

# Nonrelativistic conformality and hydrodynamics

**Yusuke Nishida (Tokyo Tech)**

**International Conference on Recent  
Progress in Many-Body Theories XXI**

**Sep. 12 - 16 (2022) @ Chapel Hill, NC**

# Plan of this talk

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## 1. Introduction

- Scale and conformal symmetries  
in nonrelativistic systems

## 2. Physical consequences

- Breathing mode & bulk viscosity

## 3. Recent progress

- Bulk viscosity as contact correlation
- Bose-Fermi duality in 1D

K. Fujii & Y. Nishida, Phys. Rev. A 98, 063634 (2018);

Phys. Rev. A 102, 023310 (2020); Phys. Rev. A 103, 053320 (2021)

Y. Nishida, Ann. Phys. 410 (2019) 167949

T. Tanaka & Y. Nishida, arXiv:2206.07848

# 1. Nonrelativistic CFT

Y. Nishida & D. T. Son, PRD (2007); arXiv:1004.3597

# Nonrelativistic CFT

**Maximal spacetime symmetries of**

$$S_{\text{free}} = \int dt d^d \vec{x} \psi^\dagger \left( i\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \psi$$

U. Niederer, HPA (1972)

C. R. Hagen, PRD (1972)

- Translations in time and space
- Galilean boosts
- Scale transformation
- Spatial rotations
- Phase rotation

$$\vec{x} \rightarrow e^{-s} \vec{x}, \quad t \rightarrow e^{-2s} t, \quad \psi \rightarrow e^{(d/2)s} \psi$$

- Conformal transformation

$$\vec{x} \rightarrow \frac{\vec{x}}{1 - ct}, \quad t \rightarrow \frac{t}{1 - ct},$$

$$\psi \rightarrow (1 - ct)^{d/2} \exp \left( i \frac{c}{1 - ct} \frac{m}{2} \vec{x}^2 \right) \psi$$

# Nonrelativistic CFT

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# Nonrelativistic CFT

- **Scale transformation (infinitesimal)**

U. Niederer, HPA (1972)  
C. R. Hagen, PRD (1972)

$$\begin{aligned}\delta_s \psi &= s \left( \frac{d}{2} + \vec{x} \cdot \vec{\nabla} + 2t\partial_t \right) \psi \\ &= -is [D - 2tH, \psi]\end{aligned}$$

$$D \equiv \int d^d \vec{x} \, \vec{x} \cdot \psi^\dagger (-i \vec{\nabla}) \psi$$

- **Conformal transformation**

$$\vec{x} \rightarrow \frac{\vec{x}}{1 - ct}, \quad t \rightarrow \frac{t}{1 - ct},$$

$$\psi \rightarrow (1 - ct)^{d/2} \exp \left( i \frac{c}{1 - ct} \frac{m}{2} \vec{x}^2 \right) \psi$$

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$$D \equiv \int d^d \vec{x} \vec{x} \cdot \psi^\dagger (-i \vec{\nabla}) \psi$$

- **Conformal transformation (infinitesimal)**

$$\begin{aligned}\delta_c \psi &= c \left( i \frac{m}{2} \vec{x}^2 - t \frac{d}{2} - t \vec{x} \cdot \vec{\nabla} - t^2 \partial_t \right) \psi \\ &= -ic [C - tD + t^2 H, \psi]\end{aligned}$$

$$C \equiv \frac{m}{2} \int d^d \vec{x} \vec{x}^2 \psi^\dagger \psi$$

~ mean square radius  
~ harmonic potential

# Nonrelativistic CFT

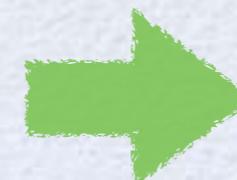
- **Scale transformation (infinitesimal)**

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$$\delta_s \psi = s \left( \frac{d}{2} + \vec{x} \cdot \vec{\nabla} + 2t\partial_t \right) \psi$$

$$= -is [D - 2tH, \psi] \equiv -is [D(t), \psi]$$

$$D \equiv \int d^d \vec{x} \vec{x} \cdot \psi^\dagger (-i \vec{\nabla}) \psi$$



$$[D, H] = 2iH$$

$$\dot{D}(t) = 0$$

- **Conformal transformation (infinitesimal)**

$$\delta_c \psi = c \left( i \frac{m}{2} \vec{x}^2 - t \frac{d}{2} - t \vec{x} \cdot \vec{\nabla} - t^2 \partial_t \right) \psi$$

$$= -ic [C - tD + t^2 H, \psi] \equiv -ic [C(t), \psi]$$

$$C \equiv \frac{m}{2} \int d^d \vec{x} \vec{x}^2 \psi^\dagger \psi$$



$$[C, H] = iD$$

$$\dot{C}(t) = 0$$

# Nonrelativistic CFT

Generators ( $D, C, H$ ) obey  $SO(2,1)$  Lie algebra

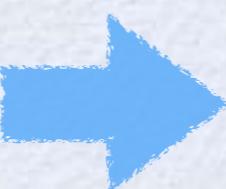
$$[D, H] = 2iH, \quad [C, H] = iD, \quad [D, C] = -2iC$$

scale invariance

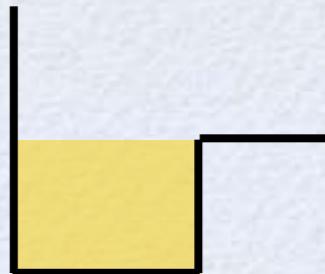
$$\dot{n} = -\vec{\nabla} \cdot \vec{j}$$

always true

$$H = H_0 + V(r)$$

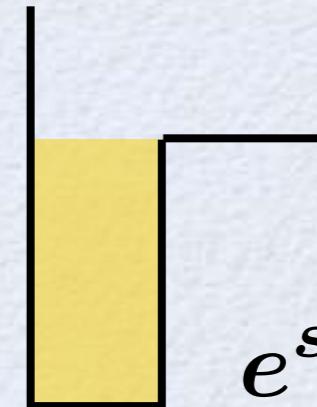


$$H' = H_0 + e^{-2s}V(e^{-s}r)$$



$r_0$  &  $a$

$$e^{-isD} H e^{isD} = e^{2s} H'$$



$e^s r_0$  &  $e^s a$

**H=H'** for zero-range ( $r_0=0$ )  
 & infinite scattering length ( $a=\infty$ ) interaction  
 (relevant to cold atom experiments)

