Stochastic Fluid Dynamics and the QCD Critical Point

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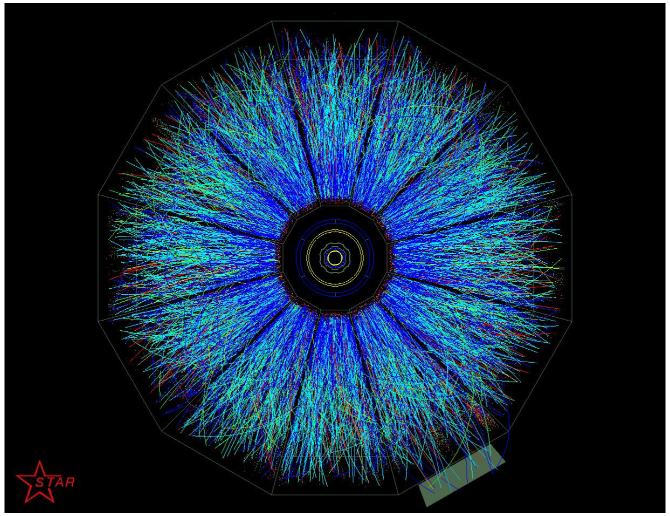
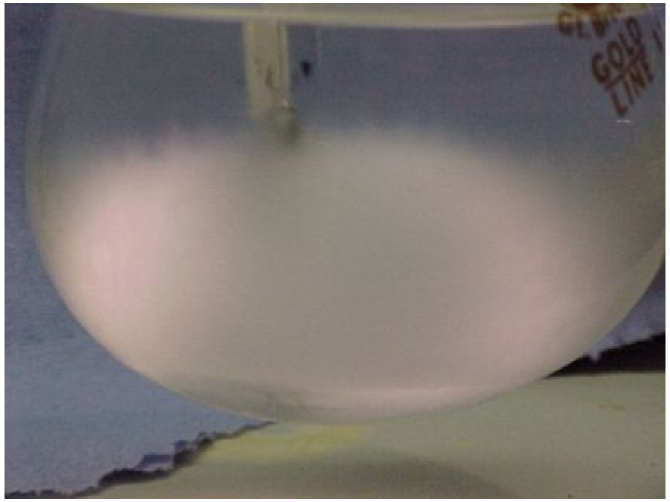
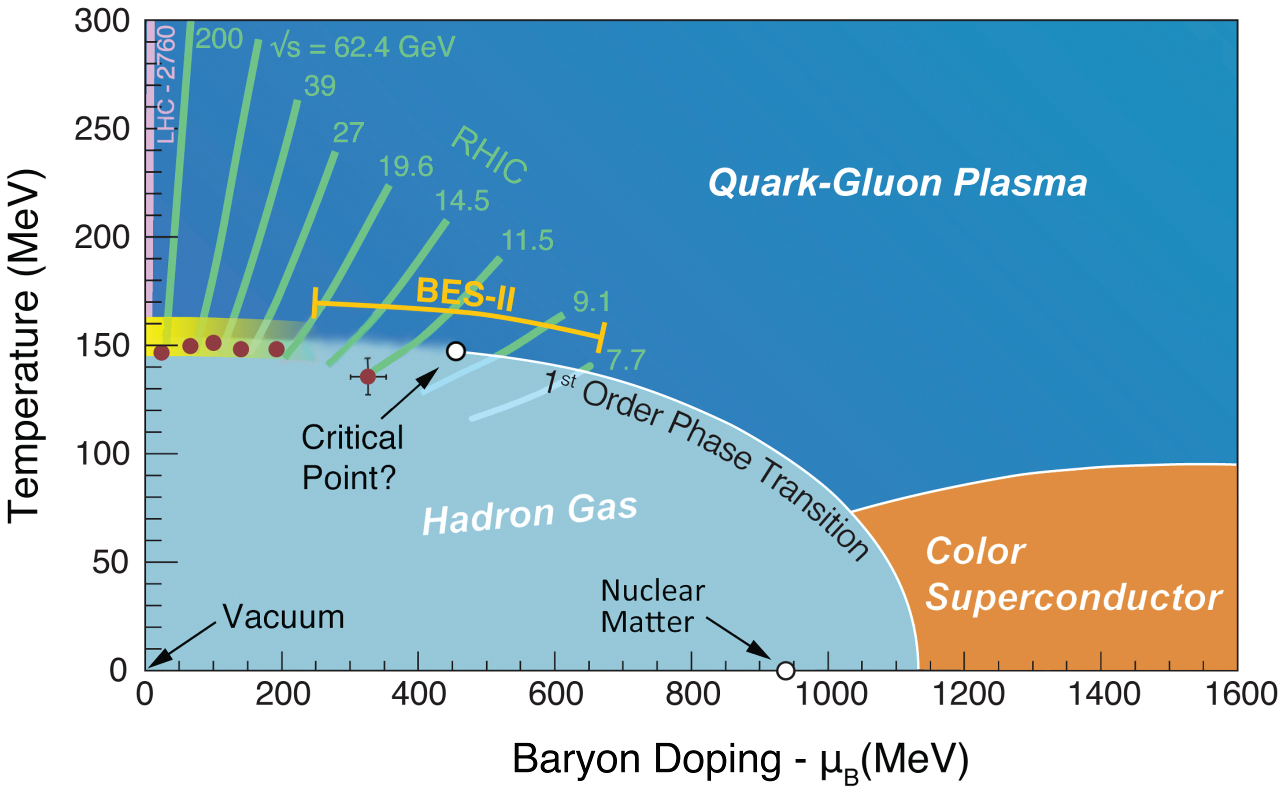


