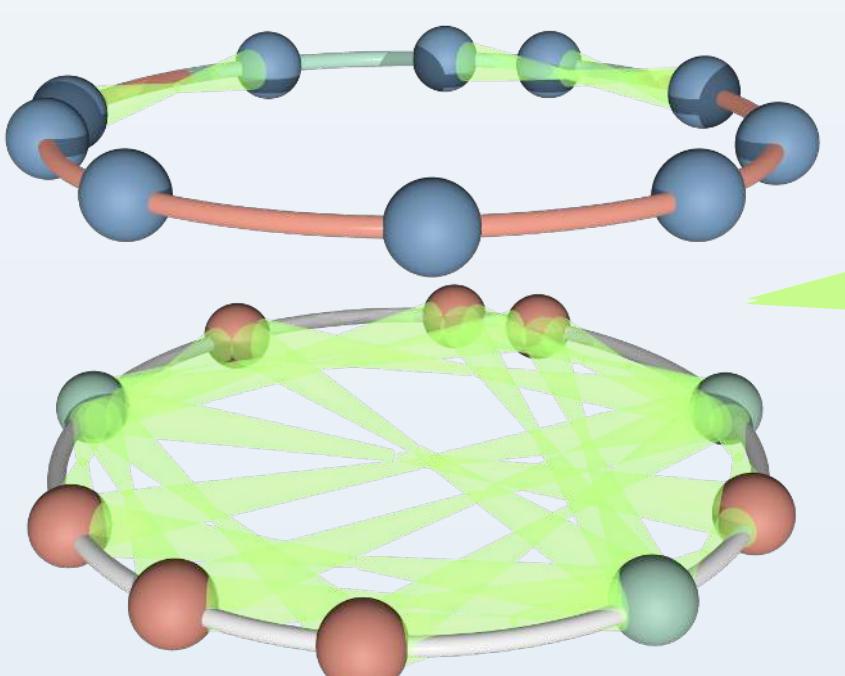


# Equivalence of Spatial and Particle Entanglement Growth After a Quantum Quench



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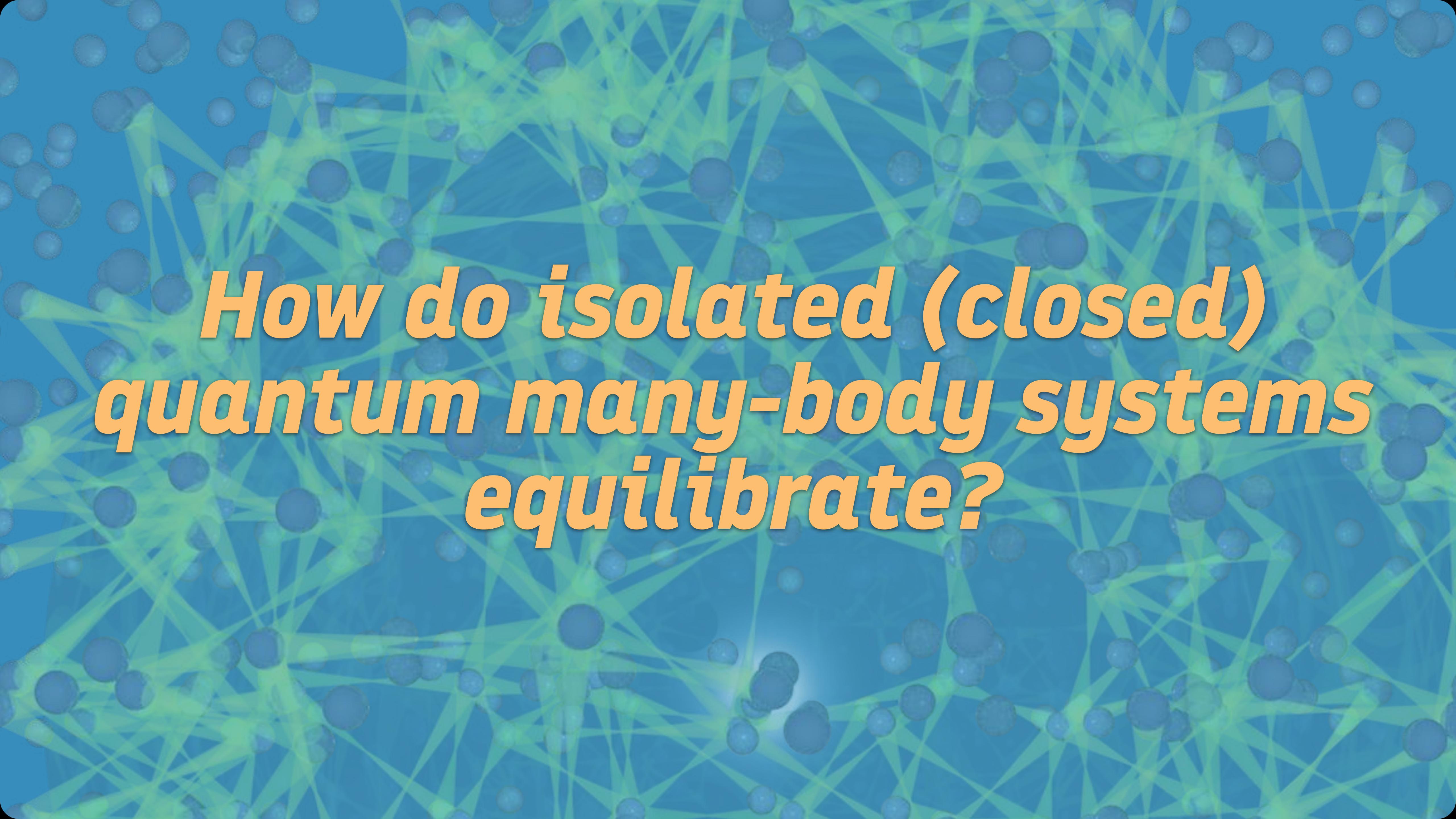


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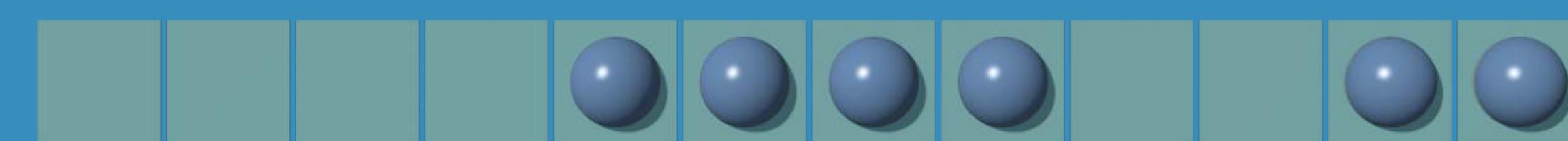
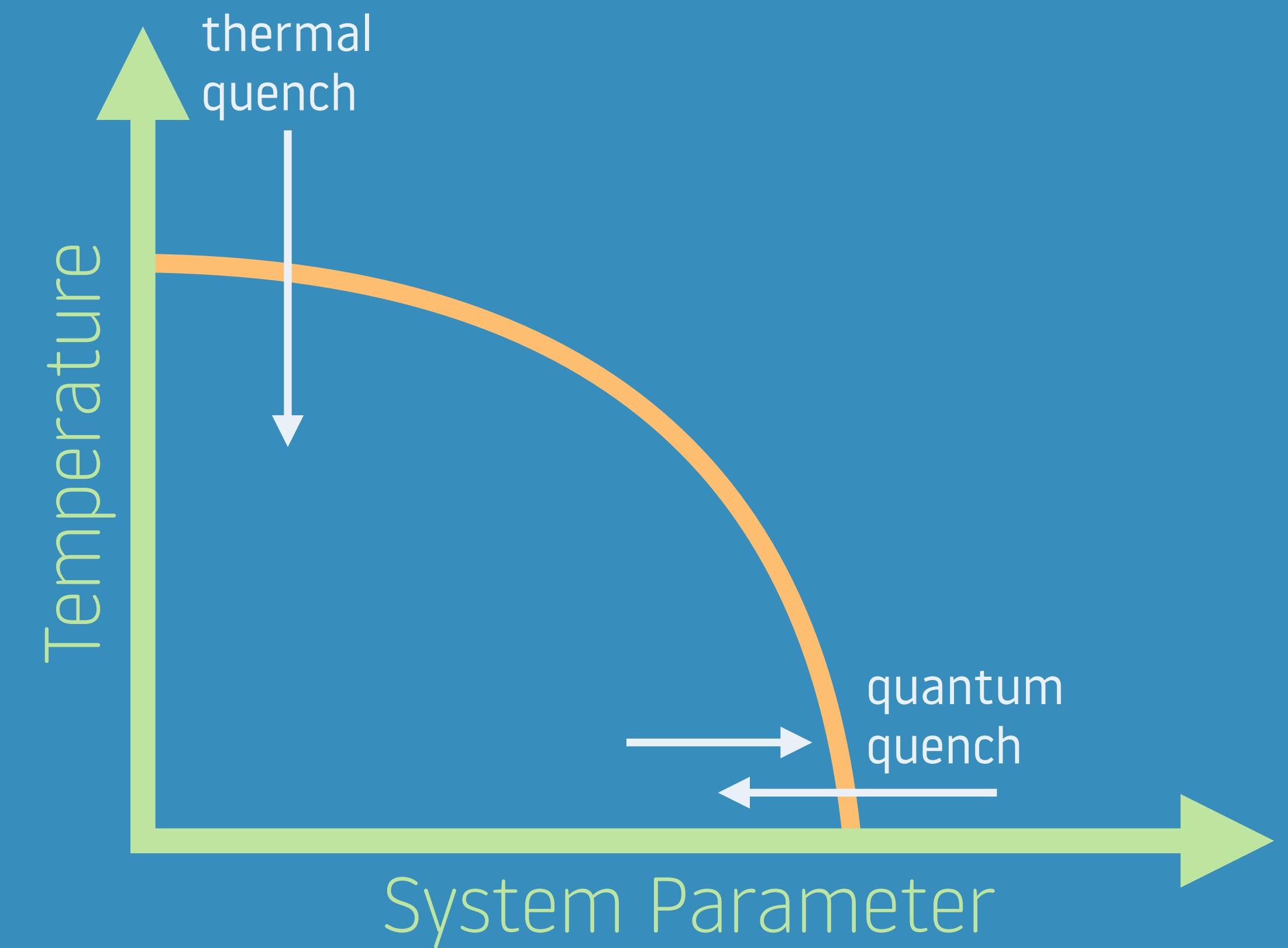
<http://delmaestro.org/adrian> • <https://github.com/DelMaestroGroup/>

AD, H. Barghathi, and B. Rosenow, Phys Rev B **104**, 195101 (2021)  
AD, H. Barghathi, and B. Rosenow, Phys Rev R **4**, L022023 (2022)  
M. Thamm, H. Radhakrishnan, H. Barghathi, B. Rosenow, A.D.  
arXiv:2206.11301



*How do isolated (closed)  
quantum many-body systems  
equilibrate?*

# Quench: A Sudden System Change



$V = 0$



$V \gg 0$

# Quantum Quench: Recipe

1. Prepare system in ground state at  $t < 0$

$$H = H_0 + \Theta(t)H'$$

$$H_0 |\Psi_0\rangle = E_0 |\Psi_0\rangle$$

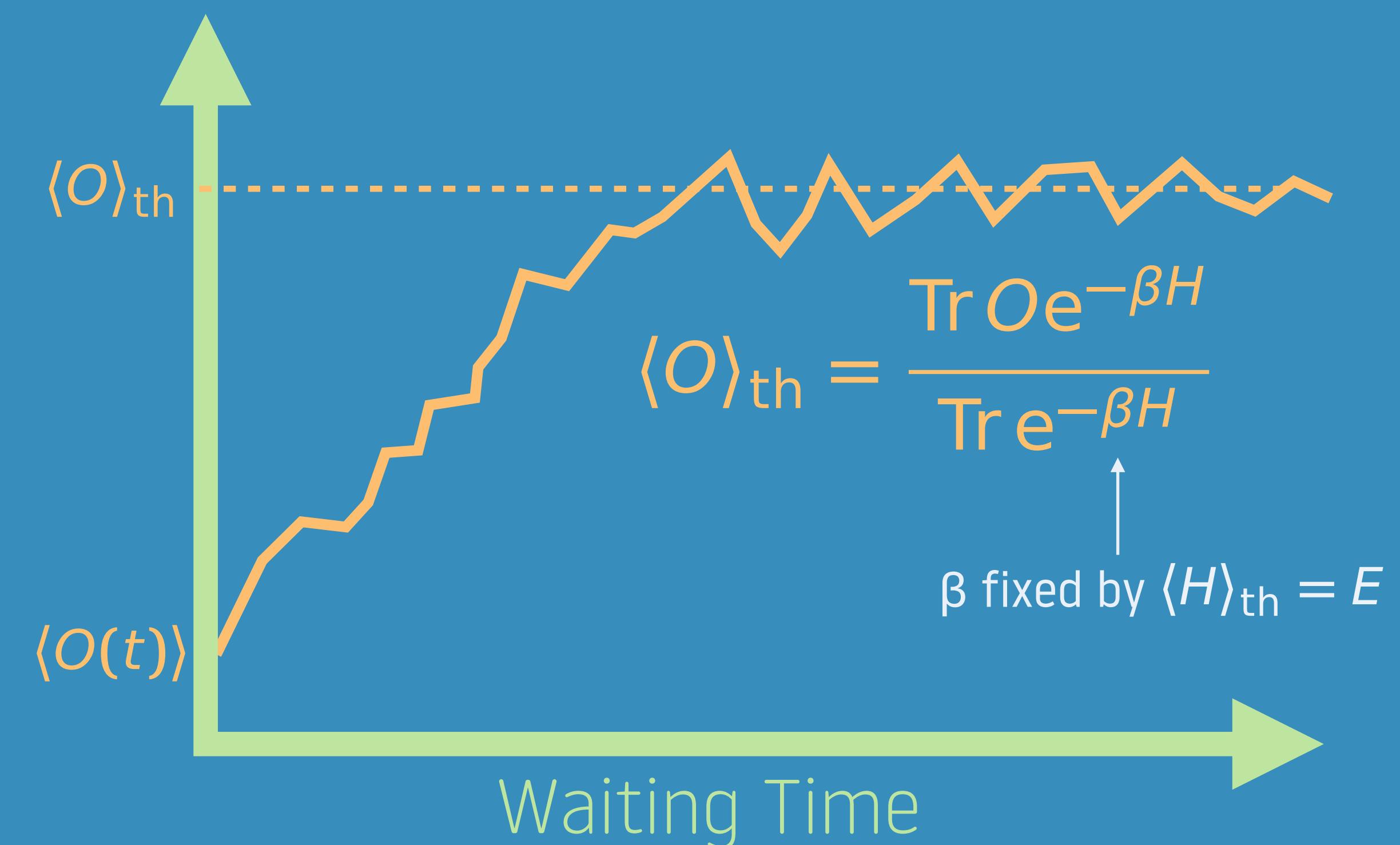
2. Unitary time evolution for  $t > 0$

$$|\Psi(t)\rangle = \sum_a e^{-iE_a t} \langle \psi_a | \Psi_0 \rangle |\psi_a\rangle$$