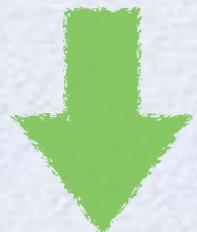
## 2. Breathing mode & bulk viscosity

Y. Nishida & D. T. Son, PRD (2007); arXiv:1004.3597

K. Fujii & Y. Nishida, Phys. Rev. A 98, 063634 (2018)

# Operator-State correspondence

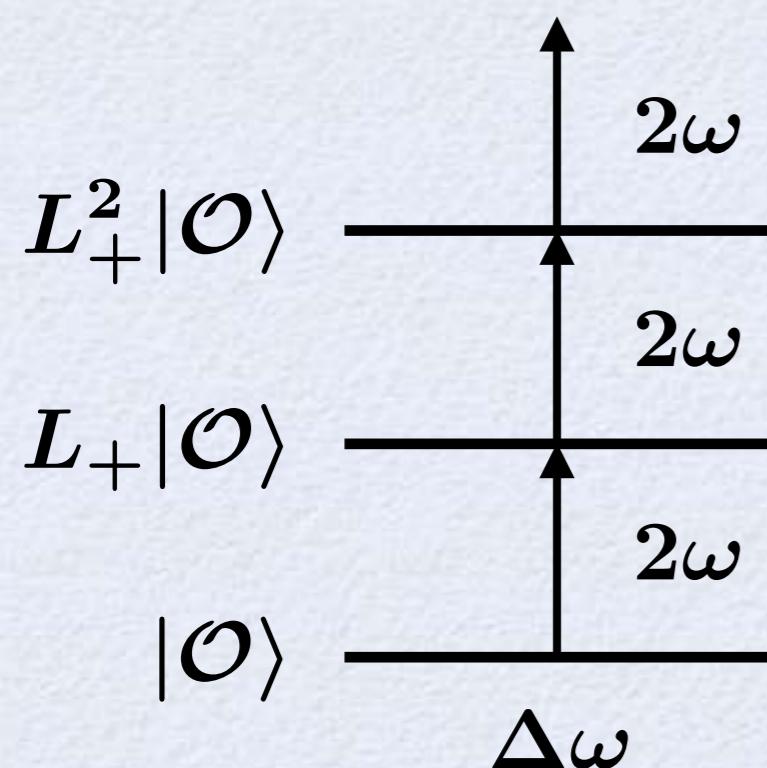
$$[D, H] = 2iH, \quad [C, H] = iD, \quad [D, C] = -2iC$$



$$H_\omega \equiv H + \omega^2 C, \quad L_\pm \equiv H - \omega^2 C \pm i\omega D$$

$$[H_\omega, L_\pm] = \pm 2\omega L_\pm, \quad [L_+, L_-] = -4\omega H_\omega$$

**raising & lowering operators**



$$L_- |\mathcal{O}\rangle = 0$$

$$H_\omega L_+^n |\mathcal{O}\rangle = (\Delta\omega + 2n\omega) L_+^n |\mathcal{O}\rangle$$

Valid for

**any scale invariant systems  
confined by harmonic potential**

Y. Nishida & D. T. Son, PRD (2007); arXiv:1004.3597

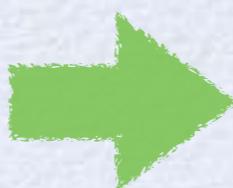
Y. Castin & F. Werner, PRA (2006); arXiv:1103.2851

# Breathing mode

Arbitrary time-evolving state  $|\Psi_t\rangle = e^{-iH_\omega t}|\Psi_0\rangle$

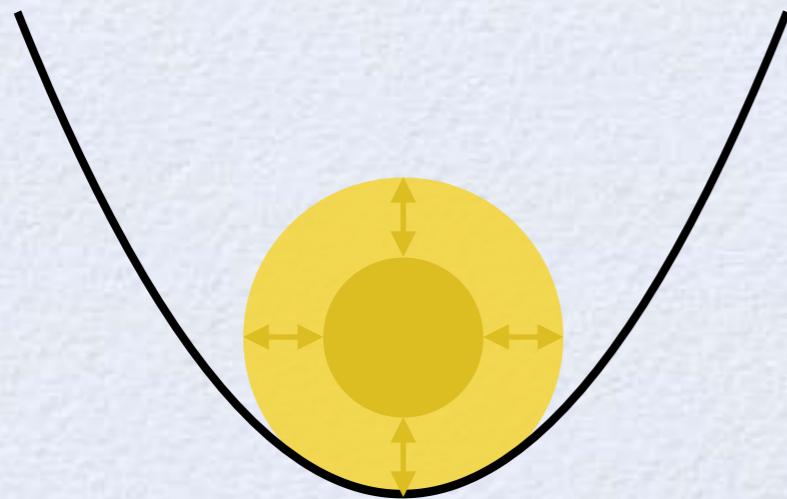
$$\begin{aligned}\langle C \rangle &= \langle \Psi_0 | e^{iH_\omega t} \frac{2H_\omega - L_+ - L_-}{4\omega^2} e^{-iH_\omega t} |\Psi_0\rangle \\ &= \langle \Psi_0 | \frac{2H_\omega - e^{i2\omega t}L_+ - e^{-i2\omega t}L_-}{4\omega^2} |\Psi_0\rangle \\ &\equiv \frac{\langle \Psi_0 | H_\omega | \Psi_0 \rangle - \cos(2\omega t + \varphi) |\langle \Psi_0 | L_+ | \Psi_0 \rangle|}{2\omega^2}\end{aligned}$$

$$\left( C \equiv \frac{m}{2} \int d^d \vec{x} \vec{x}^2 \psi^\dagger \psi \right)$$