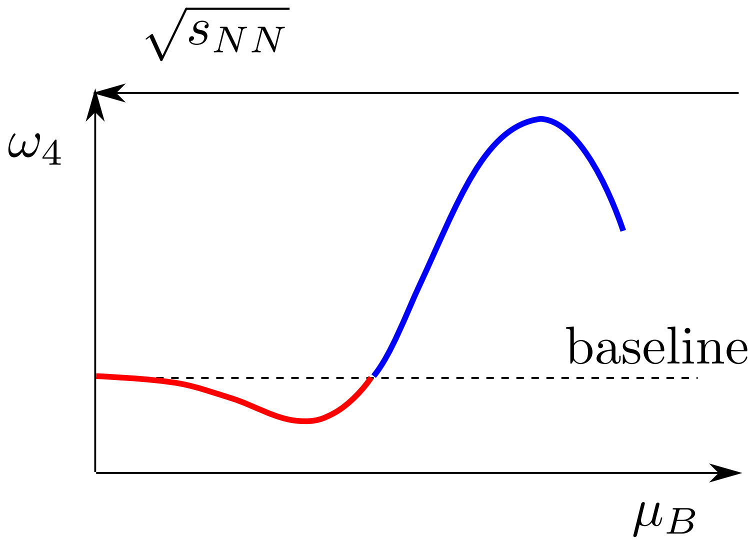


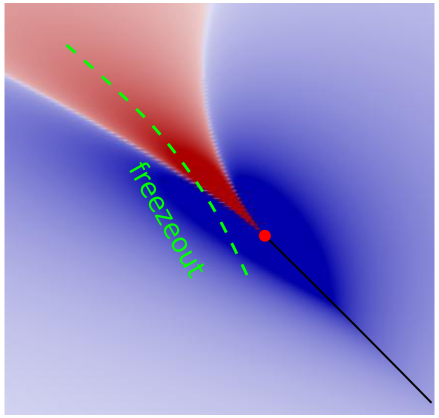
RHIC beam energy scan

Can we experimentally locate the QCD phase transition, either by detecting a critical point, or by identifying a first order transition?



Basic discovery idea: Study fluctuation observables. Expect non-monotonic variation of 4th order cumulant near Ising critical point.





