac{1}{-2\epsilon_k} \frac{4\pi a}{m}$$


- ▶ Express g in terms of a , e.g. in the gap equation ($E_k = \sqrt{(\epsilon_k - \mu)^2 + \Delta^2}$)

$$\Delta = -g \int^{\Lambda} \frac{d^3 k}{(2\pi)^3} \frac{\Delta}{2E_k} \quad \Leftrightarrow \quad \Delta = -\frac{4\pi a}{m} \int^{\Lambda} \frac{d^3 k}{(2\pi)^3} \left(\frac{\Delta}{2E_k} - \frac{\Delta}{2\epsilon_k} \right)$$

\Rightarrow now the cutoff can be removed

- ▶ Coupling constant vanishes for $\Lambda \rightarrow \infty$: $\frac{1}{g} = \frac{m}{4\pi a} - \frac{m\Lambda}{2\pi^2}$
- ▶ Keeping Λ finite would induce a finite effective range: $r_{\text{eff}} = \frac{4}{\pi\Lambda}$
- ▶ For cold atoms one should take the limit $\Lambda \rightarrow \infty$
- ▶ No Hartree field: $U_{\sigma} = g n_{-\sigma} \xrightarrow{\Lambda \rightarrow \infty} 0$
- ▶ In order to get the simplest weak-coupling correction $\frac{4\pi a}{m} n_{\uparrow} n_{\downarrow}$ to the GS energy, resummation of ladder diagrams is necessary

Low-momentum interactions with zero range

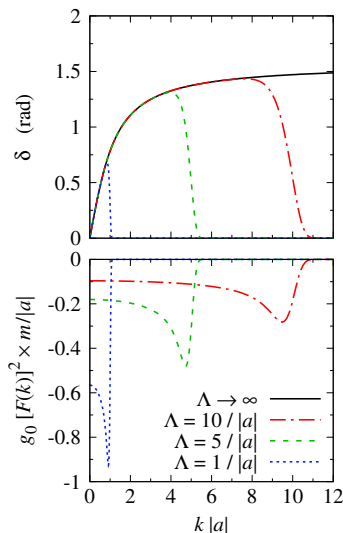
- ▶ In nuclear physics: “soft” $V_{\text{low-}k}$ or SRG interactions reproduce exactly the low-momentum scattering phase shifts of the full NN interaction below the cutoff
- ▶ Is it possible to reproduce the scattering amplitude of a contact interaction for $k < \Lambda$ with a finite cutoff Λ ? \rightarrow **Yes!**
- ▶ Explicit construction [Tabakin 1969] of a separable s-wave interaction ($V = 4\pi V_0, F(0) = 1$)

$$V_0(k, k') = g_0 F(k) F(k')$$

that gives scattering phase shifts

$$\delta(q) = R\left(\frac{k}{\Lambda}\right) \operatorname{arccot}\left(-\frac{1}{ka}\right)$$

- ▶ We use a smooth regulator $R(x) = \exp(-x^{20})$
- ▶ In the unitary limit and with a sharp regulator, i.e., $\delta(k) = \frac{\pi}{2}\theta(\Lambda - k)$, analytic expressions exist [Köhler 2007, Ruiz Arriola et al. 2017]



Hartree-Fock-Bogoliubov (HFB)

- ▶ In nuclear physics: hard core of “realistic” potentials requires explicit inclusion of short-range correlations, and nuclei are not bound in HF(B) approximation
- ▶ Soft interactions ($V_{\text{low-}k}$, SRG) much better suited for perturbative methods
- ▶ HFB with perturbative corrections can give good results for open-shell nuclei [e.g., Tichai et al. 2019] → try this method for cold atoms
- ▶ Momentum dependent mean field U_k and gap Δ_k : ($\Lambda' > \Lambda$ because of smooth cutoff)

$$U_k = \int \frac{d^3p}{(2\pi)^3} 4\pi V_0\left(\frac{\vec{p}-\vec{k}}{2}, \frac{\vec{p}-\vec{k}}{2}\right) v_p^2, \quad \Delta_k = -\frac{2}{\pi} \int_0^{\Lambda'} dp p^2 V_0(k, p) u_p v_p$$

with the usual definitions

$$u_k = \sqrt{\frac{1}{2} + \frac{\xi_k}{2E_k}}, \quad v_k = \sqrt{\frac{1}{2} - \frac{\xi_k}{2E_k}}, \quad \xi_k = \frac{k^2}{2m} + U_k - \mu, \quad E_k = \sqrt{\xi_k^2 + \Delta_k^2}$$