Mean square radius

$$\langle \vec{x}^2 \rangle = A + B \cos(2\omega t + \varphi)$$



**Undamped “breathing mode”  
with frequency right at  $2\omega$**

# Breathing mode

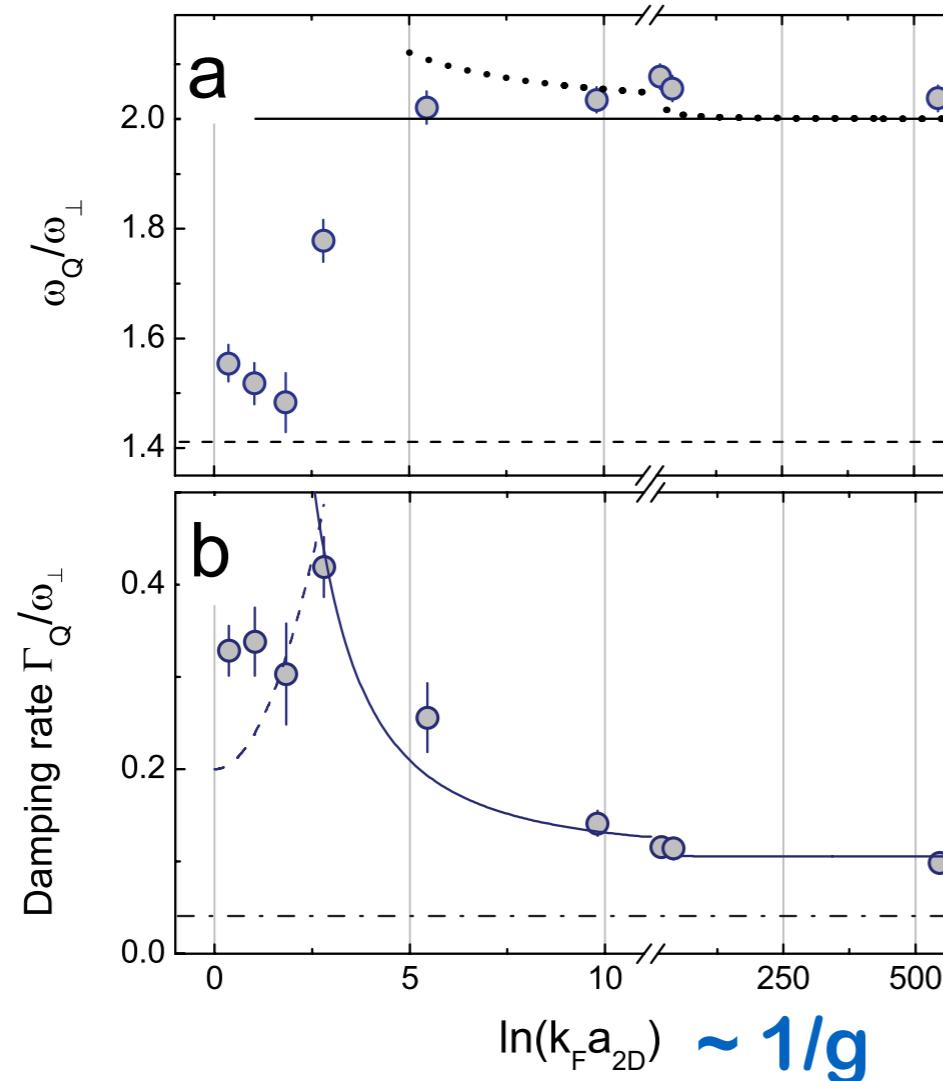
$H$  is scale invariant for  $V = -g \delta^2(\vec{r})$  in 2D ??