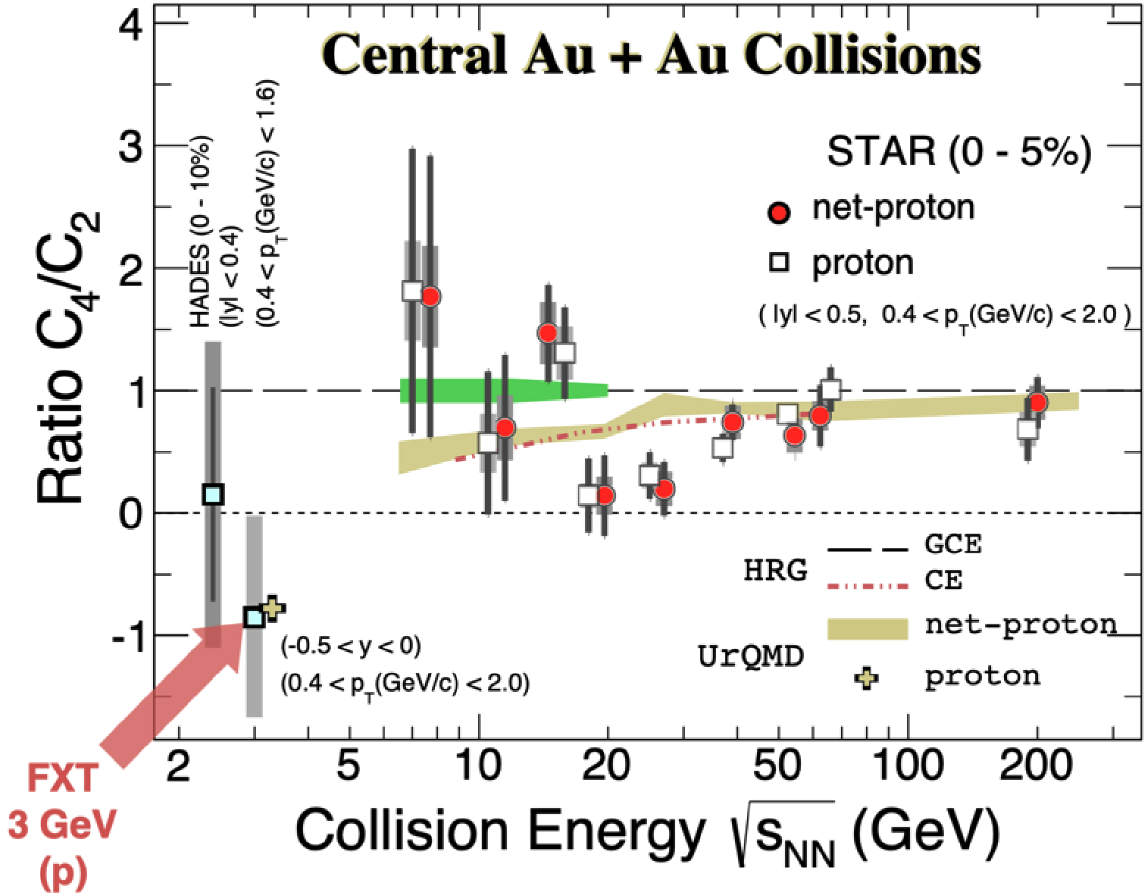
Real world may well be more complicated:

- Finite size and finite expansion rate effects.
- Non-equilibrium effects (memory, critical slowing).
- Freezeout, resonances, global charge conservation, etc.

Motivates dynamical studies.

Figure from Bzdak et al. [1906.00936]

RHIC beam energy scan, BESI



BESII data have been taken, and are being analyzed.

Dynamical Theory

What is the dynamical theory near the critical point?

The basic logic of fluid dynamics still applies. Important modifications:

- Critical equation of state.
- Stochastic fluxes, fluctuation-dissipation relations.
- Possible Goldstone modes (chiral field in QCD?)