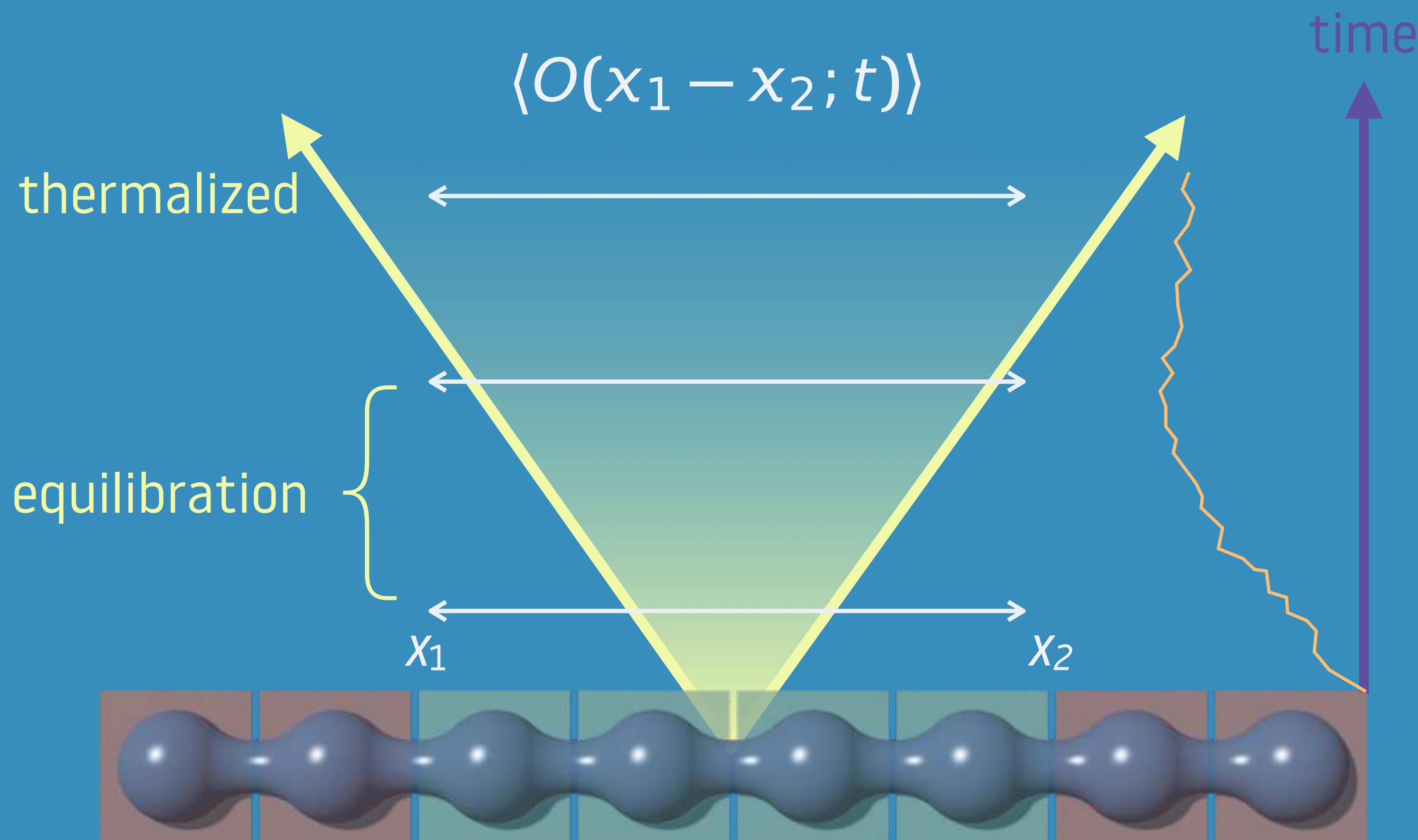
$$H |\psi_a\rangle = E_a |\psi_a\rangle$$

3. Measure time-dependent expectation values of local observables

$$\langle O(t) \rangle = \langle \Psi(t) | O | \Psi(t) \rangle$$

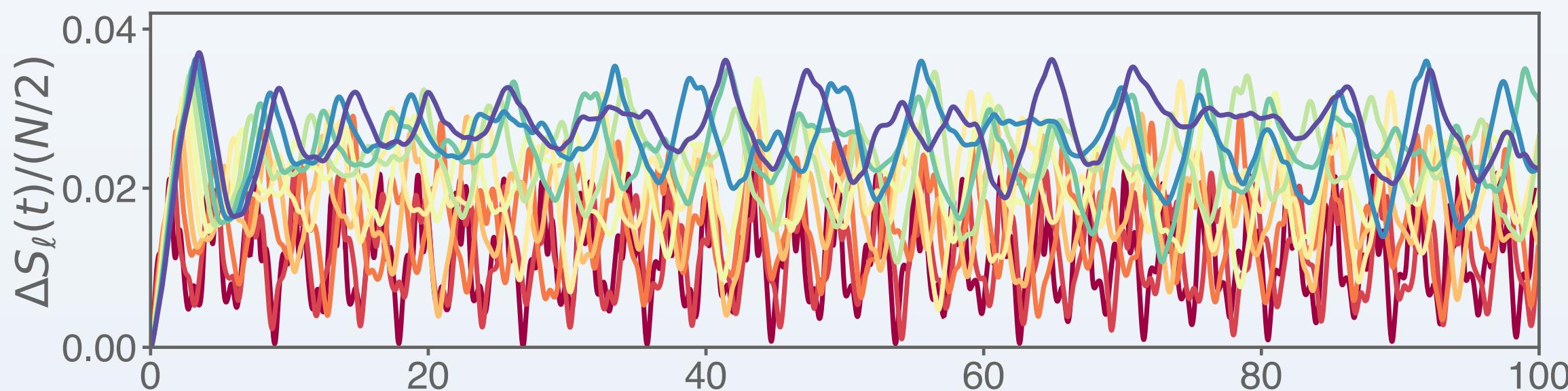
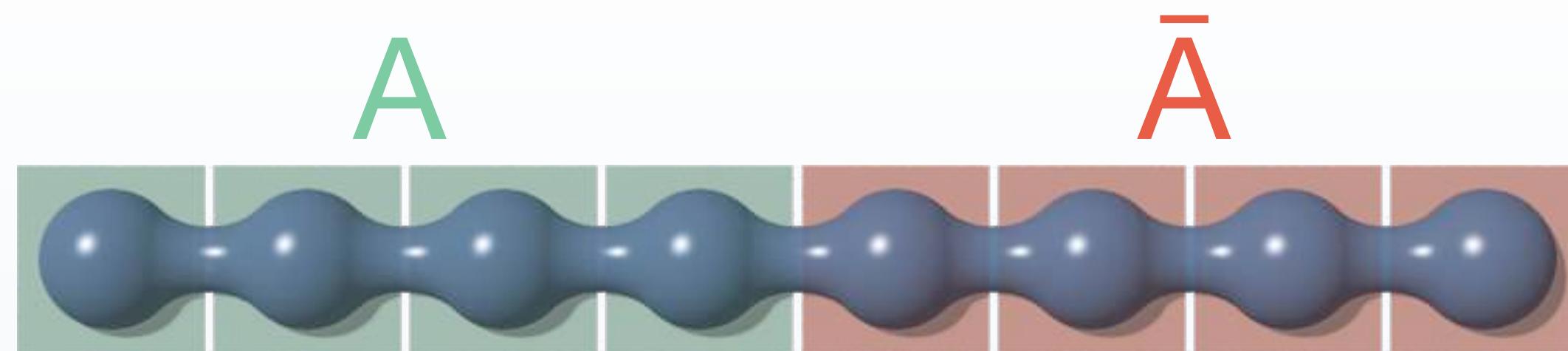