Tunable via Feshbach resonance  
with ultracold atoms

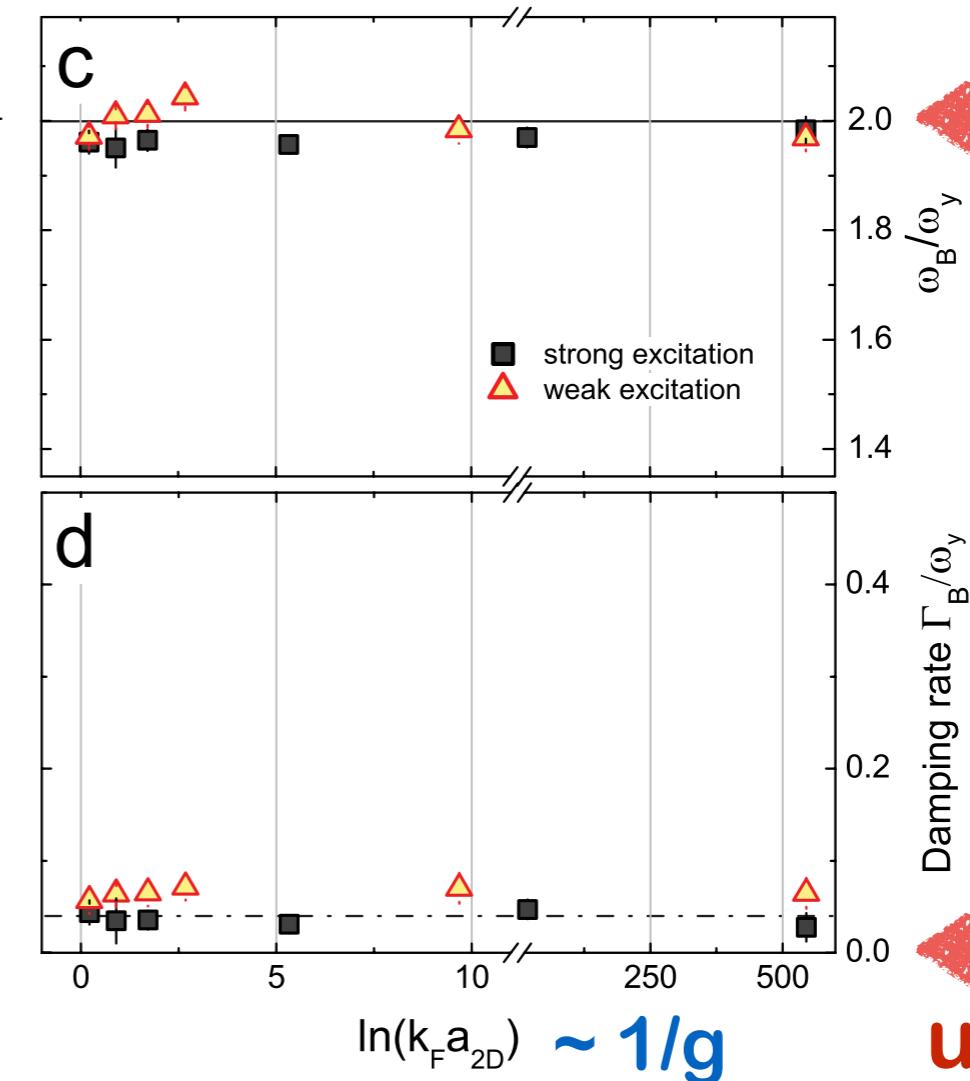
$T \sim 0.4 T_F$

M. Köhl's group, PRL (2012)

Quadrupole mode



Breathing mode



$2\omega$

undamped

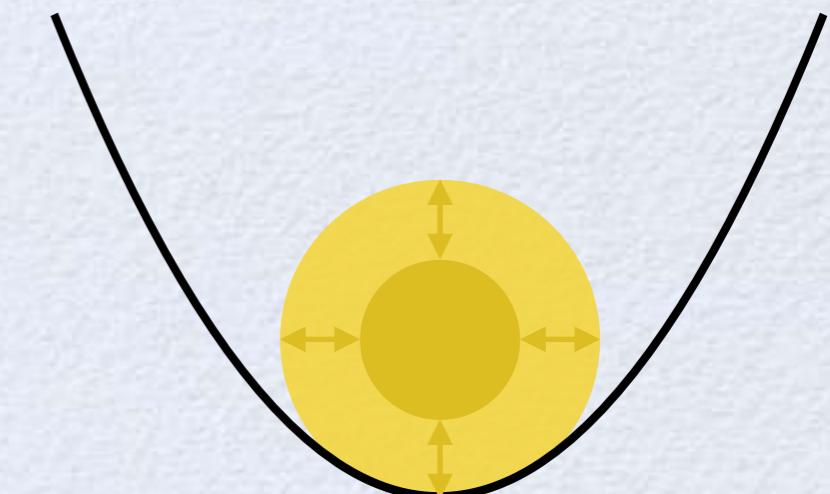
# Bulk viscosity @ $a=\infty$

D. T. Son, PRL (2007) 14/24

Undamped “breathing mode”

for any scale invariant systems  
confined by harmonic potential

→ Vanishing bulk viscosity !?



When coupled with external gauge field & metric

$$S = \int dt d^d \vec{x} \sqrt{g} \left( i\psi^\dagger \overset{\leftrightarrow}{D}_t \psi - \frac{g^{ij}}{2m} \vec{D}_i \psi^\dagger \vec{D}_j \psi + \mathcal{L}_{\text{int}} \right)$$

is invariant under

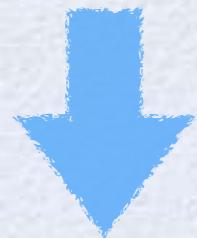
D. T. Son & M. Wingate, Ann Phys (2006)

- Gauge transformation  $\psi \rightarrow e^{i\chi(\vec{x}, t)} \psi$
- General coordinate transformation  $\vec{x} \rightarrow \vec{x}'(\vec{x}, t)$
- Conformal transformation  $t \rightarrow t'(t)$

Microscopic symmetries must be inherited by hydrodynamics

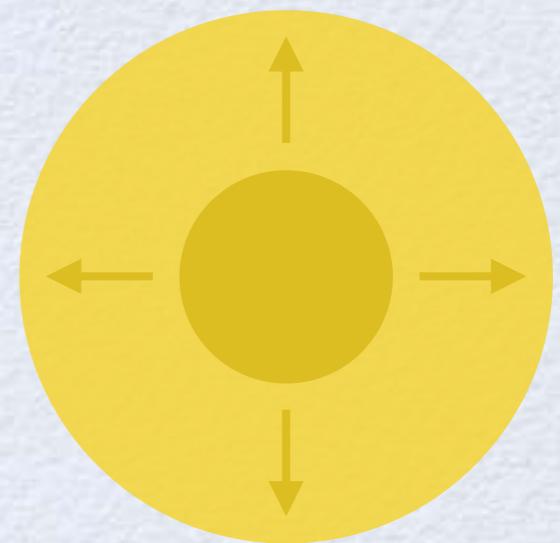
Viscous stress tensor **coupled with metric**

$$\pi_{ij} = \zeta \delta_{ij} \partial_k v^k + \text{shear}$$



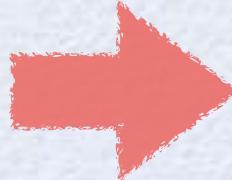
**fluid expansion**

$$\pi_{ij} = \cancel{\zeta} g_{ij} (\nabla_k v^k + \cancel{\partial_t \ln \sqrt{g}}) + \text{shear}$$