Outline:

1. Stochastic field theories: Diffusion of a conserved charge.
2. What if fluctuations are large? Functional methods, the nPI action.
3. Large fluctuations: Numerical approaches to stochastic diffusion.

1. Stochastic diffusion

Consider diffusion of a conserved charge

$$\partial_0 \psi + \vec{\nabla} \cdot \vec{j} = 0 \qquad \vec{j} = -D \nabla \psi + \ldots$$

Introduce noise and non-linear interactions

$$\partial_0 \psi = \kappa \nabla^2 \frac{\delta \mathcal{F}}{\delta \psi} + \xi$$

$$\mathcal{F} = \int d^d x \left[\frac{\gamma}{2} (\vec{\nabla} \psi)^2 + \frac{m^2}{2} \psi^2 + \frac{\lambda}{3} \psi^3 + \frac{u}{4} \psi^4 \right]$$

$$\langle \xi(x,t) \xi(x',t') \rangle = \kappa T \nabla^2 \delta(x - x') \delta(t - t') \qquad D = \kappa m^2$$

Equilibrium distribution

$$P[\psi] \sim \exp\left(-\frac{\mathcal{F}[\psi]}{k_B T} \right)$$

Stochastic Field Theory

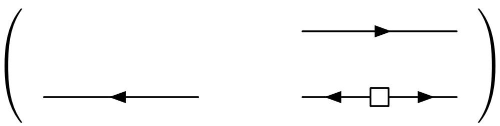
Stochastic effective lagrangian

$$\mathcal{L} = \tilde{\psi}\left(\partial_0 - D\nabla^2\right)\psi + \tilde{\psi} D T \nabla^2 \tilde{\psi} + \tilde{\psi} D \lambda \nabla^2 \psi^2 + \ldots$$

Diffusion Noise Interactions

Matrix propagator

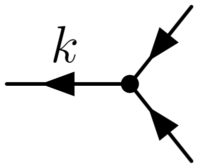
$$\begin{pmatrix} \langle \tilde{\psi}\tilde{\psi} \rangle & \langle \tilde{\psi}\psi \rangle \\ \langle \psi\tilde{\psi} \rangle & \langle \psi\psi \rangle \end{pmatrix} = \begin{pmatrix} 0 & G_R \\ G_A & G_S \end{pmatrix} =$$



Analytic structure of the Schwinger-Keldysh propagator

Interaction vertex

$D\lambda k^2$



What are the rules for constructing more general vertices?

Time reversal invariance

Stochastic theory must describe detailed balance

$$\frac{P(\psi_1 \to \psi_2)}{P(\psi_2 \to \psi_1)} = \exp\left(-\frac{\Delta\mathcal{F}}{k_B T}\right)$$

Related to T-reversal symmetry

$$\begin{aligned} \psi(t) \quad &\to \quad \psi(-t) \\ \tilde{\psi}(t) \quad &\to \quad -\left[\tilde{\psi}(-t) + \frac{\delta\mathcal{F}}{\delta\psi}\right] \end{aligned} \qquad \mathcal{L} \to \mathcal{L} + \frac{d\mathcal{F}}{dt}$$

Ward identities: Fluctuation-Dissipation relations

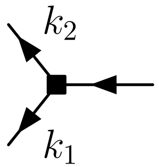
$$2\kappa\,\mathrm{Im}\left\{k^2\langle\psi(\omega,k)\tilde{\psi}(-\omega,-k)\rangle\right\} = \omega\langle\psi(\omega,k)\psi(-\omega,-k)\rangle$$

