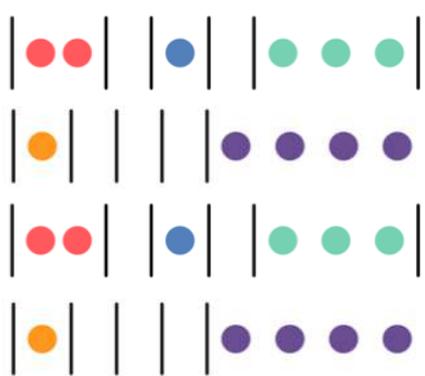


# Balls and Walls: A Compact Unary Coding for Bosonic States

Hatem Barghathi, Caleb Usadi, Micah Beck and Adrian Del Maestro



Phys. Rev. B **105**, L121116 (2022)

<https://github.com/DelMaestroGroup/papers-code-UnaryBosonicBasis>



The University of Vermont



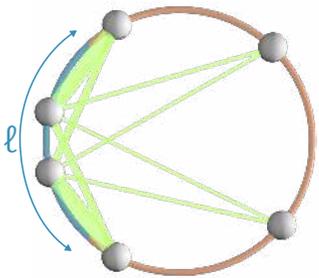
September/12/2022, RPMBT **XXI**, Chapel Hill, NC, USA

DMR-2041995

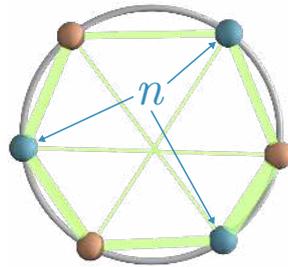
# Quantum Entanglement

## Del Maestro Group

Part of the group research focuses on Quantum Entanglement



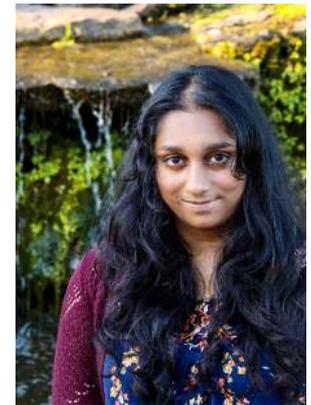
spatially reduced  
 $\rho_\ell$



particle reduced  
 $\rho_n$



Emanuel Casiano-Diaz



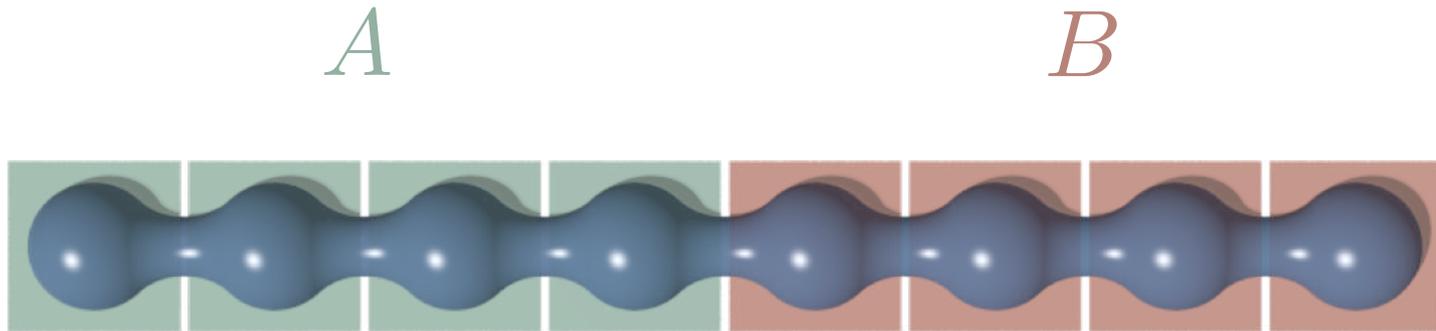
Harini Radhakrishnan

Different degrees of freedom

<https://www.delmaestro.org/adrian/>

# Mode Entanglement

Quantum information shared between different partitions of a state



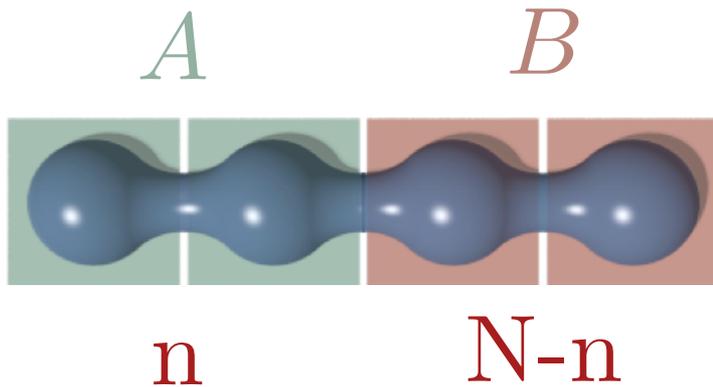
Full density matrix  $\rho = |\Psi\rangle\langle\Psi|$

$$\rho_A = \text{Tr}_B \rho$$

$$S(\rho_A) = -\text{Tr} \rho_A \ln \rho_A$$

von Neumann entropy

# Symmetry resolved entanglement



H. M. Wiseman and J. A. Vaccaro,  
PRL 91, 097902 (2003)

$$|\Psi_N\rangle = \sum_n C_n |\psi_{n, N-n}\rangle$$

$$|\Psi_N\rangle = \sum_n C_n \sum_{i,j} |\psi_{n,i}\rangle_A \otimes |\psi_{N-n,j}\rangle_B$$

$$\rho_A = \sum_n P_n \rho_{A_n}$$

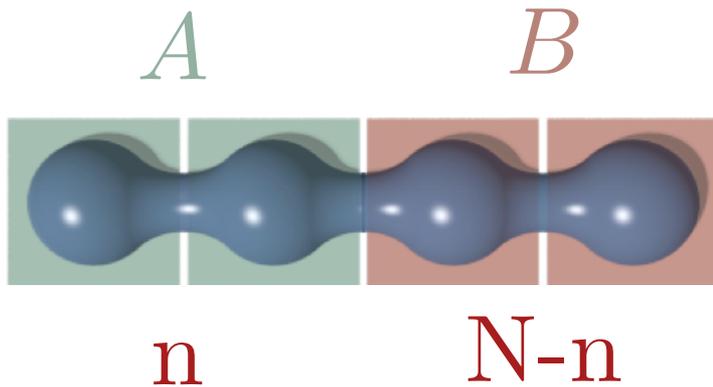
$$\rho_{A_n} = \frac{\Pi_{A_n} \rho_A \Pi_{A_n}}{P_n} \quad P_n = \text{Tr} \Pi_{A_n} \rho_A \Pi_{A_n}$$