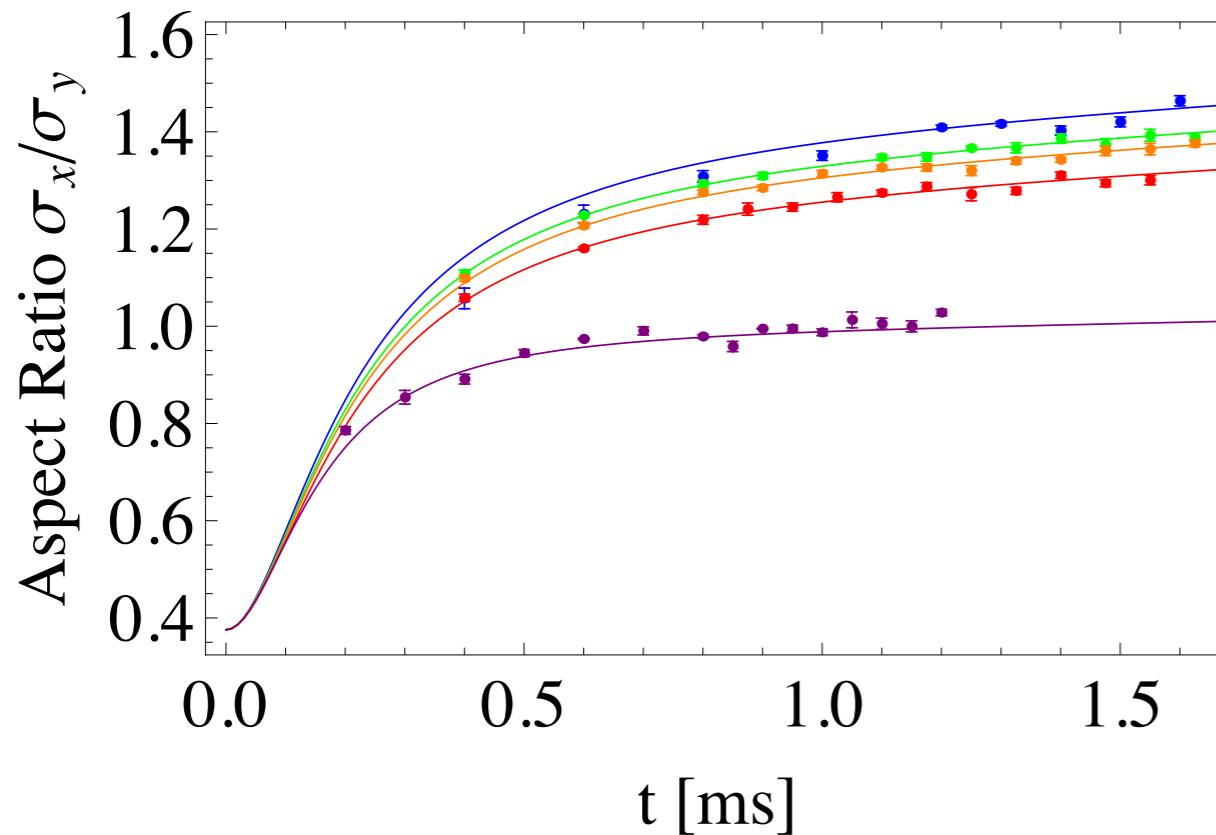
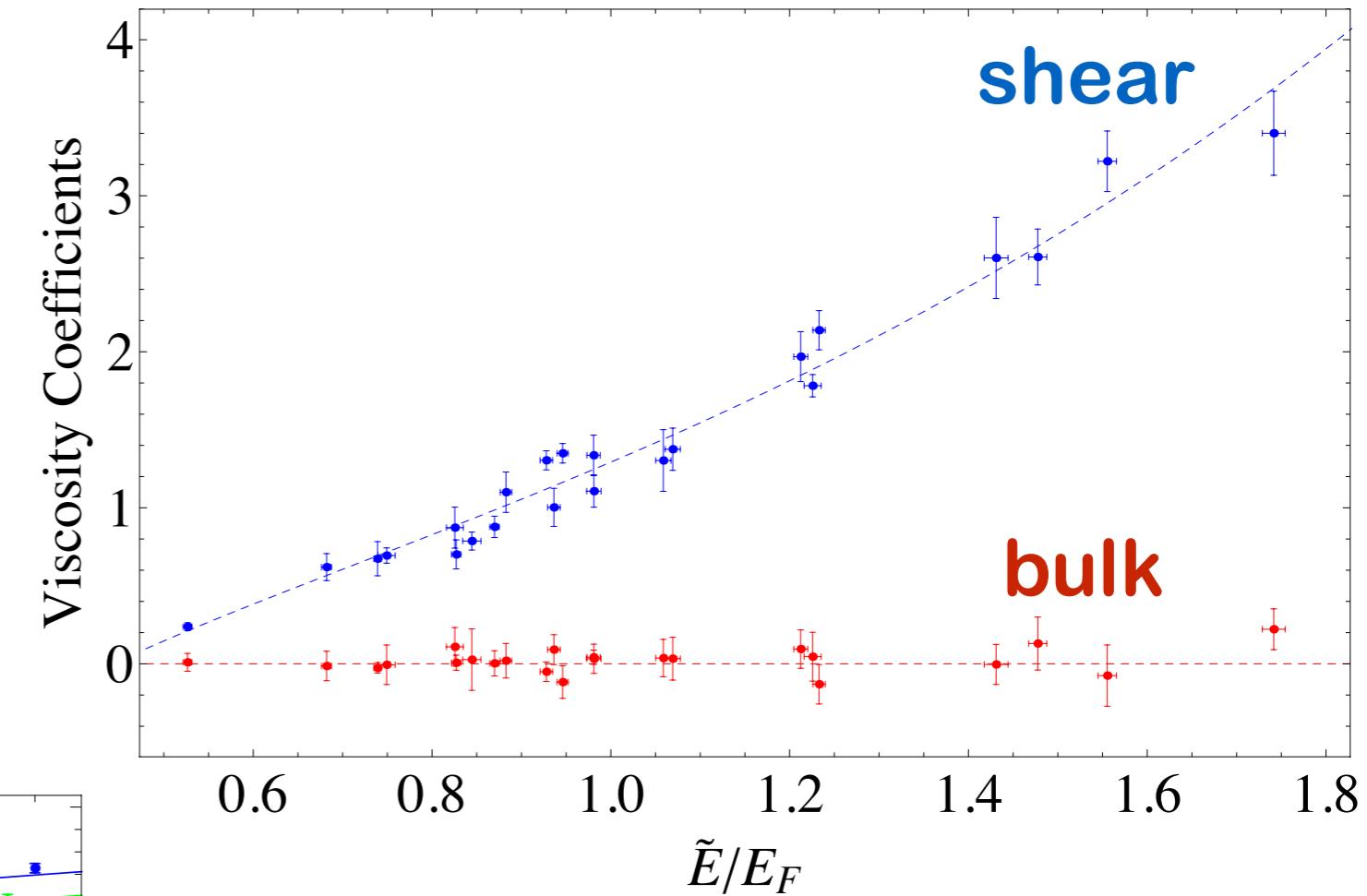