- ▶ Choice of the cutoff Λ : as small as possible to make the interaction perturbative, but without cutting the physically relevant states
- ▶ Hartree-Fock requires $\Lambda \geq k_F$, with pairing we need somewhat higher cutoff

Cutoff dependence of HFB results

(a) Weak coupling ($1/k_F a = -5$):

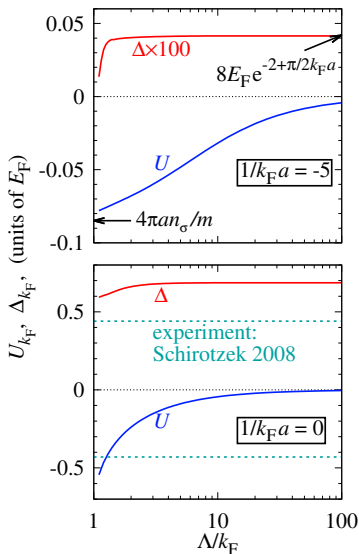
- ▶ Hartree shift $U_{k_F} \approx \frac{4\pi a}{m} n_\sigma$ for $\Lambda \rightarrow k_F$, but $U_{k_F} \rightarrow 0$ for $\Lambda \rightarrow \infty$
- ▶ Gap Δ_{k_F} reaches rapidly (for $\Lambda \gtrsim 1.5k_F$) the usual BCS result $8E_F \exp(-2 + \frac{\pi}{2k_F a})$

(b) At unitarity ($1/k_F a = 0$):

- ▶ Hartree shift $U_{k_F} \sim -0.5E_F$ at small Λ , but again $U_{k_F} \rightarrow 0$ for $\Lambda \rightarrow \infty$
- ▶ Gap Δ_{k_F} less cutoff dependent needs larger Λ ($\sim 3k_F$) to reach asymptotic value

Physical quantities should be cutoff independent!

- ▶ If perturbative corrections to HFB converge in a range of cutoffs, the corrected results should be cutoff independent in this range



Bogoliubov Many-Body Perturbation Theory (BMBPT)

- Express $\hat{K} = \hat{H} - \mu \hat{N}$ in terms of quasiparticle (QP) operators

$$\beta_{\vec{k}\uparrow} = u_k a_{\vec{k}\uparrow} - v_k a_{-\vec{k}\downarrow}^\dagger, \quad \beta_{\vec{k}\downarrow} = u_k a_{\vec{k}\downarrow} + v_k a_{-\vec{k}\uparrow}^\dagger$$

- With the HFB solution for u_k and v_k , we can write

$$\hat{K} = \mathcal{E}_{\text{HFB}} + \sum_{\vec{k}\sigma} E_k \beta_{\vec{k}\sigma}^\dagger \beta_{\vec{k}\sigma} + :\hat{V}:$$

$$:\hat{V}: = V_{04} \beta \beta \beta \beta + V_{13} \beta^\dagger \beta \beta \beta + V_{22} \beta^\dagger \beta^\dagger \beta \beta + V_{31} \beta^\dagger \beta^\dagger \beta^\dagger \beta + V_{40} \beta^\dagger \beta^\dagger \beta^\dagger \beta^\dagger$$

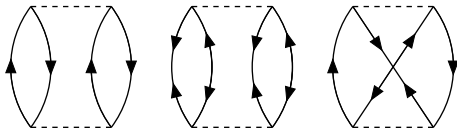
- Treat $:\hat{V}:$ as a perturbation
- Example: leading correction to GS energy is second order

$$\mathcal{E}_2 = -\frac{1}{4!} \sum_{ijkl} \frac{|\langle ijkl | \hat{V}_{40} | \text{HFB} \rangle|^2}{E_i + E_j + E_k + E_l} \quad \text{with} \quad |ijkl\rangle = \beta_i^\dagger \beta_j^\dagger \beta_k^\dagger \beta_l^\dagger | \text{HFB} \rangle$$

BMBPT at second and third order

- ▶ In practice, \mathcal{E}_2 has three terms corresponding to three diagrams:

upwards going line u_k^2
downwards going line v_k^2
anomalous line $u_k v_k$
horizontal dashed line $V_0(q, q')$



- ▶ Summation over intermediate 4 QP states:

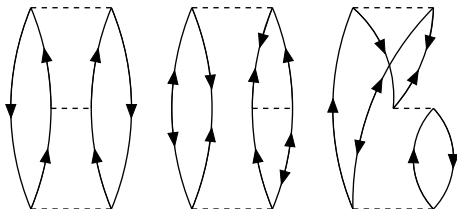
12 (four momenta) -3 (momentum conservation) -3 (rotational invariance)
 $= 6$ dimensional integral, evaluated with MC integration with importance sampling

- ▶ Third-order correction \mathcal{E}_3 has 27 terms (only three examples shown):

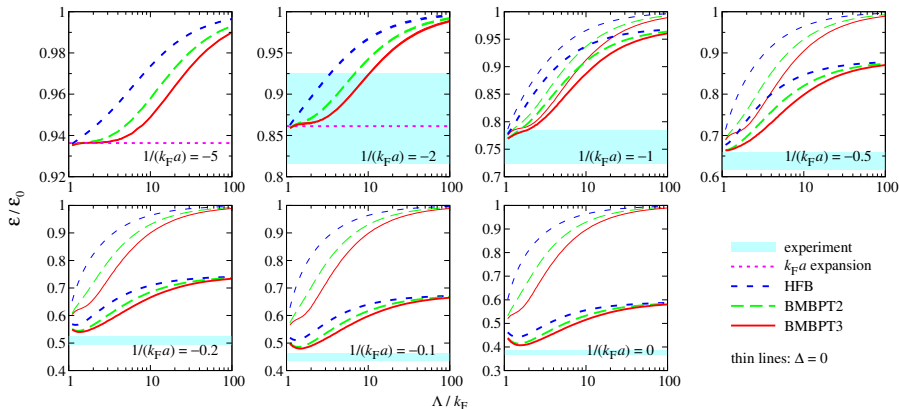
- ▶ Use Mathematica to automatically generate all terms

- ▶ Number of integrations:

18 (six momenta)
 -6 (momentum conservation)
 -3 (rotational invariance)
 $= 9$ dimensions



Cutoff dependence of HF(B)+(B)MBPT GS energy



[exp.: Horikoshi et al. (2017), $k_F a$ expansion: Wellenhofer et al. (2021)]

- Approximate cutoff independence reached in a region of small cutoffs ($\Lambda \leq 3k_F$) at weak coupling
- Inclusion of pairing (thick vs thin lines) very important at stronger coupling
- For $\Lambda \simeq 1.5 - 2k_F$, results are close to experimental ones

Discussion

- ▶ BMBPT3 weakens cutoff dependence but is not enough to remove it

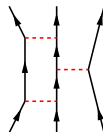
What is missing?

- ▶ Higher orders of BMBPT

Is it efficient to expand about the HFB GS although we know that screening reduces the gap?

- ▶ Induced three-body force (3BF) and higher-body forces:

even if there is no 3BF in the limit $\Lambda \rightarrow \infty$, at finite Λ there will be an effective 3BF to compensate for the contributions of loop momenta above Λ in diagrams like this one



- ▶ BMBPT expands the perturbed GS in terms of fermionic QP states:
approaching the BEC regime, bosonic degrees of freedom (Bogoliubov-Anderson mode) become progressively more important which require resummation of (Q)RPA diagrams

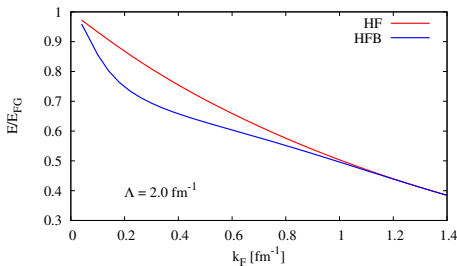
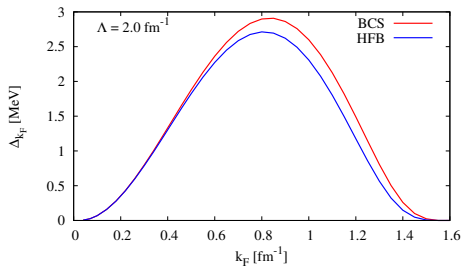
Differences between cold atoms and neutron matter

The nn interaction is more complicated:

- ▶ Even at the lowest relevant densities, the finite range of the nn interaction is not negligible
- ▶ Not only s -wave, but also higher partial waves:
in practice, we include waves up to $L = 6$
- ▶ Coupling between different L due to tensor force
- ▶ We use $V_{\text{low-}k}$ matrix elements [Bogner et al. 2007] generated from AV18 or chiral interactions (both give almost identical results)
- ▶ Although it is relatively weak in pure neutron matter, the 3BF (neglected here) could play a role at higher densities