# *Thermalization of Local Observables*



# Entanglement and Entropy

quantifying uncertainty in many-body systems

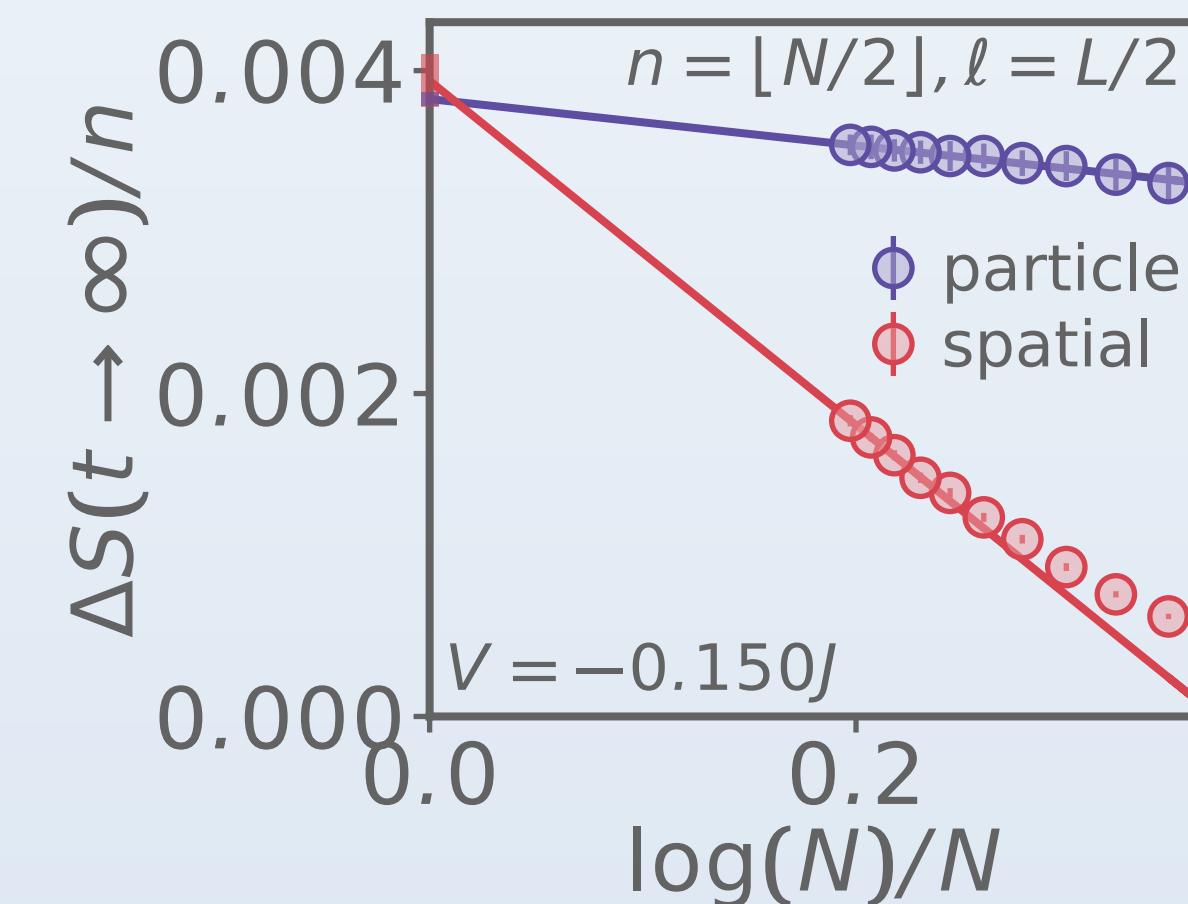


# Finite Size Scaling & Equivalence

Steady-state entanglement for different bipartitions

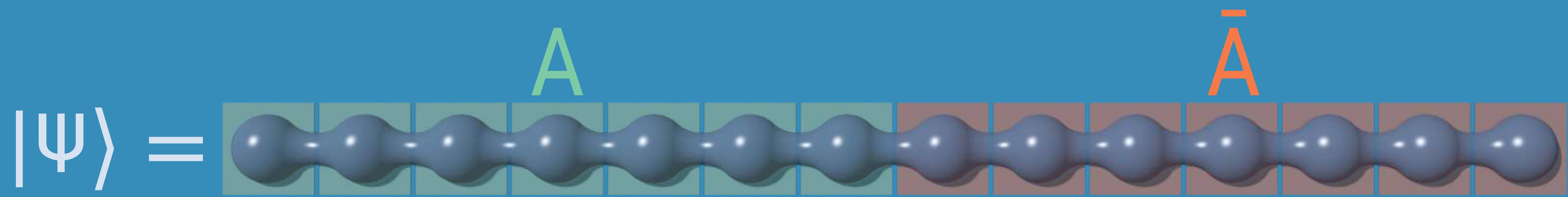
# Entanglement Dynamics

Evolution & Growth after a Quench



# Entanglement

quantum information that is encoded  
non-locally in the joint state of a system

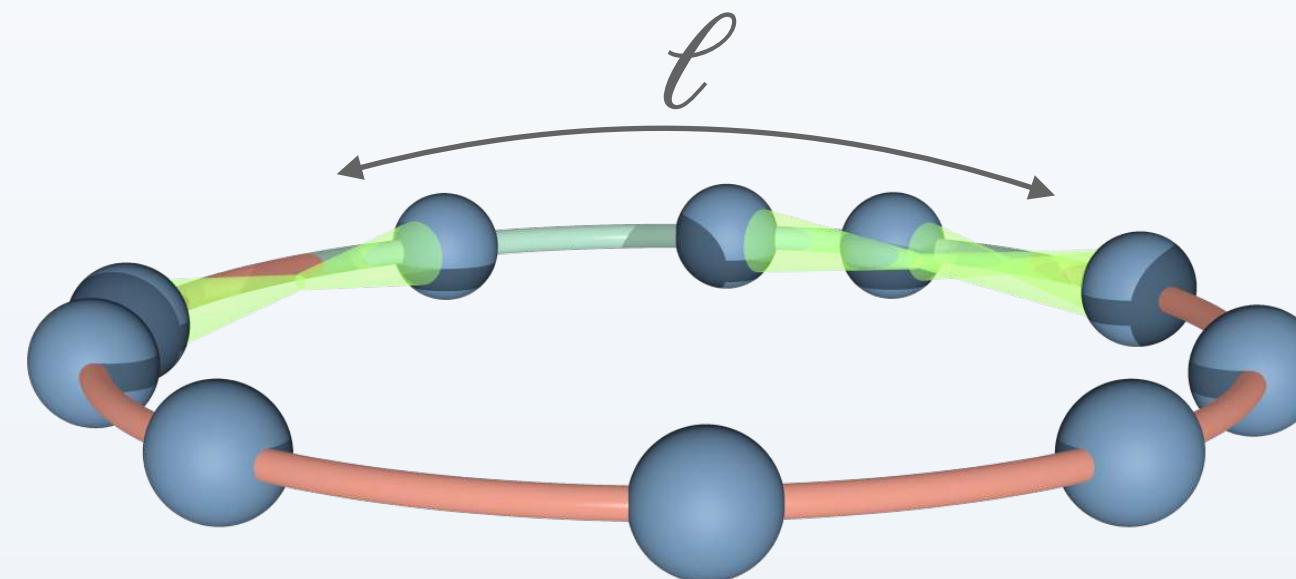


$$|\Psi\rangle = |\alpha\rangle_A \otimes |\beta\rangle_{\bar{A}}$$

$$\rho_A = \text{Tr}_{\bar{A}} |\Psi\rangle \langle \Psi| \rightarrow S = -\text{Tr} \rho_A \ln \rho_A$$

# Entanglement: Mode vs. Particle Bipartition

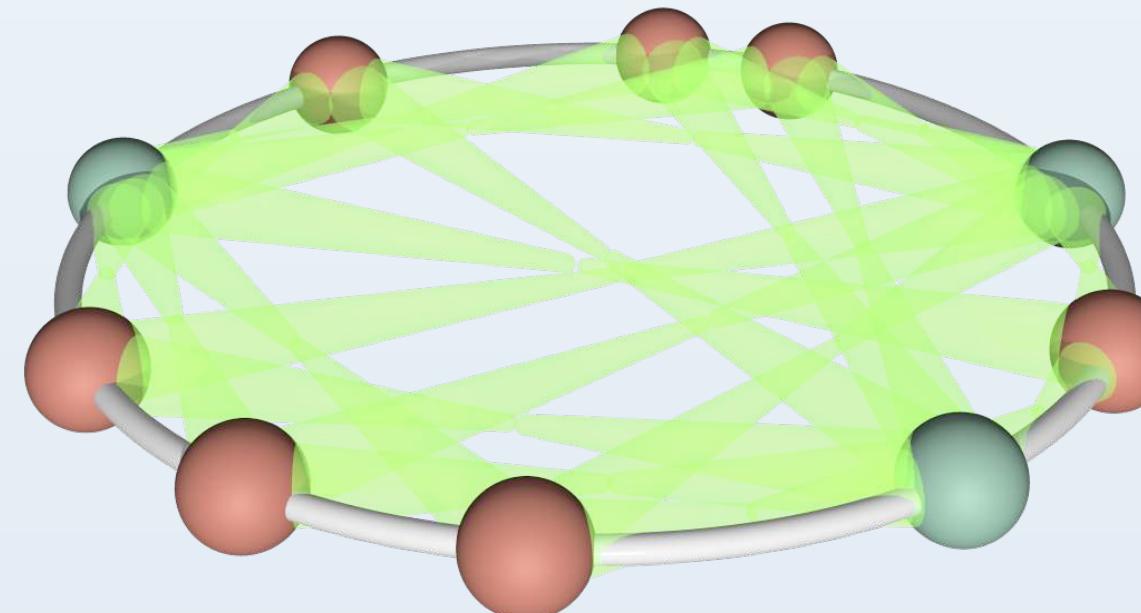
**spatial bipartition:**



constructed from  
the **Fock space** of  
single particle  
modes

$$|\Psi\rangle = \sum_{n_A, n_{\bar{A}}} C_{n_A, n_{\bar{A}}} |n_A\rangle \otimes |n_{\bar{A}}\rangle$$
$$\rho_A \rightarrow S(\rho_A)$$

**particle bipartition:**



reduced density  
matrix is the  
**n-body RDM**

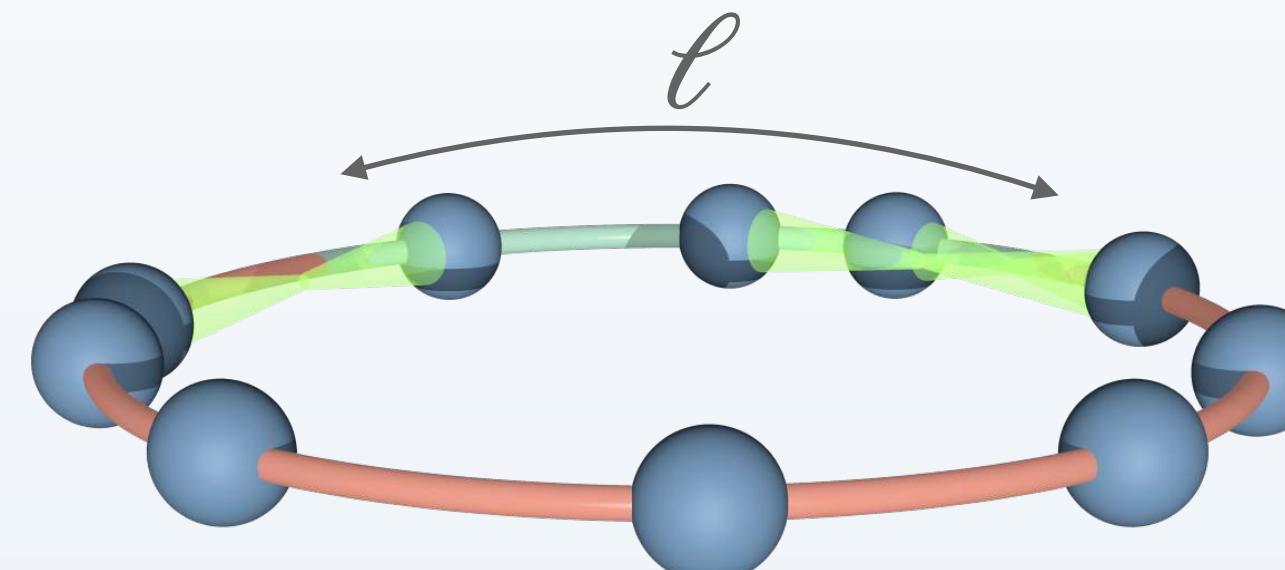
$$n_A = n$$

$$n_{\bar{A}} = N - n$$

$$|\Psi\rangle = |i_1, \dots, i_N\rangle$$
$$\rho_n \rightarrow S(n)$$

# Entanglement: Mode vs. Particle Bipartition

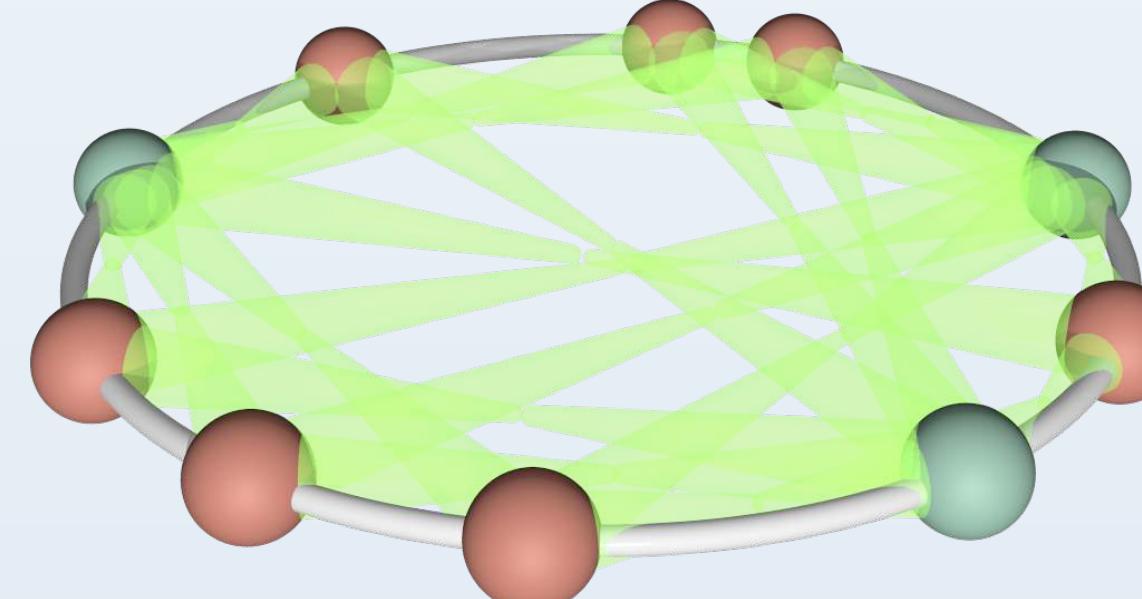
spatial bipartition:



trace out spatial modes in  $\bar{A}$

$$S(\ell) = -\text{Tr} \rho_{\bar{A}} \ln \rho_{\bar{A}}$$

particle bipartition:



trace out positions of  $n$ - particles

$$n_A = n$$

$$n_{\bar{A}} = N - n$$

M. Haque, O. S. Zozulya, and K. Schoutens, J. Phys. A 42, 504012 (2009)

1d fermionic critical systems

$$S(\ell, L) \approx \frac{c}{3} \ln \left[ \frac{L}{\pi} \sin \left( \frac{\pi \ell}{L} \right) \right] + c_1$$