**Accessible Entanglement**

$$S^{\text{acc}}(\rho_A) = \sum_{n=0}^N P_n S(\rho_{A_n})$$

**Particle number conservation limits the amount of entanglement that can be physically accessed**

# Symmetry resolved entanglement



H. M. Wiseman and J. A. Vaccaro,  
PRL 91, 097902 (2003)

$$\rho_A = \begin{array}{|c|c|} \hline P_0 \rho_{A_0} & 0 \\ \hline P_1 \rho_{A_1} & 0 \\ \hline 0 & P_n \rho_{A_n} \\ \hline \end{array}$$

## Accessible Entanglement

$$S^{\text{acc}}(\rho_A) = \sum_{n=0}^N P_n S(\rho_{A_n})$$

**Particle number conservation limits the amount of entanglement that can be physically accessed**

# Particle Partition Entanglement

System of  $N$  indistinguishable particles

Full density matrix  $\rho = |\Psi\rangle\langle\Psi|$

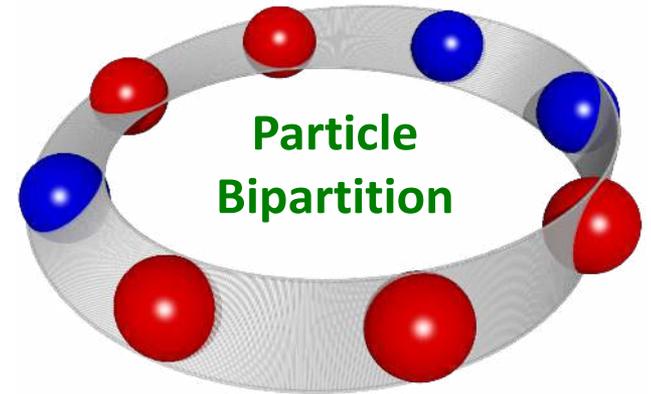
$$N = n_A + n_B = n + (N - n)$$

$$\rho_A = \text{Tr}_B \rho$$

$$\rho_A^{\{i_k\}_1^n, \{j_k\}_1^n} = \sum_{\{i_k\}_{n+1}^N} \Psi^\dagger(\{i_k\}_1^n, \{i_k\}_{n+1}^N) \Psi(\{j_k\}_1^n, \{i_k\}_{n+1}^N)$$

$$S(\rho_A) = -\text{Tr} \rho_A \ln \rho_A$$

von Neumann entropy



# Computational Tools

## Fermions

❖ Exact diagonalization of interacting fermions including time after a quantum quench

❖ Generalization of Correlation Matrix method targeting Symmetry resolved entanglement.

❖ AFQMC: Finite temperature auxiliary field quantum Monte Carlo in the canonical ensemble.



❖ DMRG using ITensor library.



Tong Shen



Brenda Rubenstein



Bernd Rosenow



Matthias\_Thamm



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# Computational Tools

## Fermions

- ❖ Exact diagonalization of interacting fermions including time after a quantum quench

$$L = 28$$

$$N = 14$$

$$n = 7$$



5-7 days

$$O\left(\frac{1}{L^2}\right)$$

>10 years

- ❖ AFQMC: Finite temperature auxiliary field quantum Monte Carlo in the canonical ensemble.



- ❖ DMRG using ITensor library.



Tong Shen



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Matthias\_Thamm



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# Computational Tools

## Fermions

- ❖ Exact diagonalization of interacting fermions including time after a quantum quench

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Tong Shen

Brenda Rubenstein

Bernd Rosenow

Matthias\_Thamm

Finite temperature auxiliary field quantum Monte Carlo in the canonical ensemble

Poster sessions



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- ❖ DMRG using ITensor library.

# Computational Tools

## Bosons

- ❖ PIGSFLI: A Path Integral Ground State Monte Carlo Algorithm for Entanglement of Lattice Bosons, [arXiv:2207.11301](https://arxiv.org/abs/2207.11301)

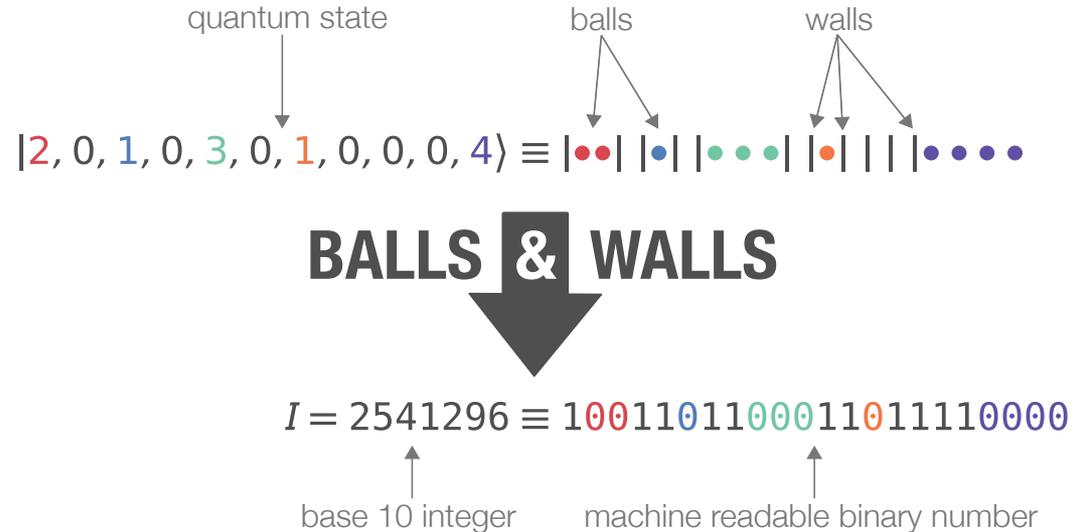


Chris Herdman



Middlebury  
College

- ❖ Exact diagonalization of interacting Bosonic



Phys. Rev. B **105**, L121116 (2022)