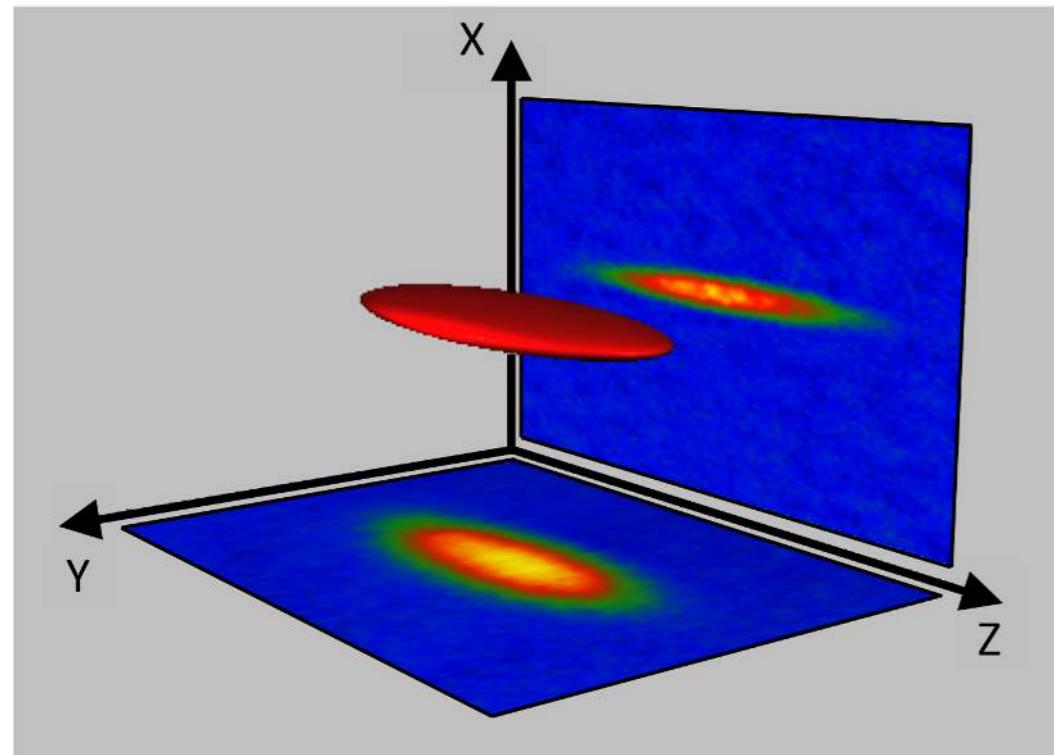
**volume expansion**