New and non-classical interactions

At this order (Ψ^3, ∇^2) there is one more interaction

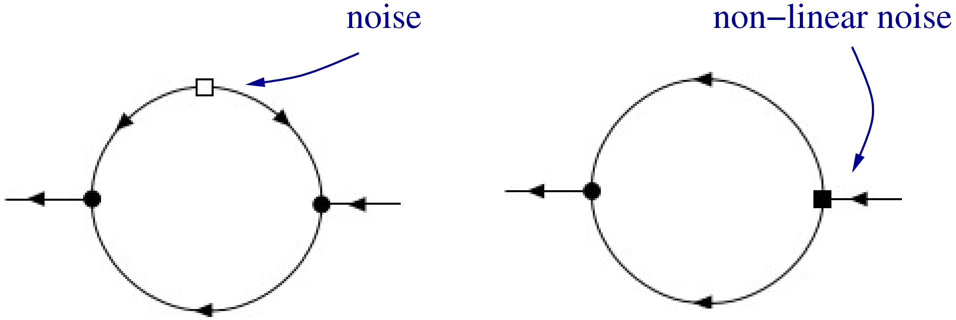
Multiplicative noise : $\quad \mathcal{L} \sim D\lambda_D \psi(\vec{\nabla}\tilde{\psi})^2$

$D\lambda_D k_1 \cdot k_2$



Non-linear noise vertex

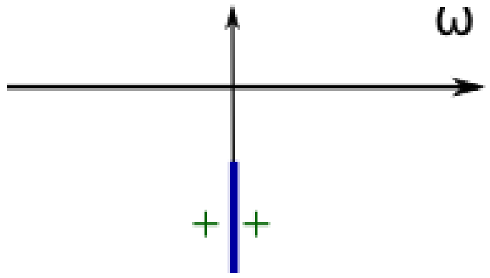
Retarded self energy



Contribute to (non-critical) order parameter relaxation

$$\Sigma(\omega, k) = \frac{\lambda'}{32\pi}\left(i\lambda'\omega k^2 + \lambda_D\left[i\omega - Dk^2\right]k^2\right)\sqrt{k^2 - \frac{2i\omega}{D}}$$

Analytical structure



Diffusive cut dominates over (split) diffusive pole.

Even higher order: Non-linear noise with no contribution to constitutive equations.

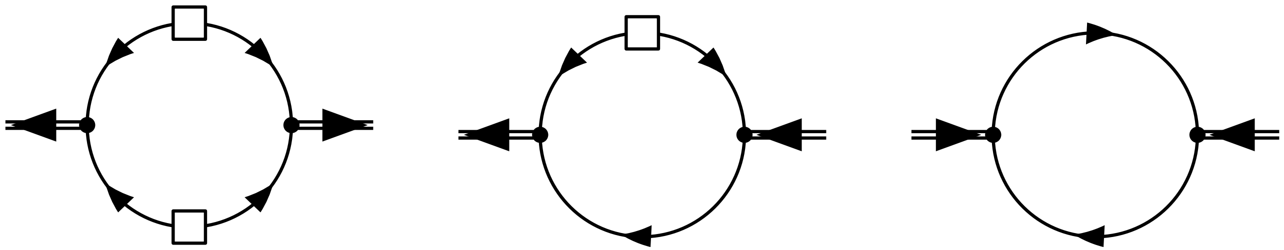
Chao, T.S. [2008.01269], see also Chen-Lin et al. [1811.12540] and Jain & Kovtun [2009.01356]

2. 1PI effective action

Consider 1PI effective action

$$\Gamma[\Psi, \tilde{\Psi}] = W[J, \tilde{J}] - \int dt\, d^3x \left(J\Psi + \tilde{J}\tilde{\Psi} \right) \qquad \frac{\delta W}{\delta J} = \langle \psi \rangle = \Psi \,,$$

Loop expansion



"Classical" equation of motion

$$(\partial_t - D\nabla^2)\Psi - \frac{D\lambda^2}{2}\nabla^2\Psi^2 + \int d^3x\, dt\, \Psi(x', t')\Sigma(x, t; x', t') = 0$$