B. Q. Jin and V. E. Korepin, J Stat. Phys 116, 79 (2004)  
P. Calabrese and J. Cardy, JSTAT P06002 (2004)  
J. Cardy and P. Calabrese, JSTAT P04023. (2010)

$$\rho_n^{i_1, \dots, i_n; j_1, \dots, j_n} = \sum_{i_{n+1}, \dots, i_N} \Psi^*(i_1, \dots, i_n, i_{n+1}, \dots, i_N)$$

$$\times \Psi(j_1, \dots, j_n, i_{n+1}, \dots, i_N)$$

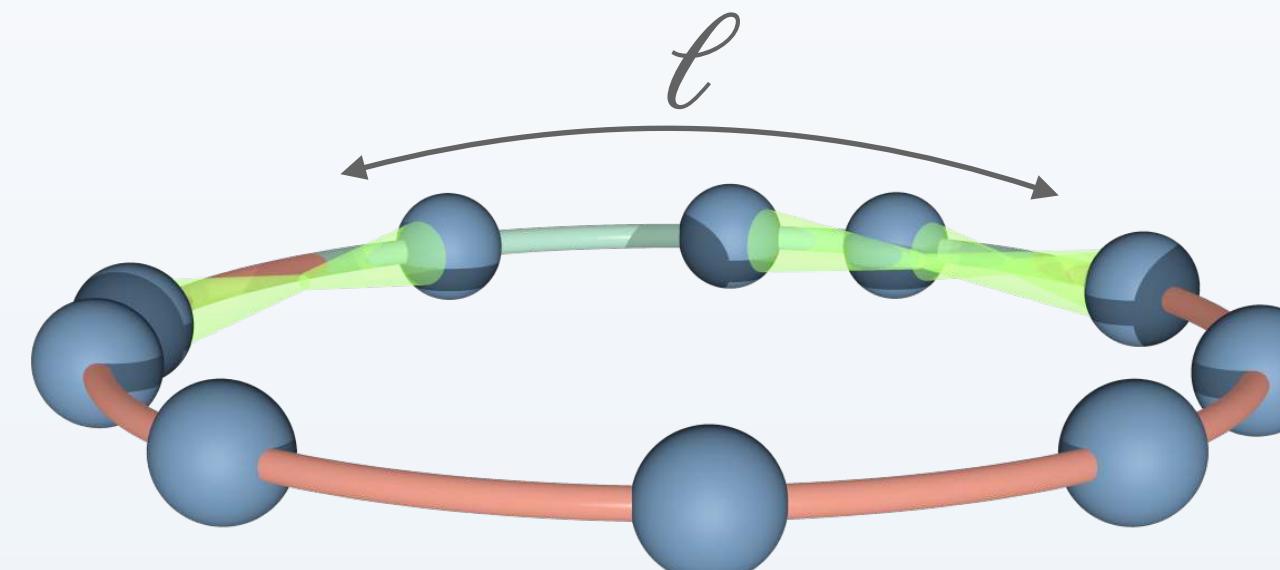
$$S(n, N) = \ln \binom{N}{n} + \dots$$

Slater determinant has equal eigenvalues

M. Thamm, H. Radhakrishnan, H. Barghathi, B. Rosenow, A.D. arXiv:2206.11301  
H. Barghathi, E. Casiano-Diaz, and AD, JSTAT. 2017, 083108 (2017)  
C. Herdman and A.D., Phys. Rev. B, 91, 184507 (2015)

# Entanglement: Mode vs. Particle Bipartition

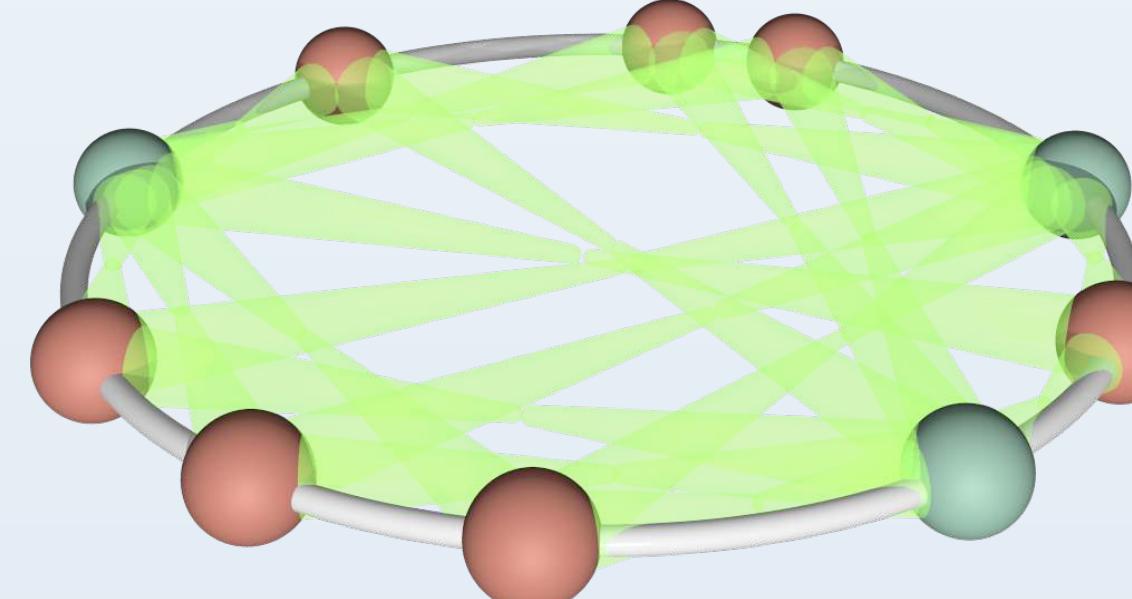
**spatial bipartition:**



trace out spatial  
modes in  $\bar{A}$

$$S(\ell) = -\text{Tr} \rho_{\bar{A}} \ln \rho_{\bar{A}}$$

**particle bipartition:**



trace out  
positions of  
 $n$ - particles

$$n_A = n$$

$$n_{\bar{A}} = N - n$$

M. Haque, O. S. Zozulya, and K. Schoutens, J. Phys. A 42, 504012 (2009)

**1d bosonic critical systems**

$$S(\ell, L) \approx \frac{c}{3} \ln \left[ \frac{L}{\pi} \sin \left( \frac{\pi \ell}{L} \right) \right] + c_1$$

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P. Calabrese and J. Cardy, JSTAT P06002 (2004)

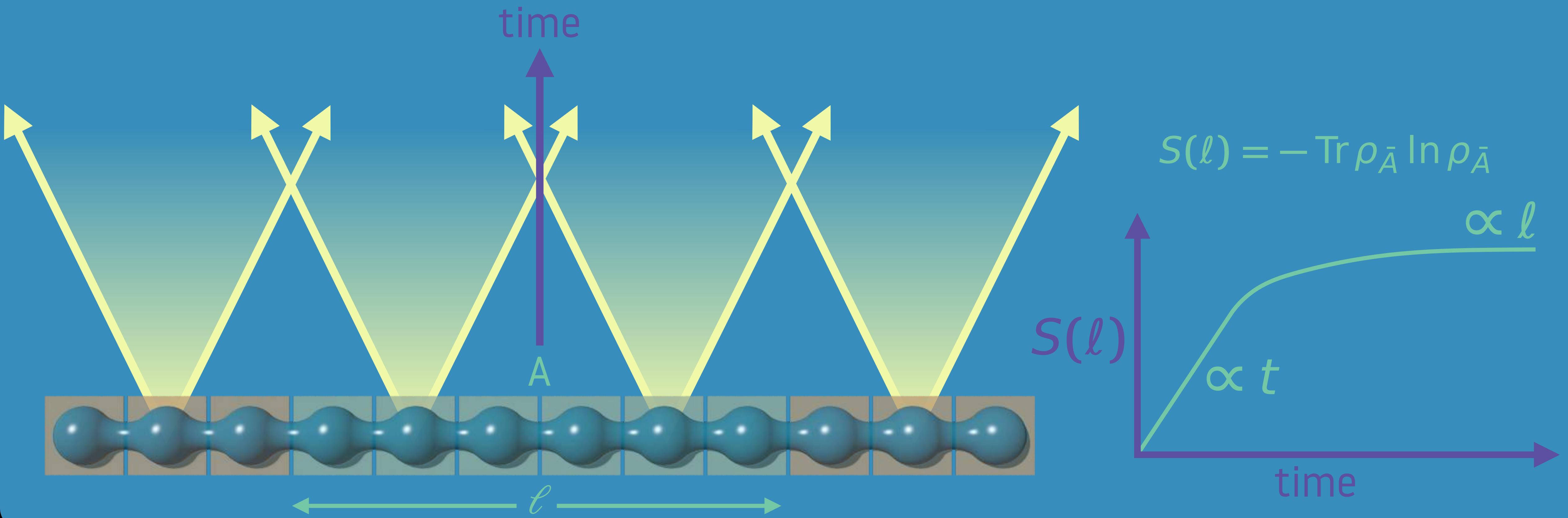
J. Cardy and P. Calabrese, JSTAT P04023. (2010)

$$\rho_n^{i_1, \dots, i_n; j_1, \dots, j_n} = \sum_{i_{n+1}, \dots, i_N} \Psi^*(i_1, \dots, i_n, i_{n+1}, \dots, i_N) \\ \times \Psi(j_1, \dots, j_n, i_{n+1}, \dots, i_N)$$

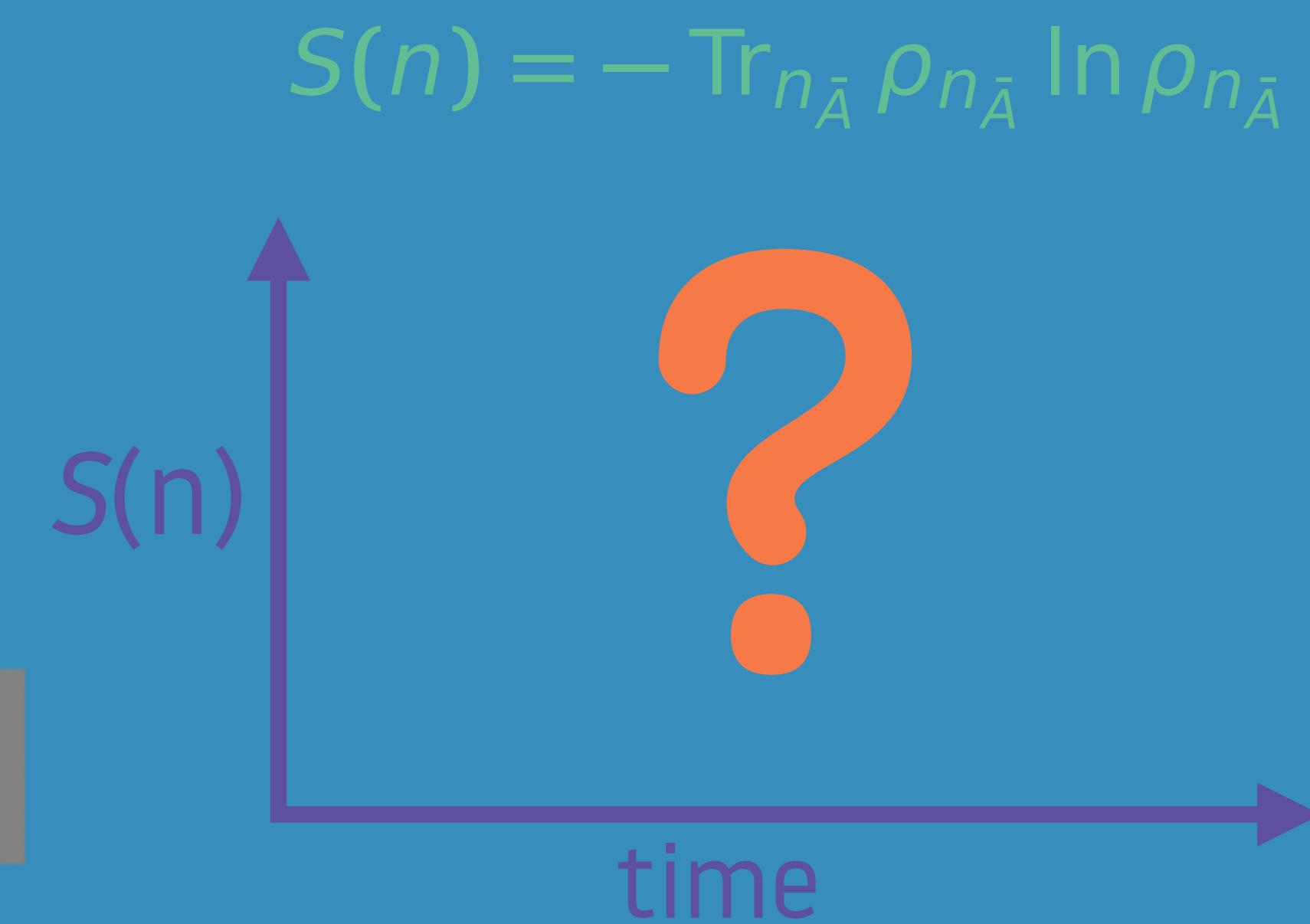
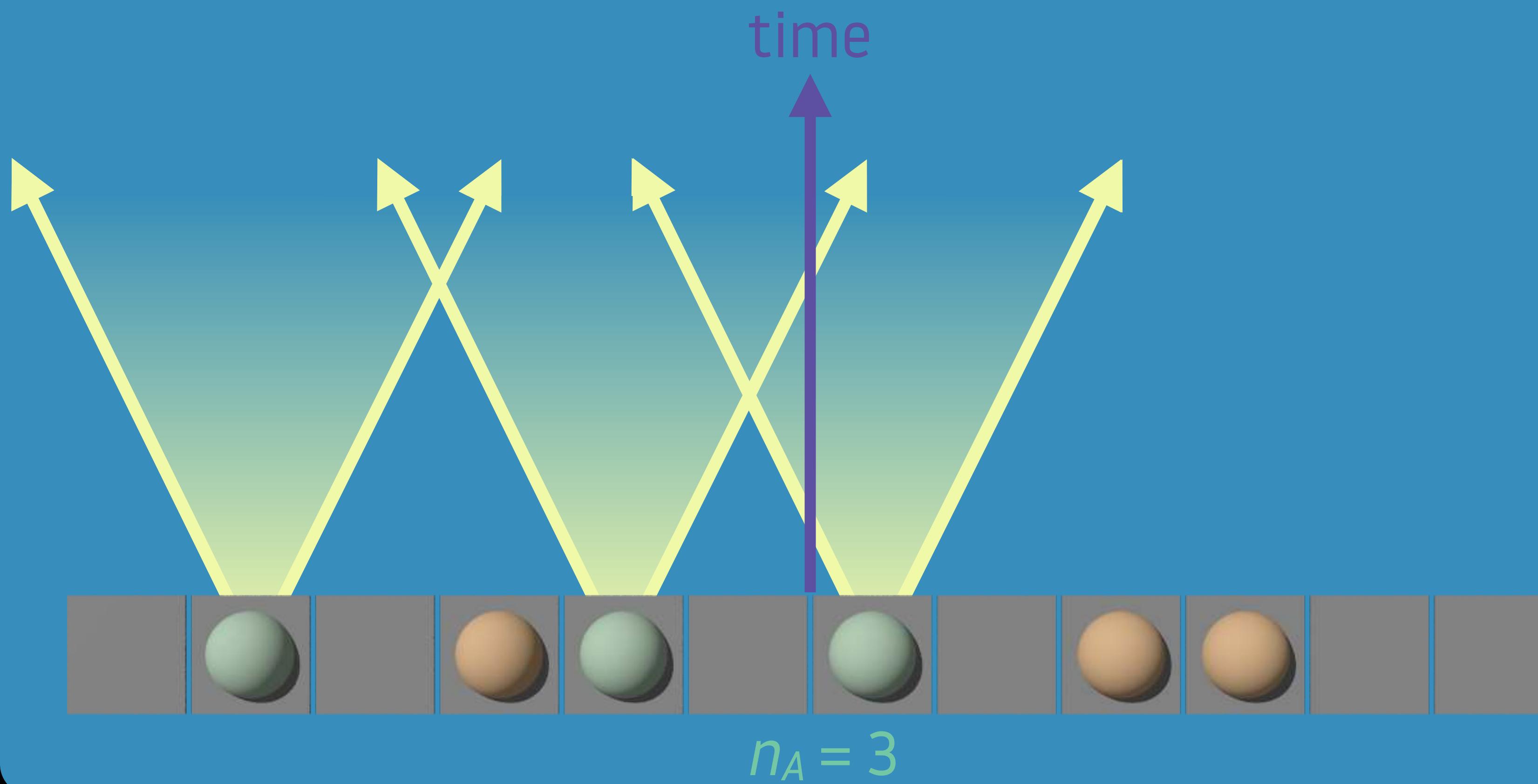
$$S(n) = \frac{n}{K} \ln N + \dots$$