is invariant under general coordinate transformation  
but is **NOT** under conformal transformation



**Vanishing bulk viscosity @  $a=\infty$**

# Bulk viscosity @ $a=\text{infinite}$



$$\frac{1}{N} \int d^3 \vec{x} (\eta, \zeta)$$

$$T/T_F = 0.2 \sim 0.6$$

# Bulk viscosity @ $a=\text{finite}$

Scattering length explicitly breaks scale invariance

because  $S(a) \rightarrow S(e^s a)$

$$\left( \vec{x} \rightarrow e^{-s} \vec{x}, \quad t \rightarrow e^{-2s} t, \quad \psi \rightarrow e^{(d/2)s} \psi \right)$$

Scale invariance is “formally” **recovered** if

$$a(\vec{x}, t) \rightarrow a'(\vec{x}', t') = e^{-s} a(x, t)$$

**spurion field** (spacetime-dependent)

Microscopic symmetries must be  
inherited by hydrodynamics

$$\pi_{ij}^{\text{bulk}} = \zeta g_{ij} [(\nabla_k v^k + \partial_t \ln \sqrt{g})]$$

is **NOT** invariant under conformal transformation

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K. Fujii & Y. Nishida, PRA (2018)

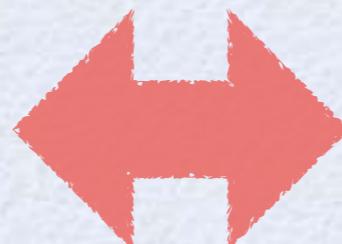
$$\pi_{ij}^{\text{bulk}} = \zeta g_{ij} [(\nabla_k v^k + \partial_t \ln \sqrt{g}) - d (\partial_t \ln a + v^k \partial_k \ln a)]$$

is ~~NOT~~ invariant under conformal transformation

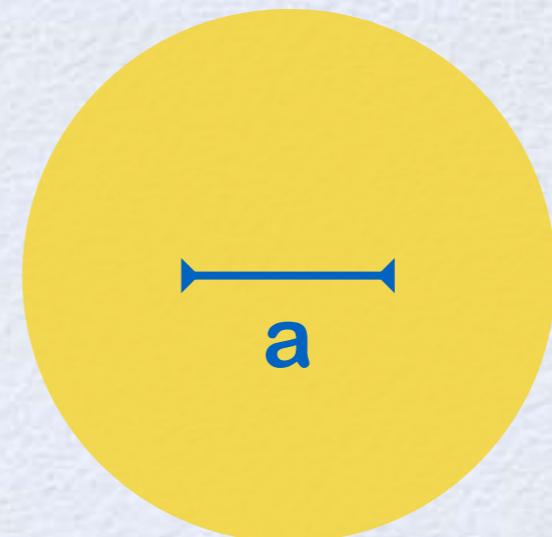
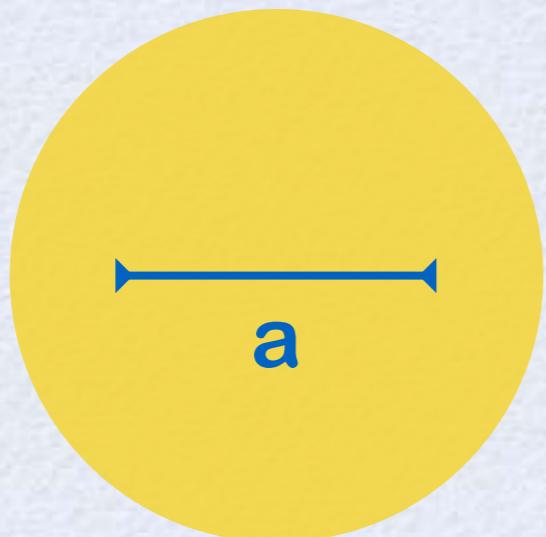
# Bulk viscosity @ $a=\text{finite}$

$$\pi_{ij}^{\text{bulk}} = \zeta g_{ij} [(\nabla_k v^k + \partial_t \ln \sqrt{g}) - d (\partial_t \ln a + v^k \partial_k \ln a)]$$