2PI effective action

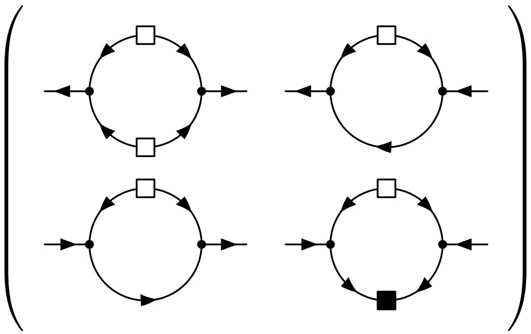
Consider 2PI effective action

$$\Gamma[\Psi_a, G_{ab}] = W[J_a, K_{ab}] - J_A \Psi_A - \frac{1}{2} K_{AB} \left[\Psi_A \Psi_B + G_{AB} \right]$$

Matrix propagator G_{ab}, Bilocal source K_{ab}

Equation of motion for Ψ_a unchanged, but Σ_{ab} satifies Dyson-Schwinger equation

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} =$$



Gap equation (mixed representation)

Consider mixed representation $\Sigma(t, k^2)$. Free propagator $G_R^0 = \Theta(t) e^{-tDk^2}$

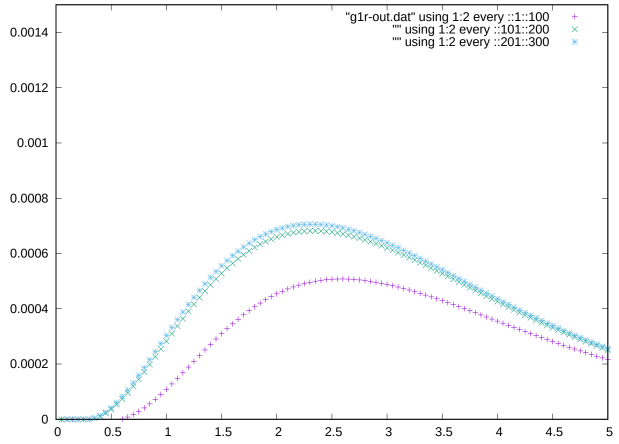
$$\Sigma(t, k^2) \sim \lambda^2 \int d^3k' \, G(t, k - k') G(t, k')$$

Have to determine G from Dyson equation (matrix structure suppressed)

$$G(t, k^2) = G_0(t, k^2) - \int dt_1 \, dt_2 \, G_0(t_1, k^2) \Sigma(t_2 - t_1, k^2) G_(t - t_2, k^2)$$

Short time singularities regulated by Pauli-Vilars "Diffuson"

Example: $G_R(t, k^2)$ for fixed k in weak coupling regime $\lambda, m \sim O(1)$



3. Stochastic diffusion

Stochastic relaxation equation ("model A")

$$\partial_t \psi = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = \Gamma T \delta(x-x')\delta(t-t')$$

Naive discretization

$$\psi(t+\Delta t) = \psi(t) + (\Delta t)\left[-\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \sqrt{\frac{\Gamma T}{(\Delta t) a^3}}\theta\right] \qquad \langle \theta^2 \rangle = 1$$

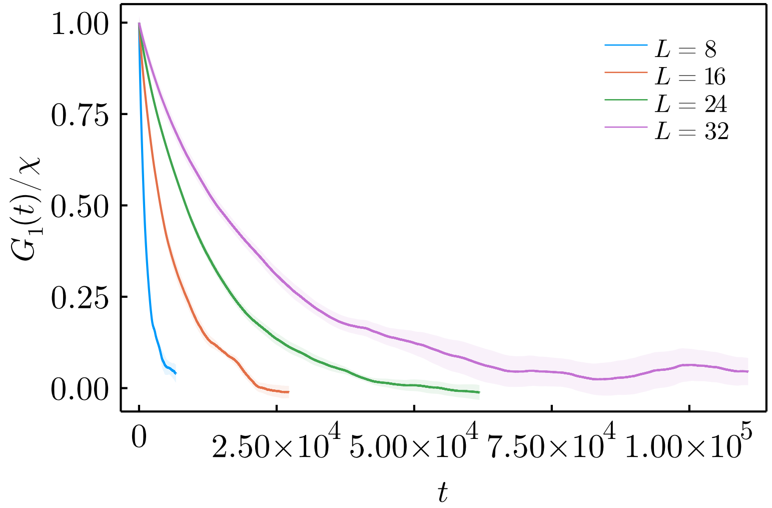
Noise dominates as $\Delta t \to 0$, leads to discretization ambiguities in the equilibrium distribution.