M. Thamm, H. Radhakrishnan, H. Barghathi, B. Rosenow, A.D. arXiv:2206.11301  
H. Barghathi, E. Casiano-Diaz, and AD, JSTAT. 2017, 083108 (2017)  
C. Herdman and A.D., Phys. Rev. B, 91, 184507 (2015)

# *Local thermalization from entanglement accumulation*



# *What is the role of n-body correlations during thermalization?*



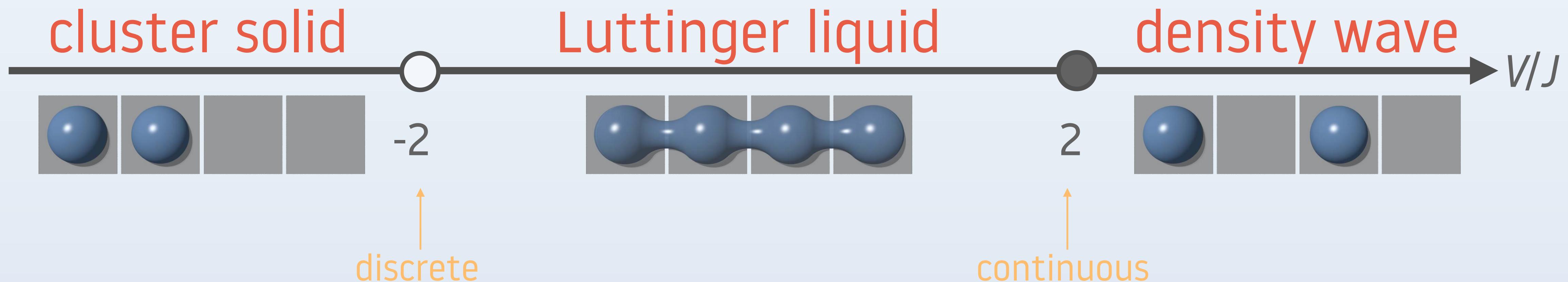
# *Microscopic model & quench*

# 1D Lattice Spinless Fermions

$N$  fermions on  $L$  sites with nearest neighbor interactions  
and periodic/anti-periodic boundary conditions:  $L = 2N$

$$H = -J \sum_{i=1}^L (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + V \sum_{i=1}^L n_i n_{i+1}$$
$$n_i = c_i^\dagger c_i$$
$$\{c_i, c_j^\dagger\} = \delta_{ij}$$

can be mapped to XXZ model  $\Rightarrow$  exactly solvable

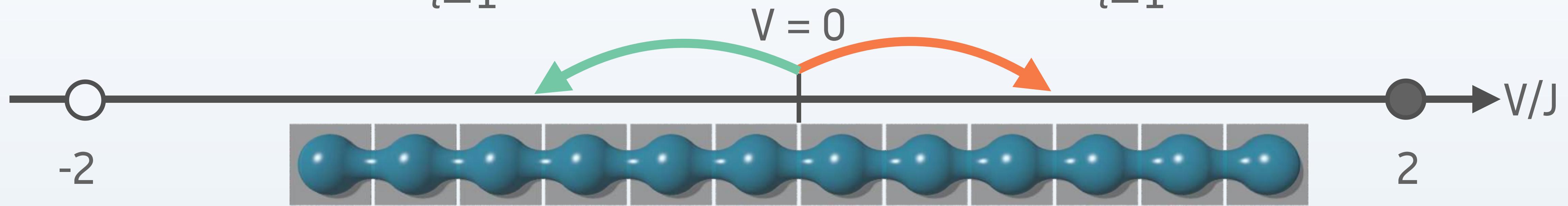


J.D. Cloizeaux, J. Math. Phys. 7 2136 (1966)

J.D. Cloizeaux and M. Gaudin, J. Math. Phys. 7 1384 (1966)

# Interaction Quench

$$H = -J \sum_{i=1}^L (c_i^\dagger c_{i+1} + h.c.) + \Theta(t) V \sum_{i=1}^L n_i n_{i+1}$$



Starting from non-interacting fermions at  $t = 0$ :

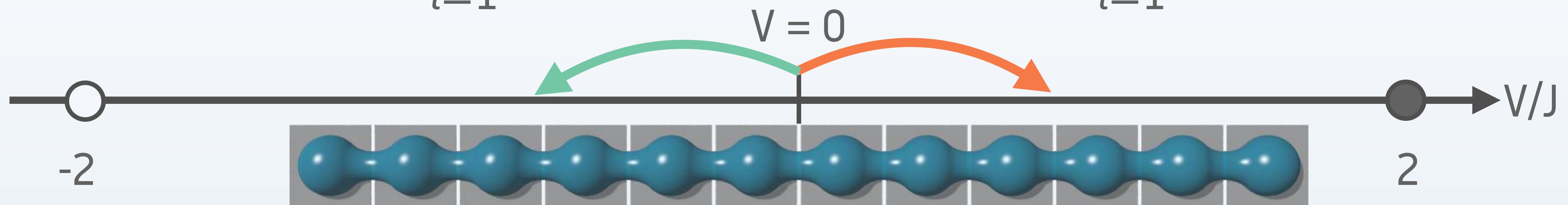
$$|\Psi_0\rangle = \prod_{k \leq k_F} c_k^\dagger |0\rangle$$

half-filling  
 $L = 2N$

turn on repulsive ( $V/J > 0$ ) or attractive ( $V/J < 0$ ) interactions.

# Interaction Quench

$$H = -J \sum_{i=1}^L (c_i^\dagger c_{i+1} + h.c.) + \Theta(t) V \sum_{i=1}^L n_i n_{i+1}$$



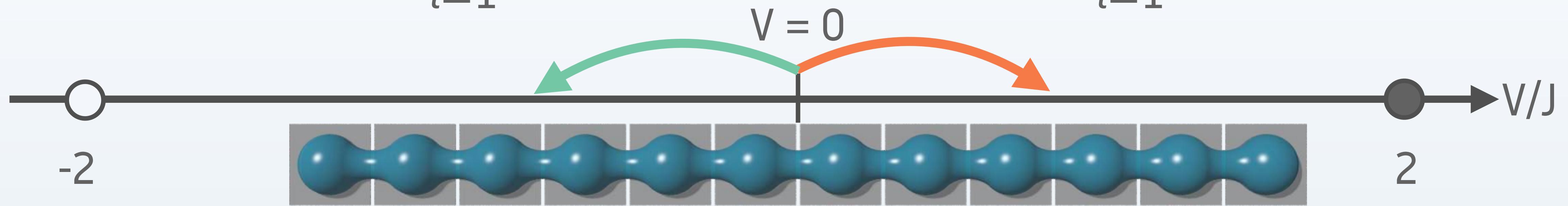
**Exact diagonalization:**

$$H |\psi_a\rangle = E_a |\psi_a\rangle \rightarrow |\Psi(t)\rangle = \sum_a e^{-iE_a t} \langle \psi_a | \Psi_0 \rangle |\psi_a\rangle$$

Exploit all symmetries to consider up to  $L = 26$  sites,  $N = 13$  particles for long times.

# Interaction Quench

$$H = -J \sum_{i=1}^L (c_i^\dagger c_{i+1} + h.c.) + \Theta(t) V \sum_{i=1}^L n_i n_{i+1}$$



## Entanglement Dynamics:

$$S_{\ell|n}(t) = -\text{Tr} \rho_{\ell|n}(t) \ln \rho_{\ell|n}(t)$$

$\rho_{\ell|n}(t) = \text{Tr}_{L-\ell|n} |\Psi(t)\rangle \langle \Psi(t)|$

← spatial ( $\ell$ ) or  
particle bipartition ( $n$ )

interested in growth  $\rightarrow$  subtract off ground state value

$$S_\ell(t=0) = a \ln \ell + b \quad \rightarrow$$

$$\Delta S_\ell(t) = S_\ell(t) - S_\ell(t=0)$$

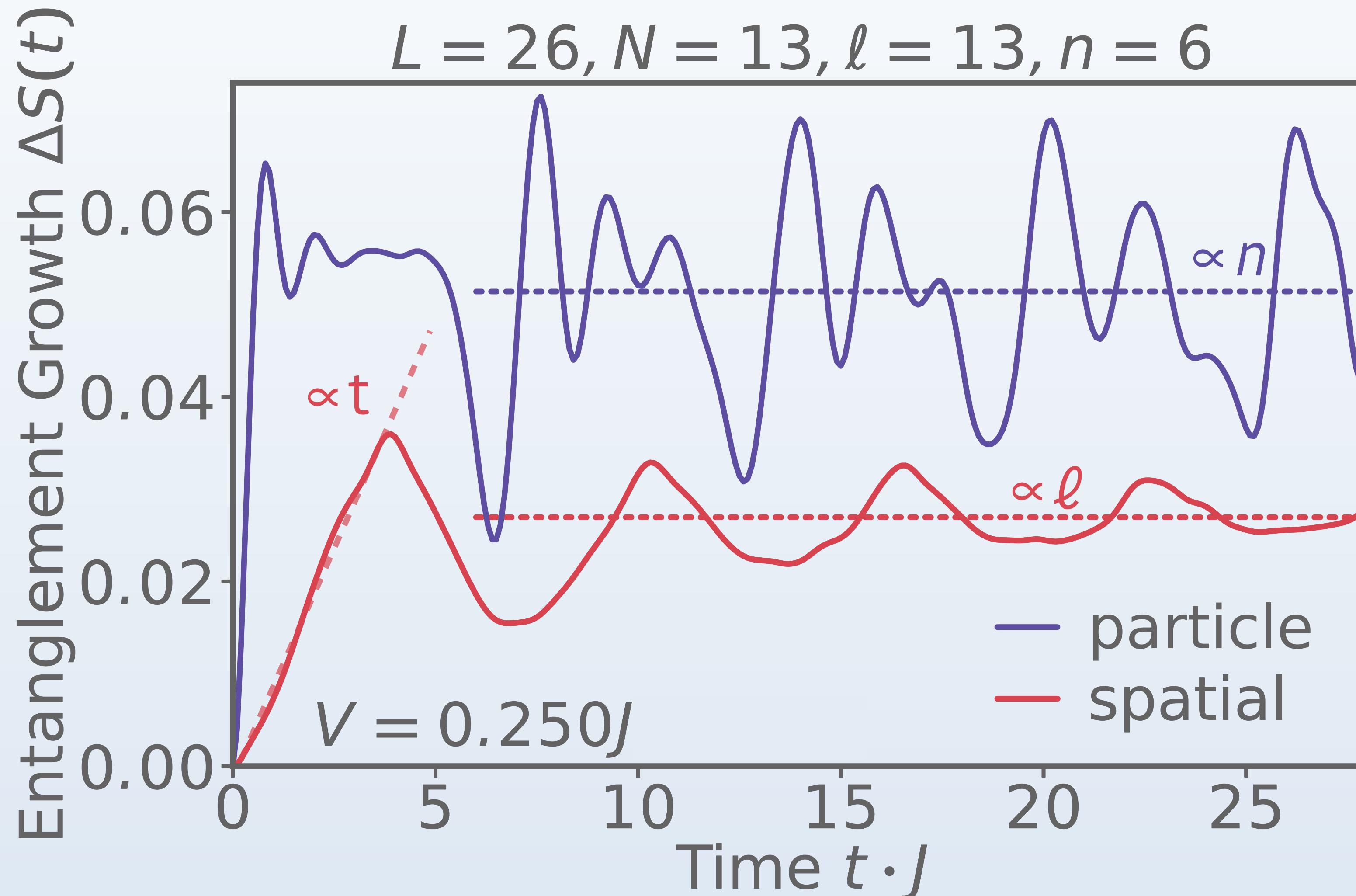
# *Exact diagonalization results*

# Time Evolution of Entanglement Density

Searching for the pre-factor of the volume-law term.