# Balls and Walls: A Compact Unary Coding for Bosonic States

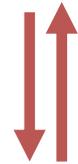
Permanent Ordering (PO)

Chem Phys 401, 208 (2012).

$L = 11$  sites

$N = 11$  bosons

$I_{PO}$



$t \sim \mathcal{O}(L)$

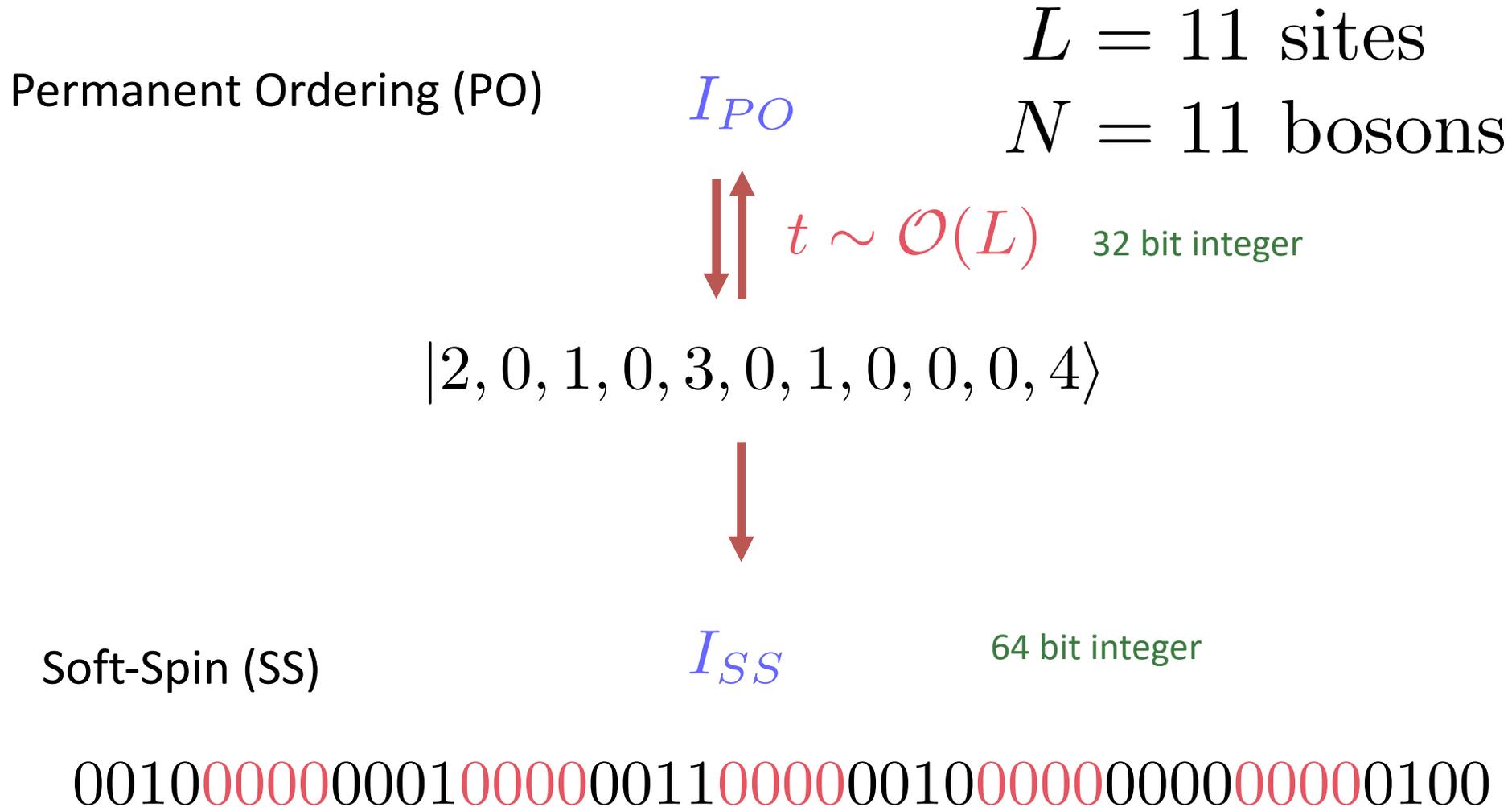
32 bit integer

$|2, 0, 1, 0, 3, 0, 1, 0, 0, 0, 4\rangle$

$m_j = 11, 11, 11, 11, 7, 5, 5, 5, 3, 1, 1.$

$$I_{PO} = 1 + \sum_{j=1}^N \binom{L - m_j}{j} = 287246$$

# Balls and Walls: A Compact Unary Coding for Bosonic States



# Balls and Walls: A Compact Unary Coding for Bosonic States

$N = 11$  bosons

$L = 11$  sites

$$|2, 0, 1, 0, 3, 0, 1, 0, 0, 0, 4\rangle \equiv |\bullet\bullet| | \bullet | | \bullet\bullet\bullet | | \bullet | | | | \bullet\bullet\bullet$$

# Balls and Walls: A Compact Unary Coding for Bosonic States

$N = 11$  bosons

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$$|2, 0, 1, 0, 3, 0, 1, 0, 0, 0, 4\rangle \equiv |\bullet\bullet| | \bullet | | \bullet\bullet\bullet | | \bullet | | | | | \bullet\bullet\bullet\bullet$$