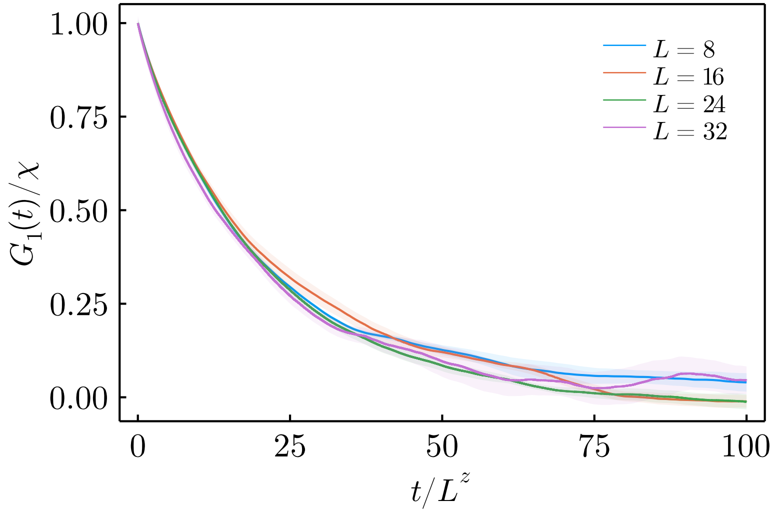
Idea: Use Metropolis update

$$\psi(t+\Delta t) = \psi(t) + \sqrt{2\Gamma(\Delta t)}\theta \qquad p = min(1, e^{-\beta \Delta \mathcal{F}})$$

Dynamic scaling (model A)