HFB results for neutron matter with usual cutoff (2 fm^{-1})

- ▶ HFB gap and GS energy (in units of non-interacting energy E_{FG}) as fct. of k_F :

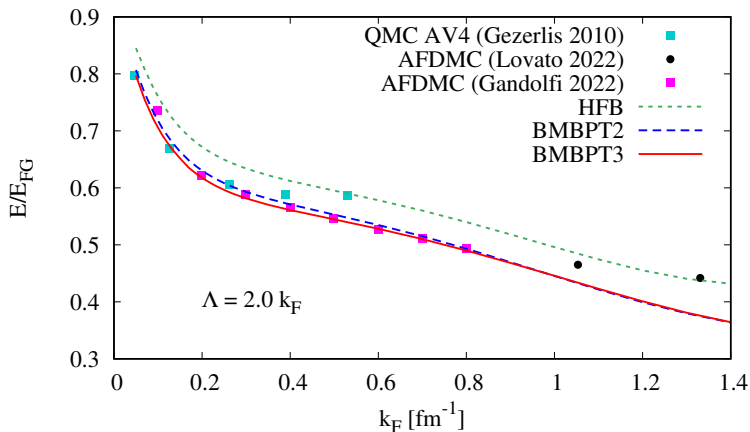


- ▶ HFB gap slightly reduced compared to the one obtained with free spectrum
- ▶ Important effect of pairing on ground-state energy at low densities
- ▶ At very low density, the HF shift is much too weak:

$$\frac{E_{HFB}}{E_{FG}} - 1 \rightarrow \frac{10mk_F V_0(0,0)}{9\pi} \approx 0.1 \times \frac{10k_F a}{9\pi}$$

- ▶ As the cold atoms case, we will use density dependent cutoffs $\Lambda \simeq 1.5 - 3k_F$

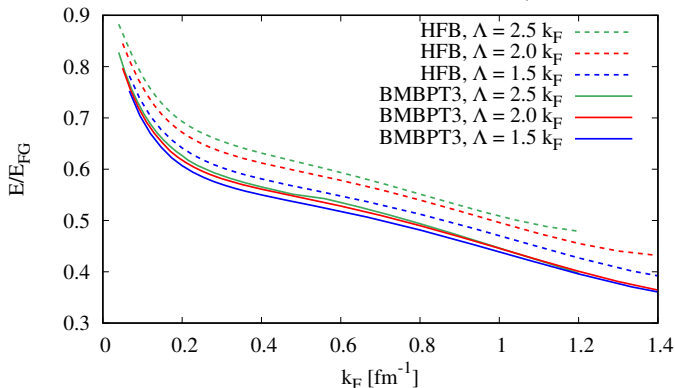
HFB+BMBPT results for neutron matter ($\Lambda = 2k_F$)



- ▶ With this low cutoff, BMBPT seems to converge rapidly
- ▶ Good agreement with QMC results at low densities
- ▶ Energies too low at high densities: missing 3-body force?

Cutoff dependence of neutron matter results

- Physical results should be independent of the ratio Λ/k_F



- Varying Λ/k_F in a reasonable range, we see that the BMBPT results show much less cutoff dependence than the HFB results
- The residual cutoff dependence indicates the necessity of including higher orders of BMBPT or induced many-body forces

Conclusions

- ▶ The HFB+BMBPT scheme with low-momentum interactions, known in nuclear structure theory, can also be applied to uniform systems
- ▶ In infinite matter, it seems natural to scale the cutoff Λ with k_F
- ▶ In cold atoms: low-momentum interactions give a HF field and hence better results already at the mean-field (HFB) level, and corrections are perturbative
- ▶ In neutron matter: BMBPT seems to converge at small cutoffs, but three-body force is missing

Outlook

- ▶ In progress: computation of perturbative corrections to U_k and Δ_k
- ▶ Missing (“genuine” and “induced”) 3BF: in-medium SRG method?
- ▶ Can IMSRG also help to solve the screening problem?
- ▶ Contribution of collective modes: work in progress
- ▶ Long-term objective: include also protons (neutron-star core)

Appendix: Tabakin's formula for the separable interaction

$$V_0(q, q) = -\frac{\sin \delta(q)}{mq} \exp \left(\frac{2}{\pi} \mathcal{P} \int_0^\infty dq' \frac{q' \delta(q')}{q^2 - q'^2} \right)$$

$$g_0 = V_0(0, 0), \quad F(q) = \sqrt{V_0(q, q)/g_0}$$