**Balls and Walls**

$$| * * | | * | | * * * | | * | | | | * * * *$$

$$\binom{L + N}{N}$$

Up to  $N$  bosons

# Balls and Walls: A Compact Unary Coding for Bosonic States

$N = 11$  bosons

$L = 11$  sites

$$|2, 0, 1, 0, 3, 0, 1, 0, 0, 0, 4\rangle \equiv |\bullet\bullet| |\bullet| |\bullet\bullet\bullet| |\bullet| || | \bullet\bullet\bullet$$

**Balls and Walls**

$$I_{UB} = 2541296 \equiv 1\color{red}00\color{blue}11\color{green}00\color{orange}11\color{purple}1110000$$

# Balls and Walls: A Compact Unary Coding for Bosonic States

$N = 11$  bosons

$L = 11$  sites

$$|2, 0, 1, 0, 3, 0, 1, 0, 0, 0, 4\rangle \equiv |\bullet\bullet| | \bullet | | \bullet\bullet\bullet | | \bullet | | | | | \bullet\bullet\bullet\bullet$$

**Balls and Walls**

$$I_{UB} = 2541296 \equiv 1\color{red}00\color{blue}11\color{green}00\color{orange}11\color{purple}1110000$$

**Fermions**  $|1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0\rangle$

$0 \equiv$  Occupied

$1 \equiv$  Empty

# Balls and Walls: A Compact Unary Coding for Bosonic States

$N = 11$  bosons

$L = 11$  sites

$$|2, 0, 1, 0, 3, 0, 1, 0, 0, 0, 4\rangle \equiv |\bullet\bullet| | \bullet | | \bullet\bullet\bullet | | \bullet | | | | | \bullet\bullet\bullet\bullet$$

**Balls and Walls**

$$\text{Fermions} \quad \binom{\bar{L}}{N} = \binom{L + N}{N}$$

$$\bar{L} = L + N$$

**Unary coding for bosonic states is as efficient as conventional binary encoding for fermionic states**



# Bose–Hubbard model

**Example:**  $L = N = 3$

**Translational symmetry:**  $[T, H] = 0$ ,  $T a_i^\dagger = a_{i+1}^\dagger T$

Eigenvalues  $e^{i2\pi q/L}$ ,  $q \in \{0, 1, -1\}$

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_0 & 0 & 0 \\ 0 & \mathcal{M}_1 & 0 \\ 0 & 0 & \mathcal{M}_{-1} \end{pmatrix}$$

**Ground state**  $T|\Psi_0\rangle = |\Psi_0\rangle$

$$|\Psi_0\rangle = \sum \alpha_i |\phi_i\rangle$$

$$T|\phi_i\rangle = |\phi_{i+1}\rangle$$

# Bose–Hubbard model

**Example:**  $L = N = 3$

**Ground state**  $|\Psi_0\rangle = \sum_i \alpha_i |\phi_i\rangle$

$$\begin{aligned} &|3, 0, 0\rangle, \quad |0, 3, 0\rangle, \quad |0, 0, 3\rangle \\ &\quad \rightarrow |\phi_1\rangle = \frac{1}{\sqrt{3}} (|3, 0, 0\rangle + |0, 3, 0\rangle + |0, 0, 3\rangle) \\ &|2, 1, 0\rangle, \quad |0, 2, 1\rangle, \quad |1, 0, 2\rangle \\ &\quad \rightarrow |\phi_2\rangle = \frac{1}{\sqrt{3}} (|2, 1, 0\rangle + |0, 2, 1\rangle + |1, 0, 2\rangle) \\ &|1, 2, 0\rangle, \quad |0, 1, 2\rangle, \quad |2, 0, 1\rangle \\ &\quad \rightarrow |\phi_3\rangle = \frac{1}{\sqrt{3}} (|1, 2, 0\rangle + |0, 1, 2\rangle + |2, 0, 1\rangle) \\ &|1, 1, 1\rangle \\ &\quad \rightarrow |\phi_4\rangle = |1, 1, 1\rangle \end{aligned}$$

# Bose–Hubbard model

**Example:**  $L = N = 3$

**Ground state**  $|\Psi_0\rangle = \sum_i \alpha_i |\phi_i\rangle$

$$\mathcal{M}_0 = -t \begin{pmatrix} -3U/t & \sqrt{3} & \sqrt{3} & 0 \\ \sqrt{3} & -U/t & 3 & \sqrt{6} \\ \sqrt{3} & 3 & -U/t & \sqrt{6} \\ 0 & \sqrt{6} & \sqrt{6} & 0 \end{pmatrix}$$

# Bose–Hubbard model

**Example:**  $L = N = 3$

$$|\phi_1\rangle = \frac{1}{\sqrt{3}} (|3, 0, 0\rangle + |0, 3, 0\rangle + |0, 0, 3\rangle)$$

# Bose–Hubbard model

**Example:**  $L = N = 3$

$$|\phi_1\rangle = \frac{1}{\sqrt{3}} (|3, 0, 0\rangle + |0, 3, 0\rangle + |0, 0, 3\rangle)$$



$$|\phi_2\rangle = \frac{1}{\sqrt{3}} (|2, 1, 0\rangle + |0, 2, 1\rangle + |1, 0, 2\rangle)$$

# Bose–Hubbard model

Example:  $L = N = 3$

$$|\phi_1\rangle = \frac{1}{\sqrt{3}} (|3, 0, 0\rangle + |0, 3, 0\rangle + |0, 0, 3\rangle)$$

1,            4,            10

Permanent Ordering (PO)

$\mathcal{O}(L^2)$

$$|\phi_2\rangle = \frac{1}{\sqrt{3}} (|2, 1, 0\rangle + |0, 2, 1\rangle + |1, 0, 2\rangle)$$