$$S_\ell(t) \sim \begin{cases} t & ; vt < \ell/2 \\ \ell & ; vt > \ell/2 \end{cases}$$

P. Calabrese and J. Cardy, JSTAT. P04010 (2005).



subtract  $t = 0$  ground state entanglement

**spatial** bipartition

$$\Delta S_\ell(t) = S_\ell(t) - S_\ell(t = 0)$$

**particle** bipartition

$$\frac{\Delta S_n(t)}{n} = \frac{1}{2} \left[ \frac{S_n(t) - \ln \binom{N}{n}}{n} + \frac{S_{N-n}(t) - \ln \binom{N}{n}}{N-n} \right]$$

parity of  $N$  is important

# Asymptotic Long Time Results

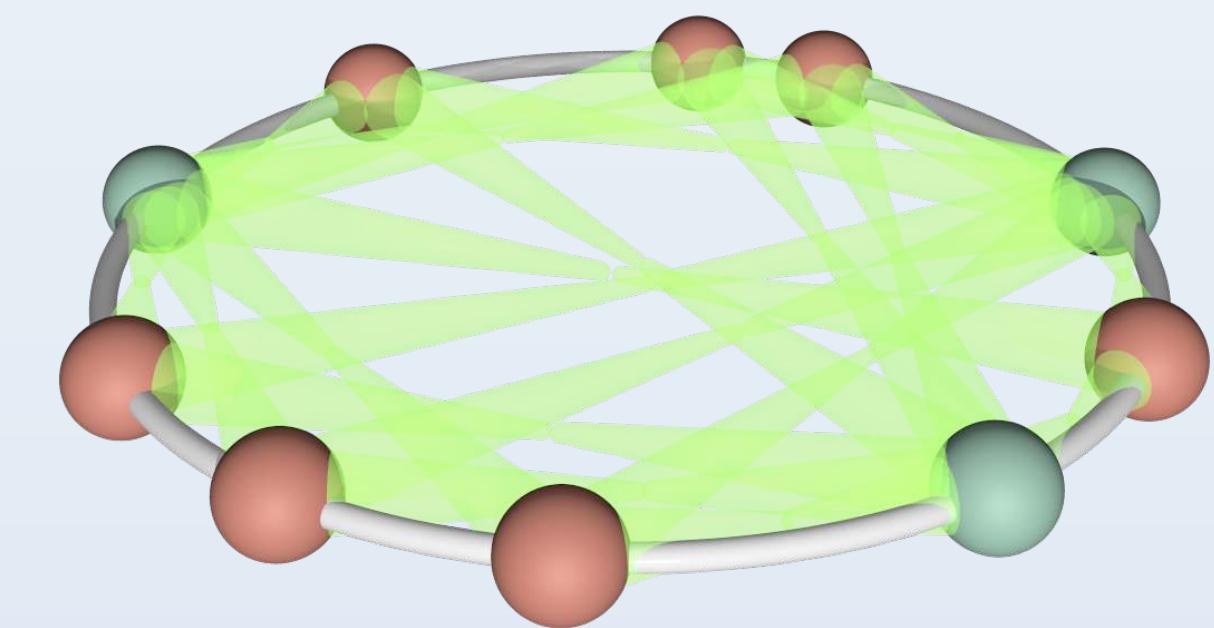
time average exact diagonalization results for  $t = N \rightarrow t_{\max}$  to gain access to saturation value of entanglement (e.g. diagonal entropy)

$$\Delta S(t \rightarrow \infty) \approx \overline{S(t)} - S(t = 0)$$
$$\sim N \cdot \text{const.}$$

M. Rigol, V. Dunjko, & M.T. Olshanii, Nature, 452, 854 (2008)  
L. F. Santos, A. Polkovnikov, & M. Rigol, PRL 107, 040601 (2011)  
A. Polkovnikov, Ann. Phys. 326, 486 (2011)

thermodynamic entropy density  
of emergent statistical ensemble

Does the transformation after a  
quantum quench between entanglement  
and thermodynamic entropy hold for  
particle entanglement?



# Finite Size Scaling of Entanglement Growth

finite size entropy density (8-26 sites)

$$n = \left\lfloor \frac{N}{2} \right\rfloor = \# \text{ particles in subregion, spatial: } \ell = L/2$$

strong finite size effects for  $S_\ell$

L. Piroli, E. Vernier, P. Calabrese, and M. Rigol, PRB 95, 054308 (2017)

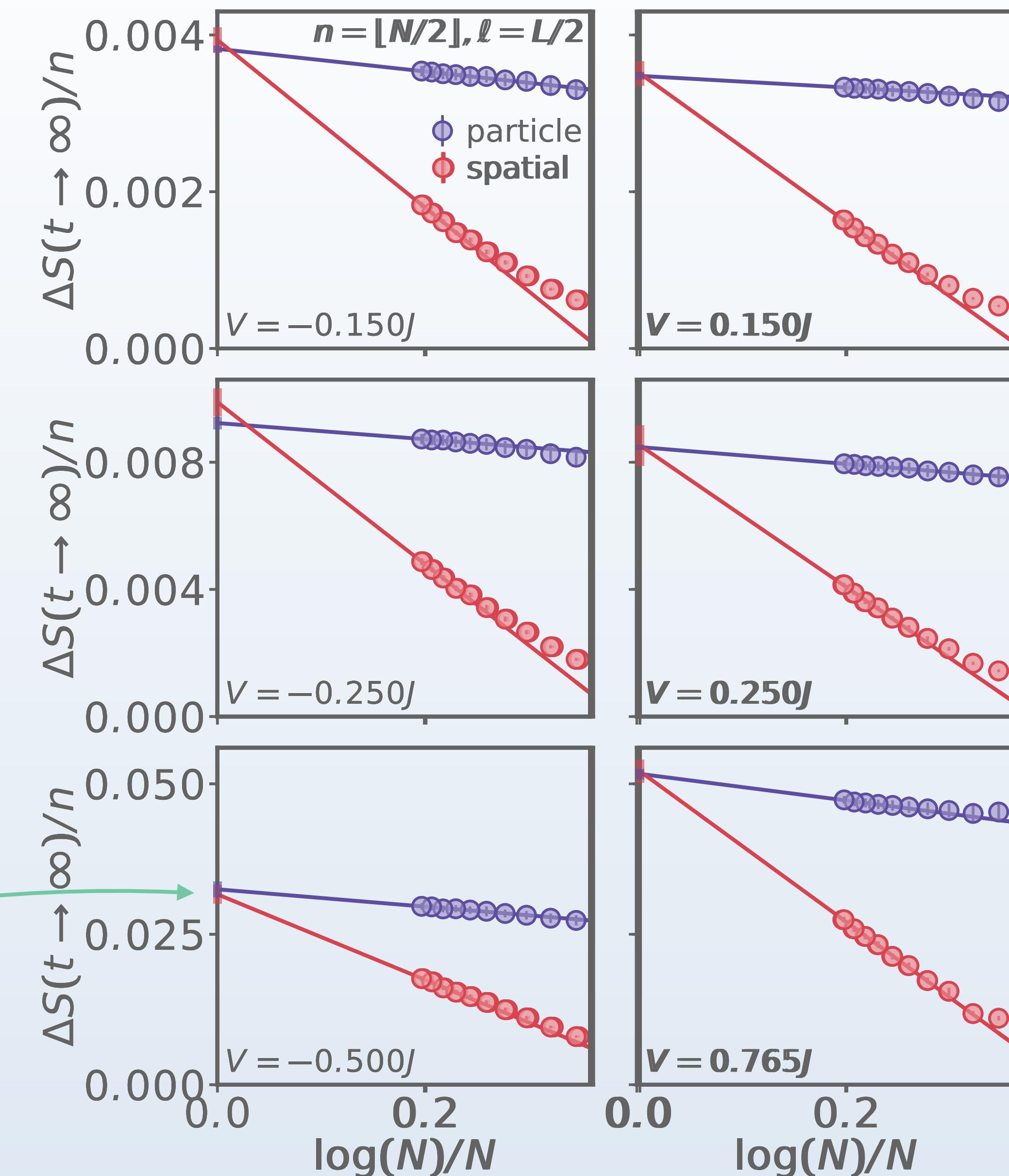
particle entanglement has weak N-dep.

$$@t=0: S_n(t) = \ln \binom{N}{n} \quad S_\ell(t) = a_1 \ln \ell + a_2$$

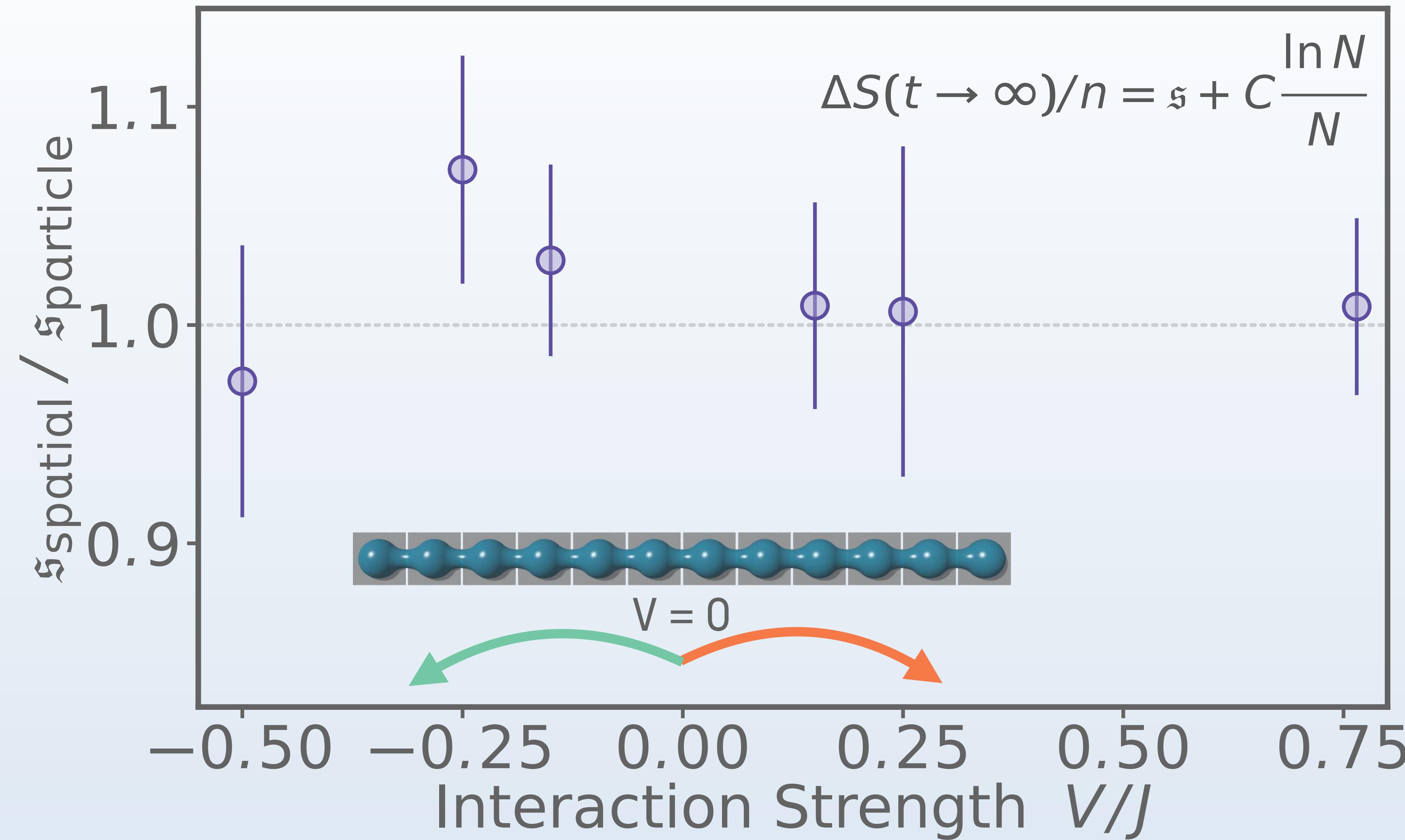
$$\frac{2}{N} \ln \binom{N}{N/2} \simeq 2 \ln 2 - \frac{\ln N}{N} + \mathcal{O}\left(\frac{1}{N}\right)$$

finite size scaling form:

$$\Delta S(t \rightarrow \infty)/n = s + C \frac{\ln N}{N}$$

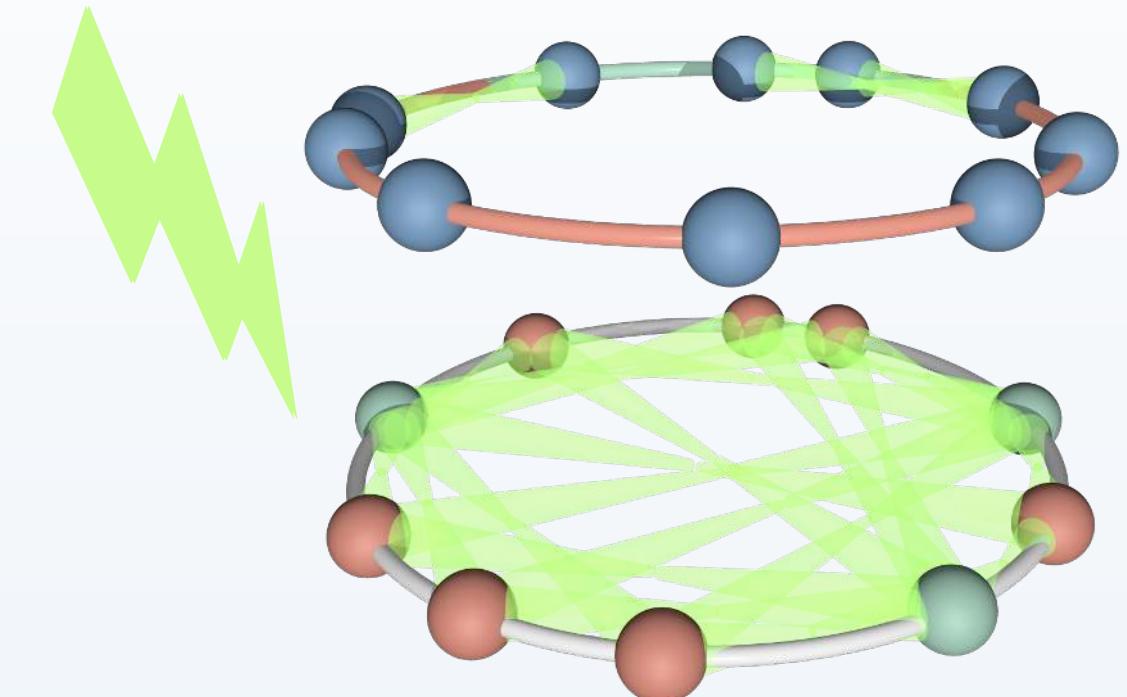


# Equivalence of Entanglement Growth



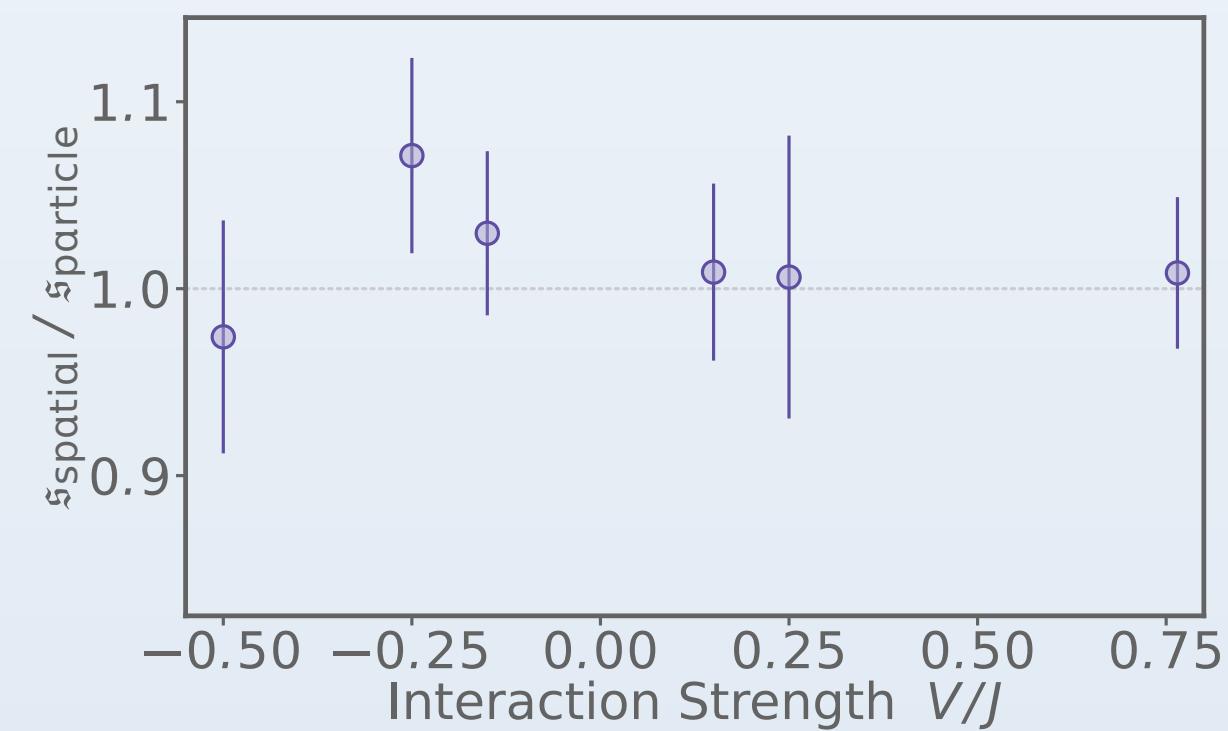
# **Entanglement transformation after quench**

Local observables after quench are controlled by a statistical ensemble with finite thermodynamic entropy.



# **Universality of entanglement conversion**

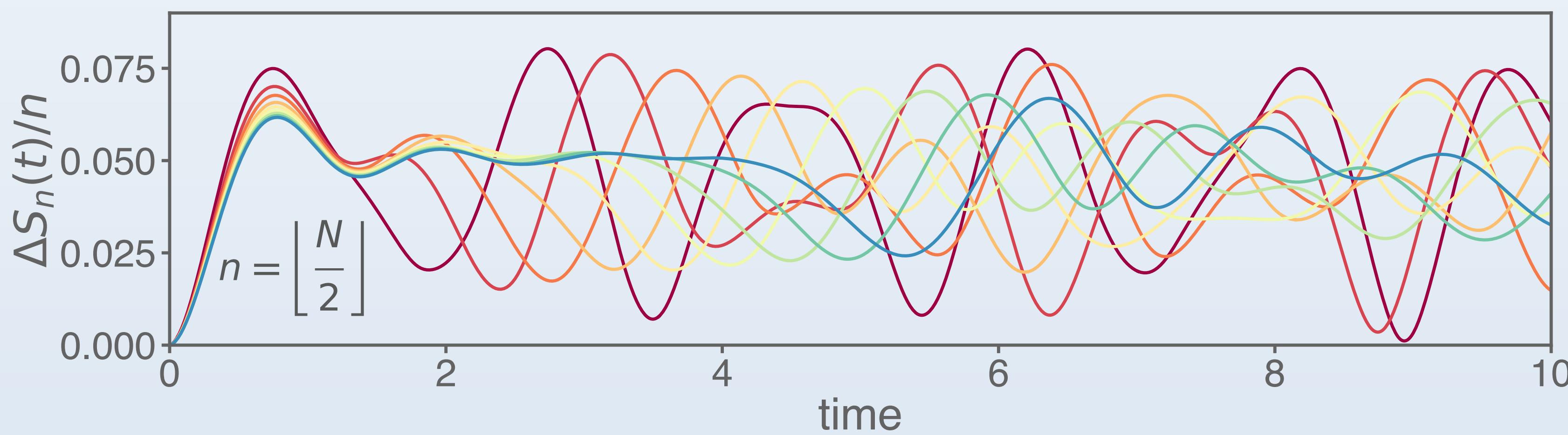
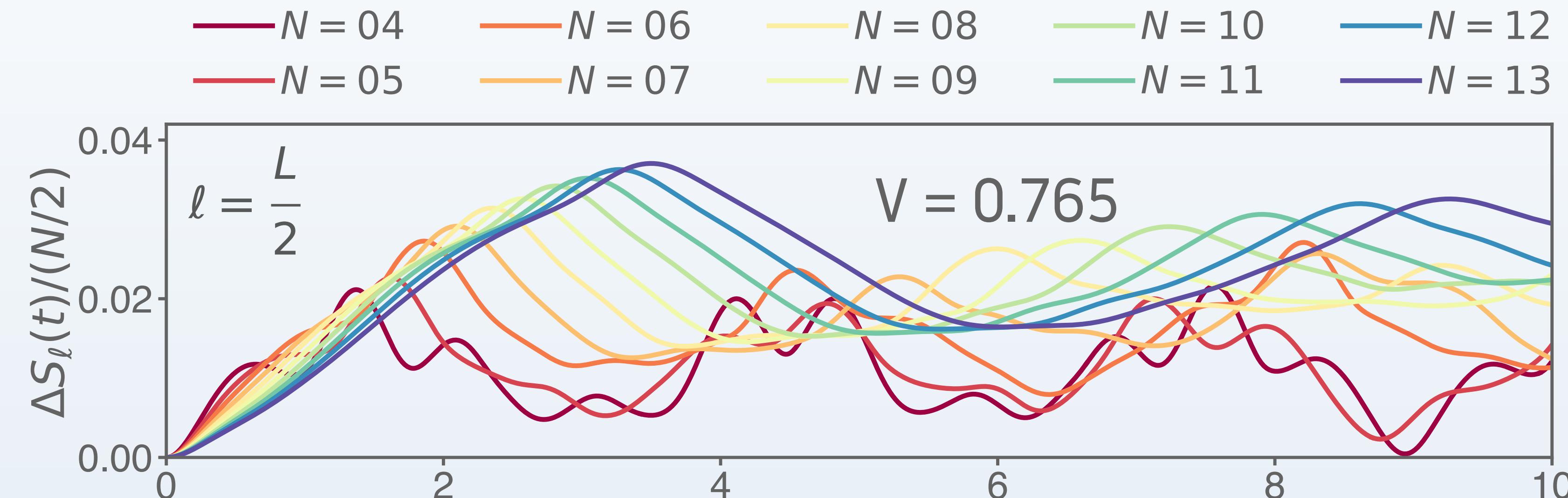
In the thermodynamic limit, the growth of both **spatial** and **particle** entanglement converge to same volume law.



# *Supporting Slides*

# Time Evolution of Entanglement Density

Searching for the pre-factor of the volume-law term.



$$S_\ell(t) \sim \begin{cases} t & ; vt < \ell/2 \\ \ell & ; vt > \ell/2 \end{cases}$$

P. Calabrese and J. Cardy, JSTAT. P04010 (2005).

subtract  $t = 0$  ground state entanglement

**spatial bipartition**

$$\Delta S_\ell(t) = S_\ell(t) - S_\ell(t=0)$$

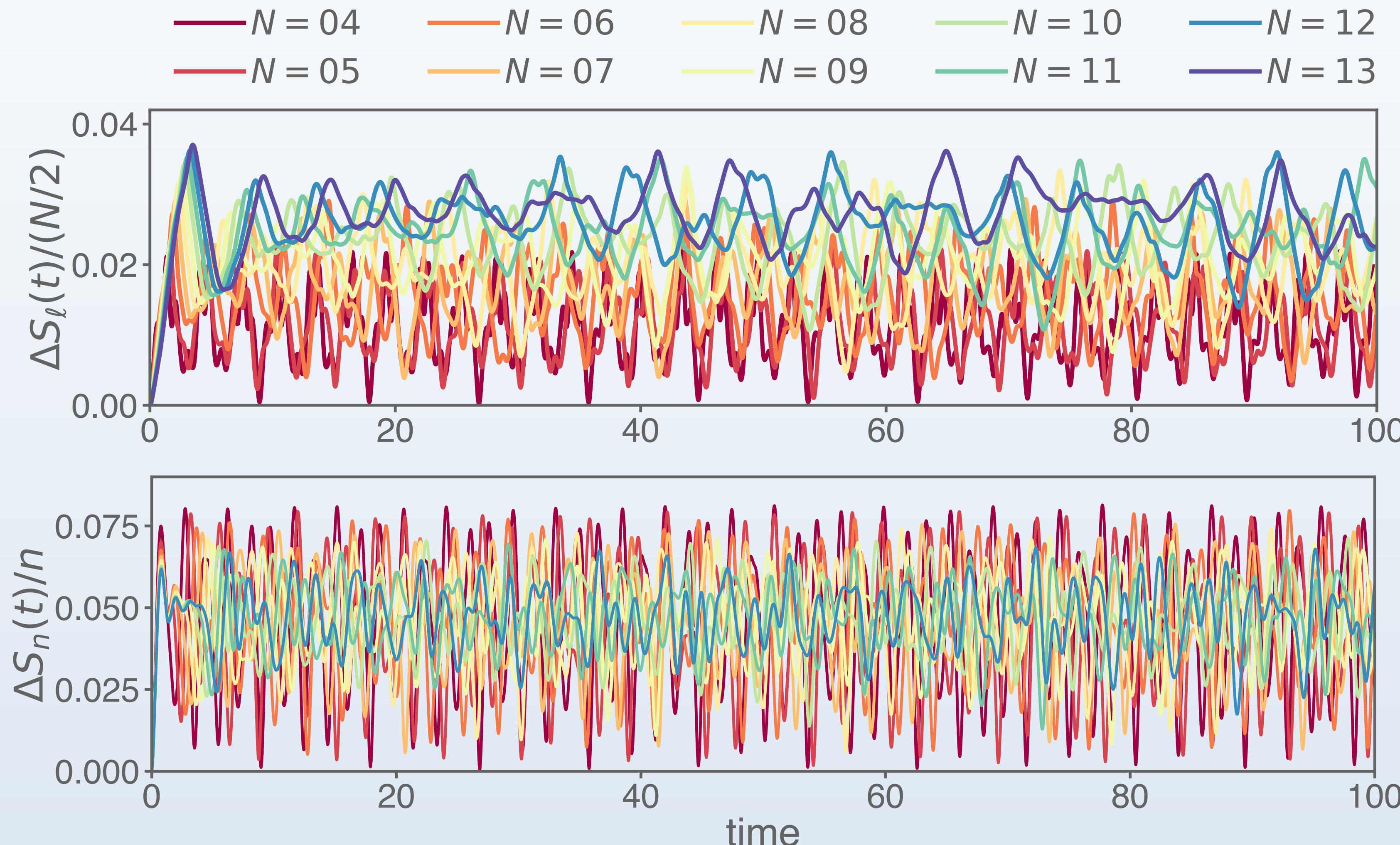
**particle bipartition**

$$\frac{\Delta S_n(t)}{n} = \frac{1}{2} \left[ \frac{S_n(t) - \ln \binom{N}{n}}{n} + \frac{S_{N-n}(t) - \ln \binom{N}{n}}{N-n} \right]$$

parity of  $N$  is important

# Time Evolution of Entanglement Density

Searching for the pre-factor of the volume-law term.