Correlation functions at T_c, $V = L^3$, $L = 8, 16, 24, 32$



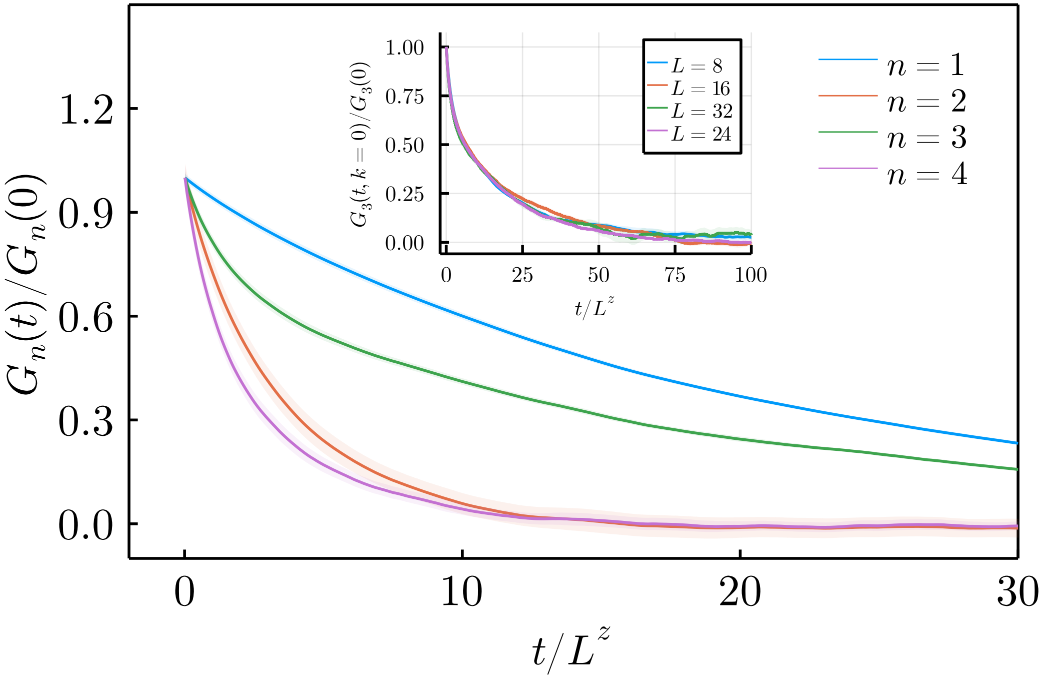


$$G_1(t) = \langle M(0)M(t)\rangle \qquad M(t) = \int d^3x\, \psi(x)$$

Dynamic critical exponent $z = 2.026(56)$.

Correlation functions of higher moments

Correlation functions at T_c



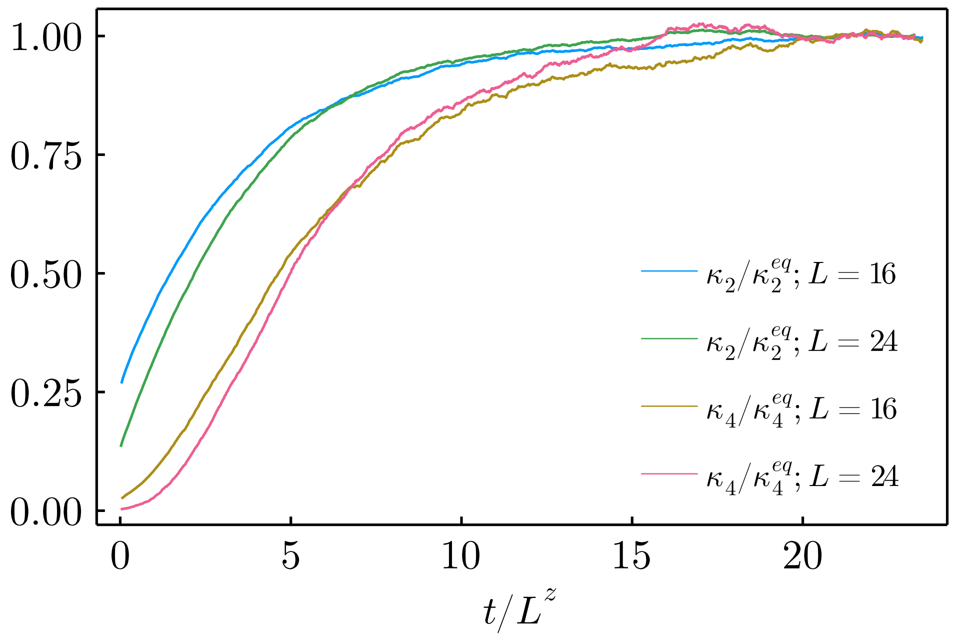
Inset: Dynamic scaling of $G_3(t)$ with $z = 2.026(56)$.

$$G_n(t) = \langle M^n(0) M^n(t) \rangle \qquad M(t) = \int d^3x\, \psi(x)$$

Dynamic scaling holds for all n, but deacy constant depends on n.

Relaxation after a quench

Thermalize at $T > T_c$. Study evolution at T_c



$$C_n(t) = \langle\langle M^n(t)\rangle\rangle_{M(0)}(n = 2, 4) \qquad M(t) = \int d^3x\, \psi(x)$$

Observe separate early ("slip") and late ("dynamical") exponents.

Summary

Dynamical evolution of fluctuations is important.

Old and new ideas about effective actions on the Keldysh contour. In principle allows systematic derivation of hydro equations for n-point functions.

Alternative approach: Direct simulation of stochastic fluid dynamics. New idea: Ignore backreaction, and use Metropolis (or heat bath?) algorithm.

Not discussed: From conserved charges to particles.