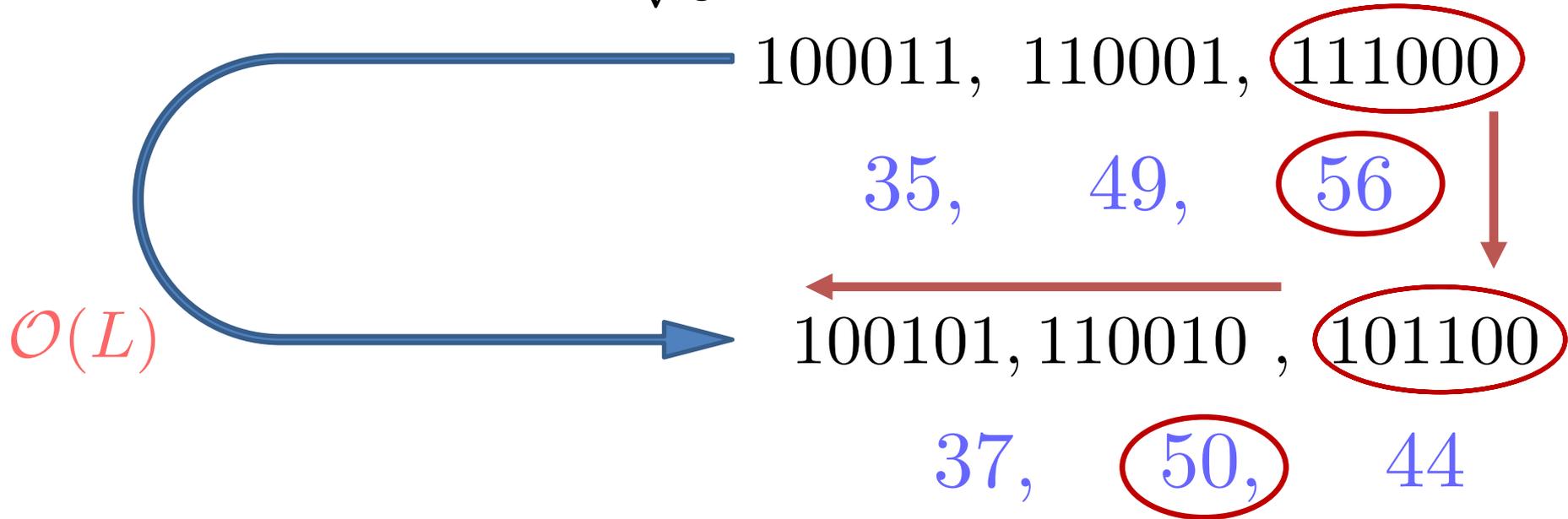
$\mathcal{O}(L)$             9,            3,            5

Max

# Bose-Hubbard model

Example:  $L = N = 3$

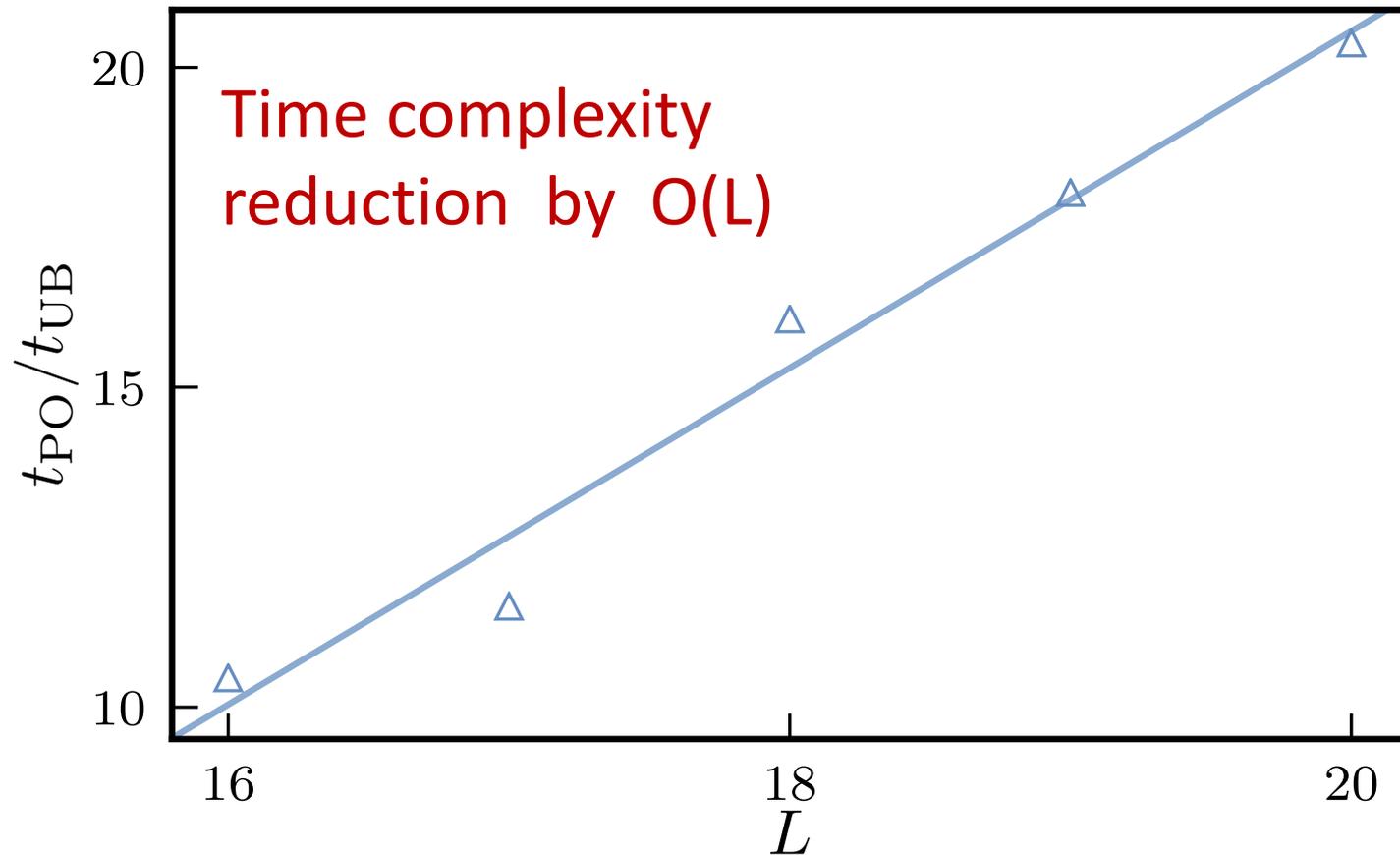
$$|\phi_1\rangle = \frac{1}{\sqrt{3}} (|3, 0, 0\rangle + |0, 3, 0\rangle + |0, 0, 3\rangle)$$