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**spatial** bipartition

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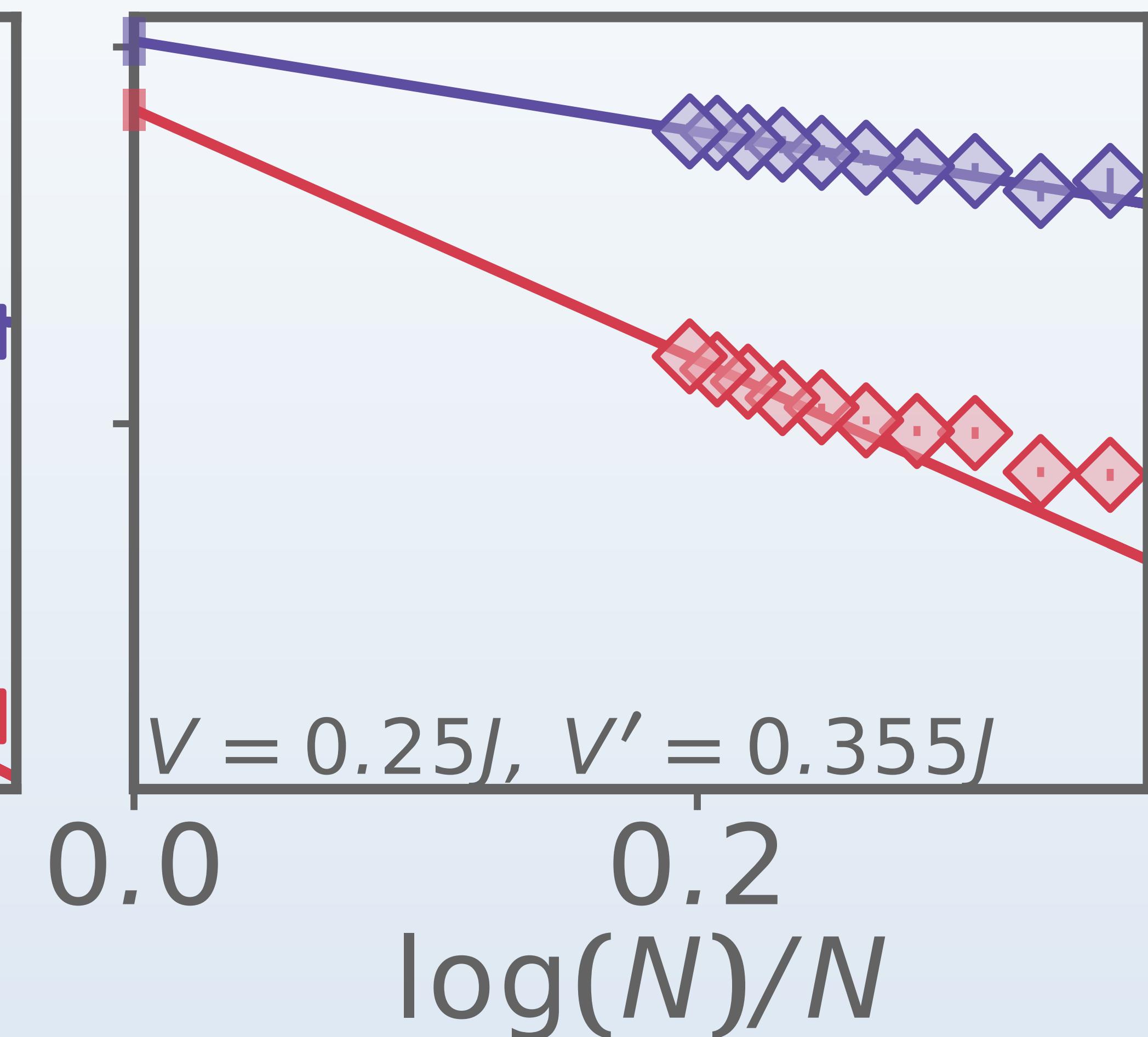
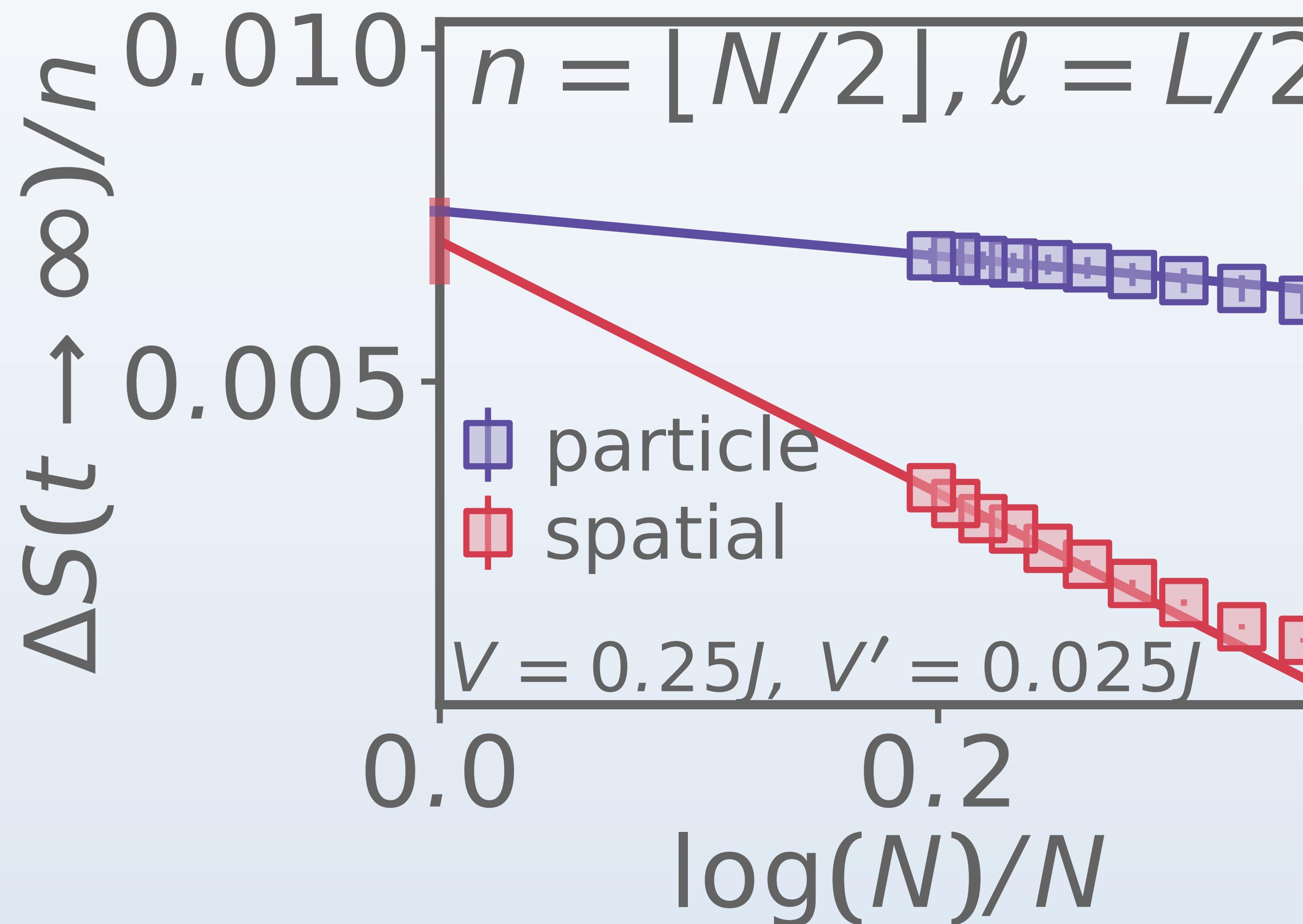
**particle** bipartition

$$\frac{\Delta S_n(t)}{n} = \frac{1}{2} \left[ \frac{S_n(t) - \ln(N)}{n} + \frac{S_{N-n}(t) - \ln(N)}{N-n} \right]$$

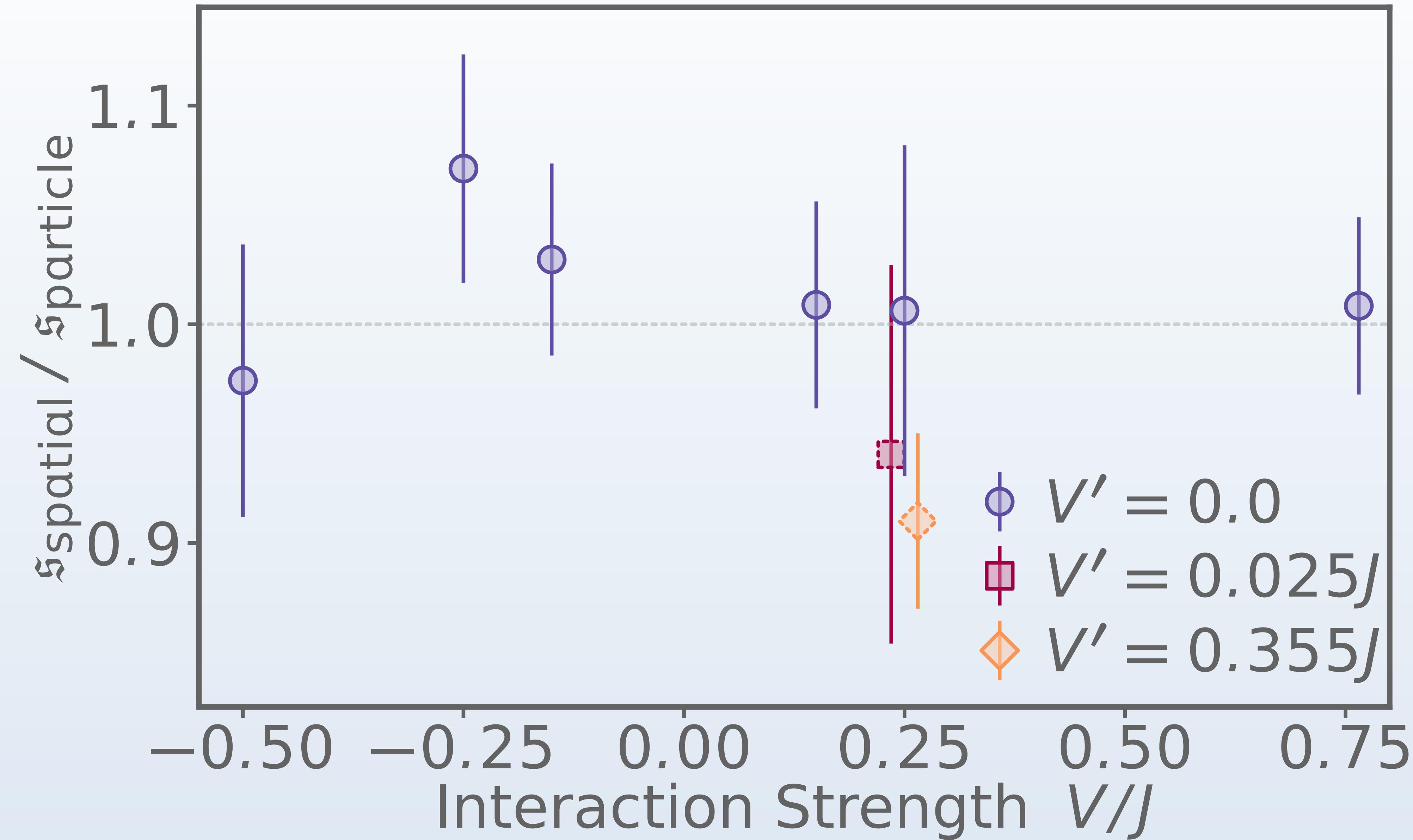
parity of  $N$  is  
important

# Entanglement Growth with Integrability Breaking

$$H = -J \sum_{i=1}^L (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + V(t) \sum_{i=1}^L n_i n_{i+1} + V'(t) \sum_{i=1}^L n_i n_{i+2}$$



# Equivalence with Integrability Breaking



# $n$ -Dependence of Particle Entanglement