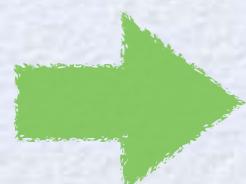
expansion  
of fluid



contraction of  
scattering length



Entropy & energy production even in stationary systems

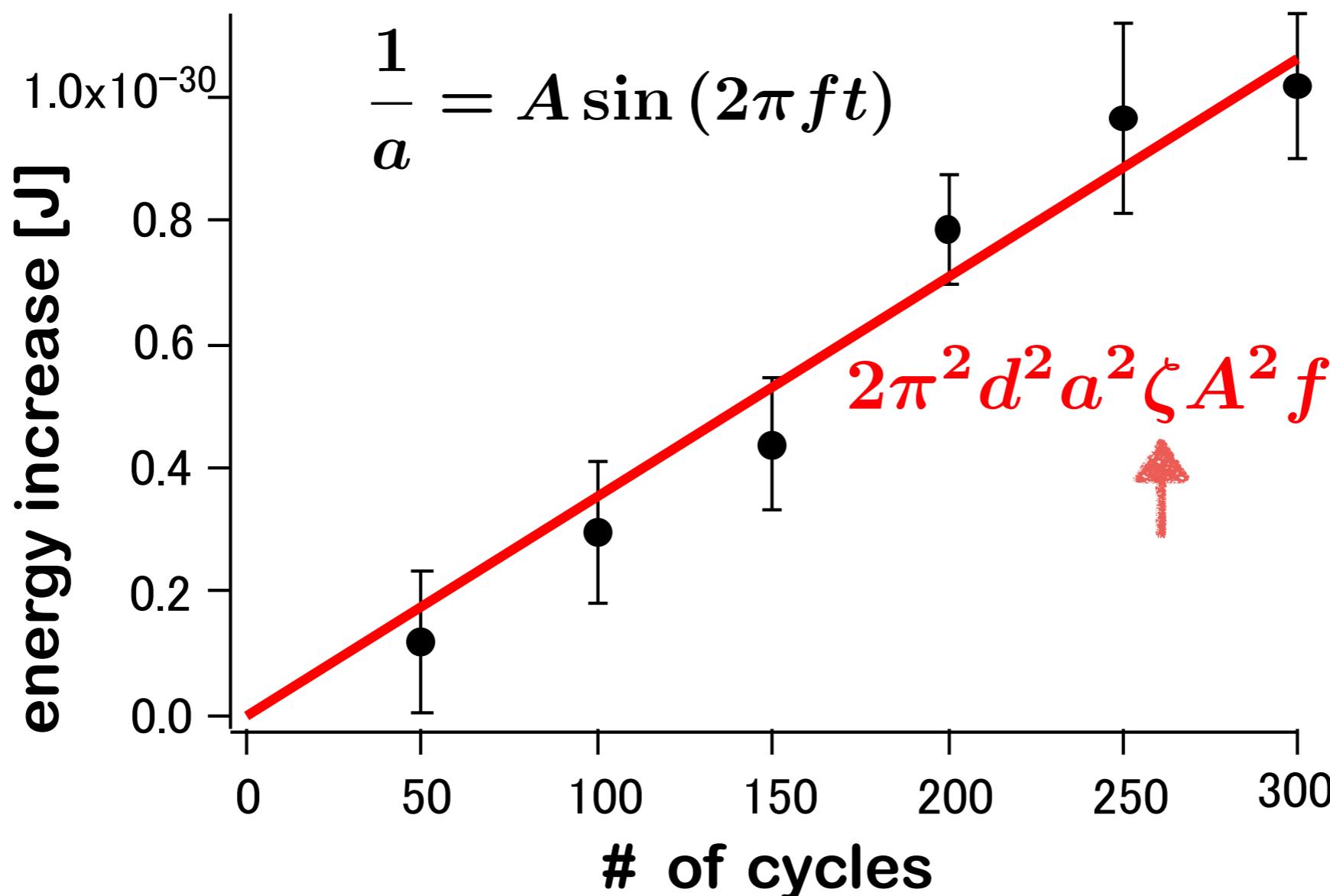


$$T\dot{S} = \frac{d^2 \zeta}{a^2} \dot{a}^2 + O(\dot{a}^3)$$

K. Fujii & Y. Nishida, PRA (2018)

$$\dot{\epsilon} = \frac{c_{\text{eq}}}{m \Omega_{d-1} a^{d-1}} \dot{a} + \frac{d^2 \zeta}{a^2} \dot{a}^2 + O(\dot{a}^3)$$

# Bulk viscosity @ $a=\text{finite}$



Ongoing experiment (DAMOP 2019)  
toward extraction of bulk viscosity

$$\dot{\epsilon} = \frac{c_{\text{eq}}}{m\Omega_{d-1}a^{d-1}\dot{a}} + \frac{d^2\zeta}{a^2}\dot{a}^2 + O(\dot{a}^3)$$

### 3. Contact correlation

- K. Fujii & Y. Nishida, Phys. Rev. A 98, 063634 (2018);  
Phys. Rev. A 102, 023310 (2020); Phys. Rev. A 103, 053320 (2021)
- Y. Nishida, Ann. Phys. 410 (2019) 167949
- T. Tanaka & Y. Nishida, arXiv:2206.07848

# Contact correlation

Kubo formula for dynamic bulk viscosity

$$\zeta(\omega) = \text{Im} \frac{i}{ZV} \int_0^\infty dt \frac{e^{i(\omega+i0^+)t}}{\omega} \frac{\langle [\hat{\Pi}_{ii}(t), \hat{\Pi}_{jj}(0)] \rangle}{d^2}$$

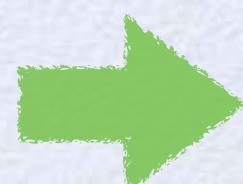
Trace of stress tensor

$$\hat{\Pi}_{ii} = 2\hat{H} + \frac{\hat{C}}{\Omega_{d-1} m a^{d-2}}$$

**Contact**  $\hat{C} \equiv \frac{(mg)^2}{2} \hat{\psi}_\sigma^\dagger \hat{\psi}_\tau^\dagger \hat{\psi}_\tau \hat{\psi}_\sigma$

**conformality breaking**

$$\Omega_{0,1,2} = 2, 2\pi, 4\pi$$



$$\zeta(\omega) = \text{Im} \frac{i}{ZV} \int_0^\infty dt \frac{e^{i(\omega+i0^+)t}}{\omega} \frac{\langle [\hat{C}(t), \hat{C}(0)] \rangle}{(d \Omega_{d-1} m a^{d-2})^2}$$

**Contact-contact correlation function**

# Contact correlation

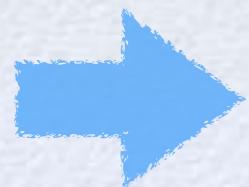
$$\zeta(\omega) = \text{Im} \frac{i}{ZV} \int_0^\infty dt \frac{e^{i(\omega+i0^+)t}}{\omega} \frac{\langle [\hat{C}(t), \hat{C}(0)] \rangle}{(d \Omega_{d-1} m a^{d-2})^2}$$

- Energy & entropy production by  $a(t)$  can be confirmed microscopically with linear response
- Easy to evaluate systematically at weak coupling or high temperature limits
  - comparison with kinetic theory results
    - C. Chafin & T. Schafer, PRA (2013); K. Dusling & T. Schafer, PRL (2013)
    - K. Fujii & T. Enss, arXiv:2208.03353
- Even strong coupling limit can be accessed in 1D thanks to the Bose-Fermi duality

# Summary of this talk

## Nonrelativistic conformality

- Undamped breathing mode right at  $2\omega$  @  $a=\text{infinite}$
- Vanishing bulk viscosity in hydrodynamics
- Spacetime-dependent scattering length naturally couples with bulk viscosity
- Fluid expansion can be simulated by modulating “a”



New experimental probe for bulk viscosity

## Bulk viscosity = contact correlation

- Systematically evaluation at weak coupling / high T replaces Boltzmann description
- Bose-Fermi duality in 1D  $\Rightarrow$  strong coupling limit