$$|\phi_2\rangle = \frac{1}{\sqrt{3}} (|2, 1, 0\rangle + |0, 2, 1\rangle + |1, 0, 2\rangle)$$

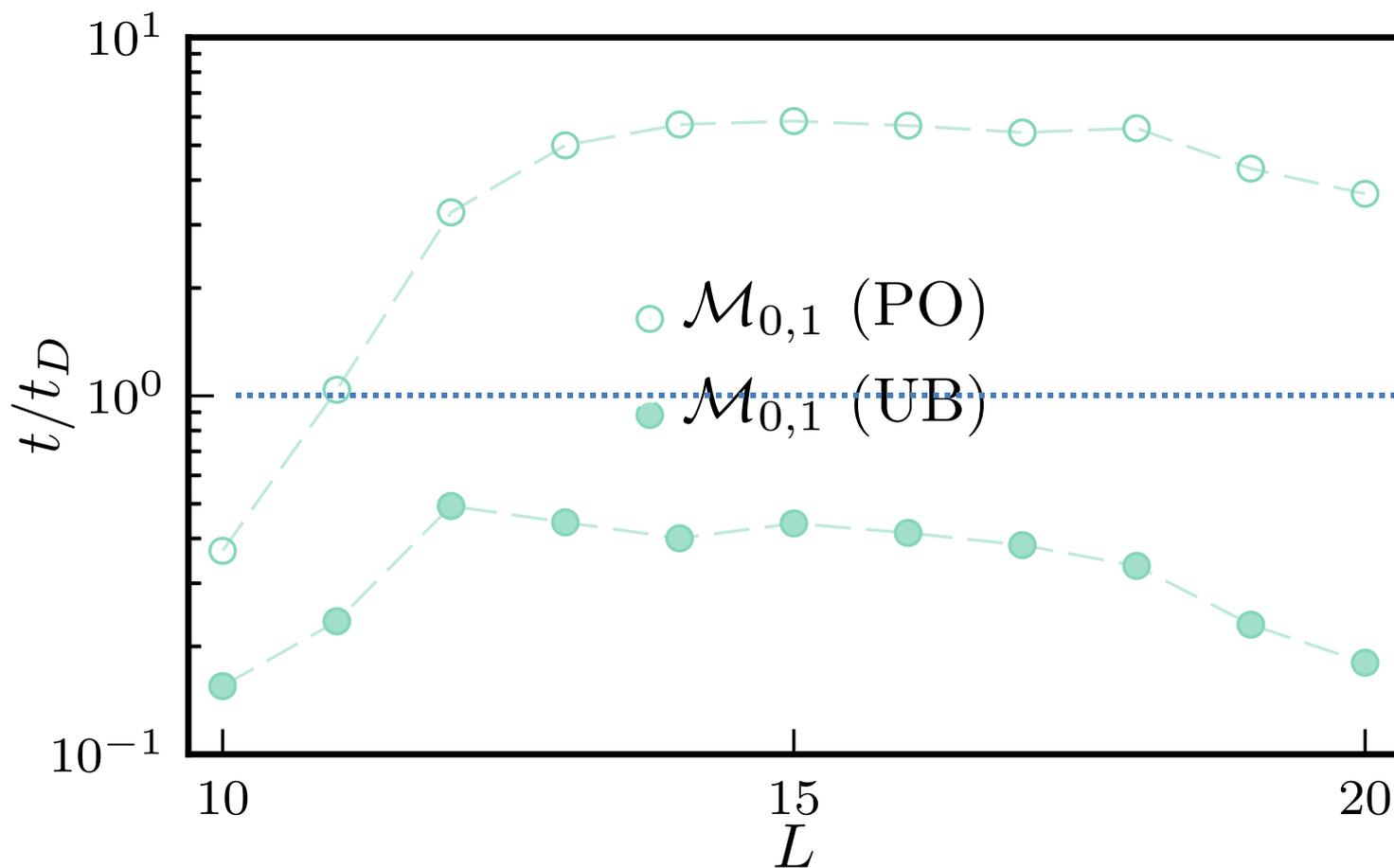
# Bose–Hubbard model

$$H = -t \sum_i \left( a_i^\dagger a_{i+1} + \text{h.c.} \right) + \frac{U}{2} \sum_i n_i (n_i - 1)$$

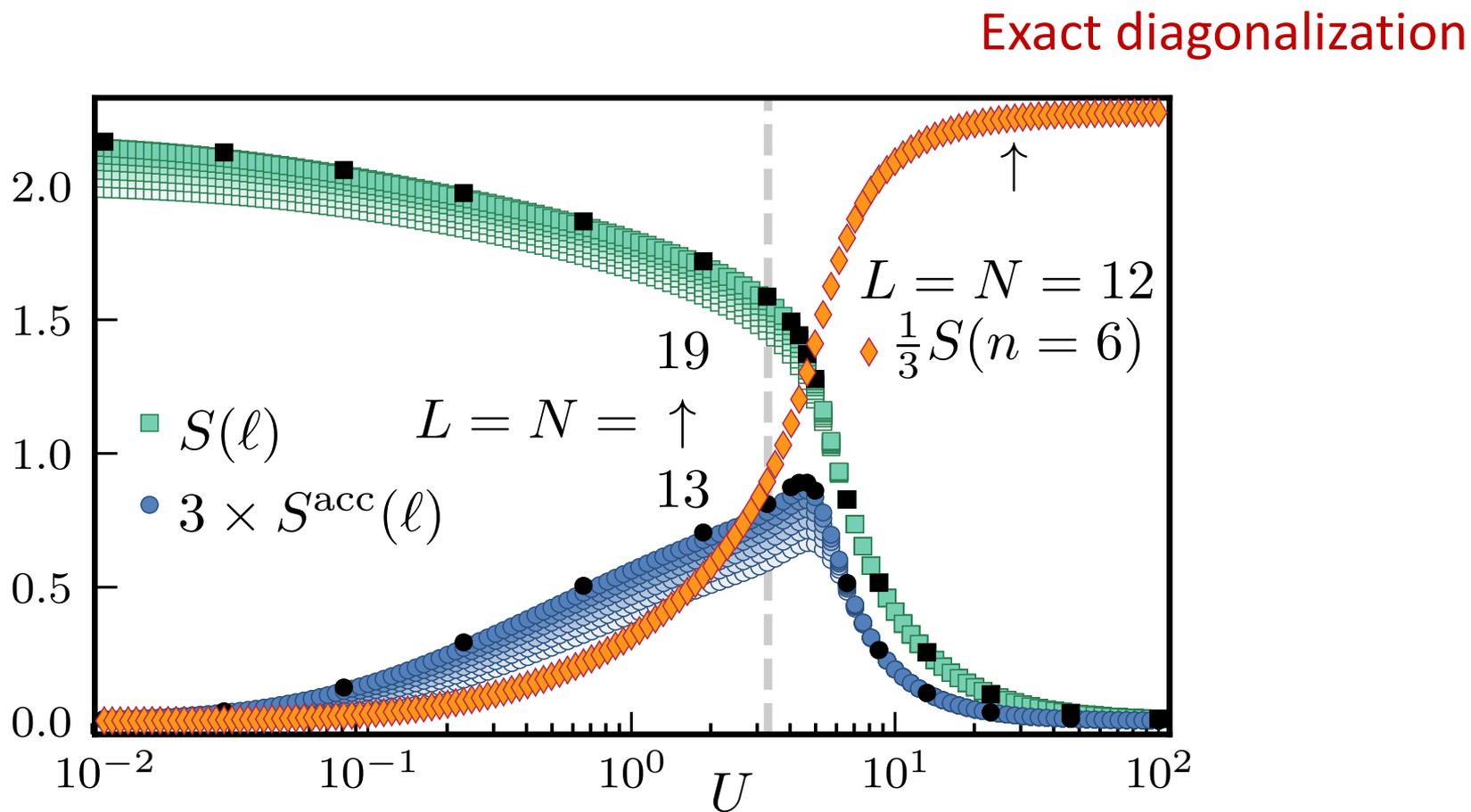


# Bose–Hubbard model

$$H = -t \sum_i \left( a_i^\dagger a_{i+1} + \text{h.c.} \right) + \frac{U}{2} \sum_i n_i (n_i - 1)$$



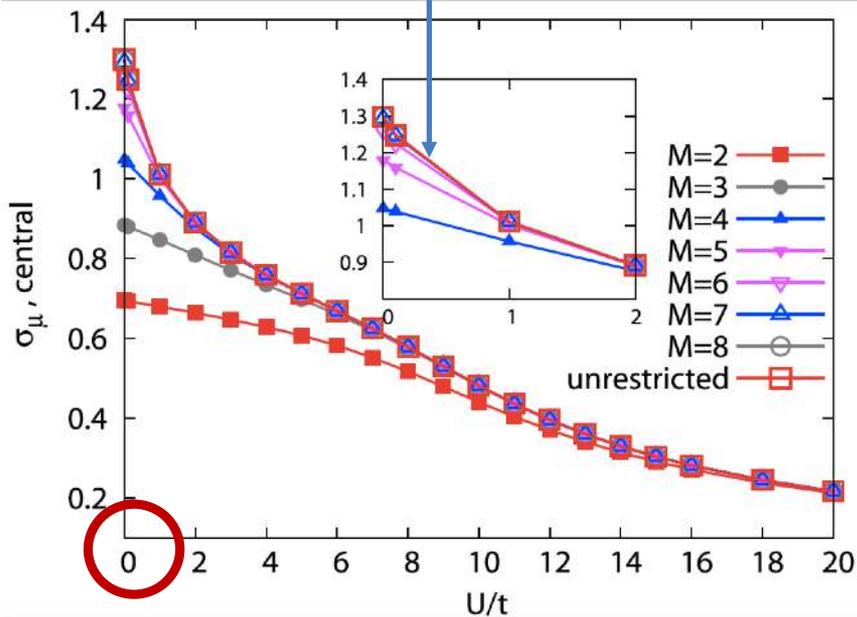
# Quantum Entanglement



# Soft-spin approximation

Setting hard cutoff on the allowed occupation  $n_{max} = 3, 4, \text{ and } 5$

$n_{max} = 6$

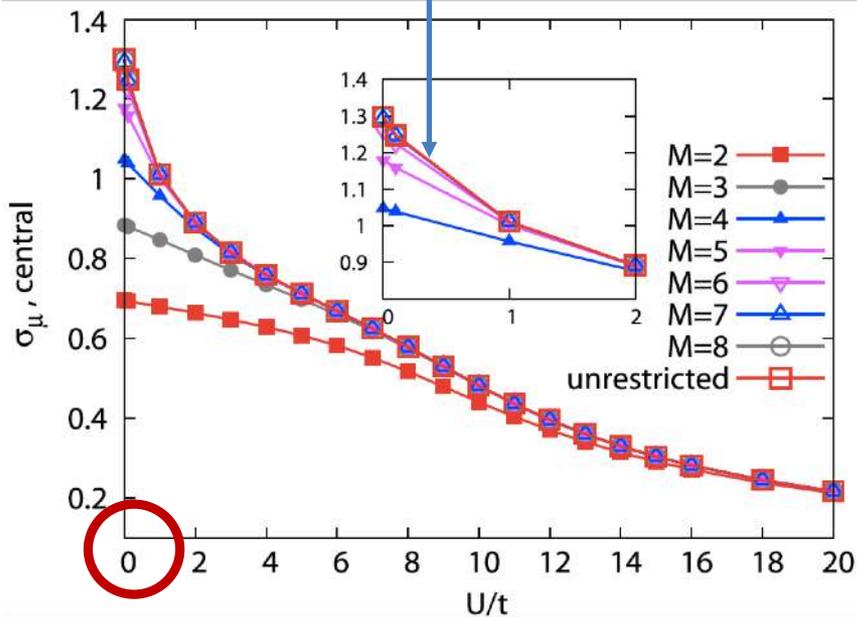


A. Szabados, P. Jeszenszki, and P.R. Surjan, Chem Phys 401, 208 (2012).

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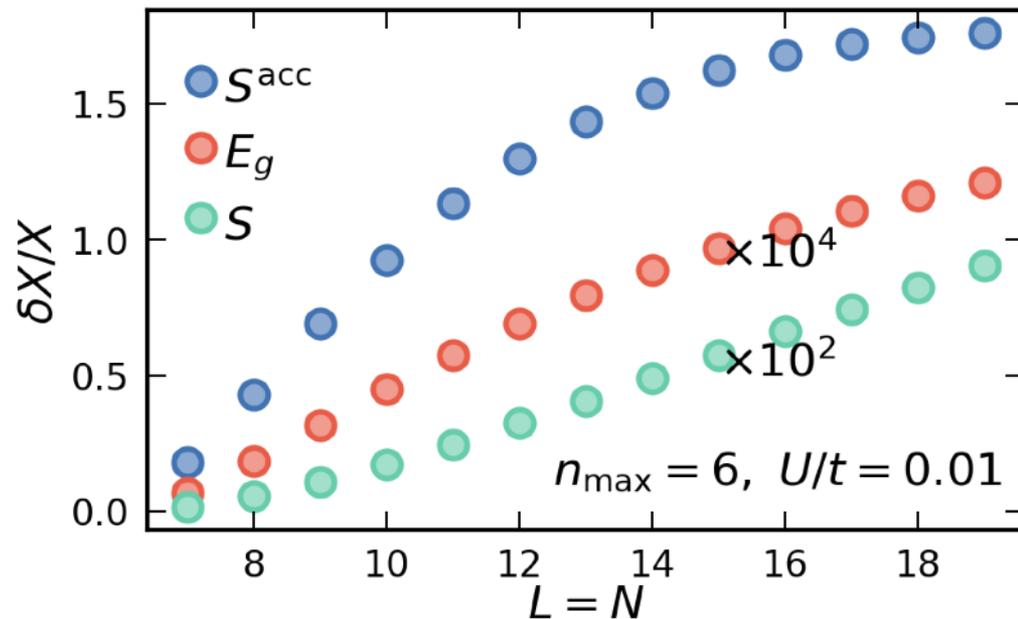
$n_{max} = 6$



A. Szabados, P. Jeszenszki, and P.R. Surjan, Chem Phys 401, 208 (2012).

Exact diagonalization

$$\frac{\delta S^{\text{acc}}}{S^{\text{acc}}} \approx 176\%$$



The relative error in accessible entanglement increases with the system size

# Relative error scaling

Free bosons

$$S^{\text{acc}}(\ell) = 0$$

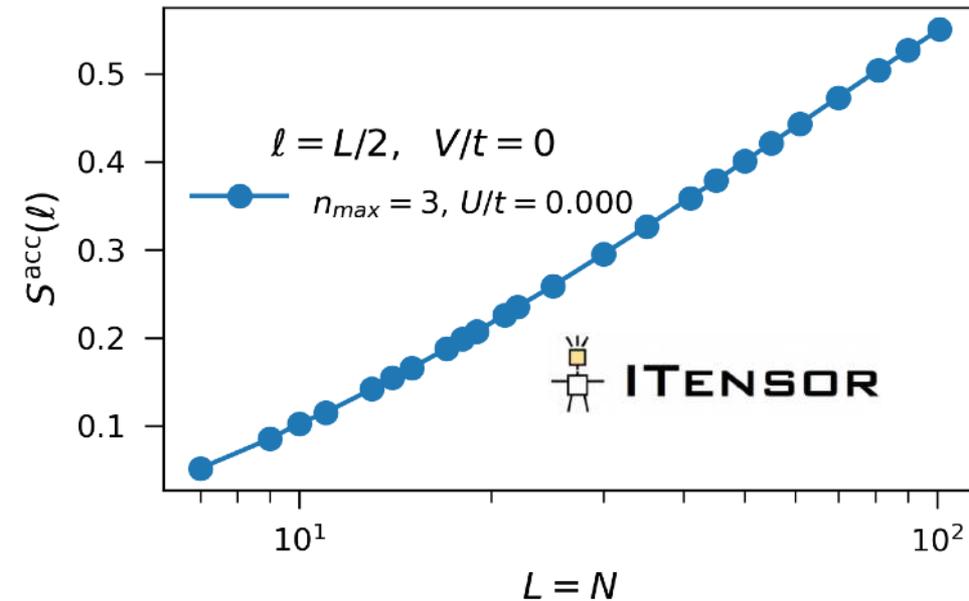
Accessible Entanglement

Interacting bosons

$$S^{\text{acc}}(\ell) \leq S(\ell) \sim \ln \ell$$

$$\ell = L/2$$

Area law



$$n_{\max} = 3$$

$$U_i = U' n_i (n_i - 1) \dots (n_i - n_{n_{\max}})$$

$$U' \gg 1$$

# Particle Entanglement

Free bosons

$$S(n) = 0$$

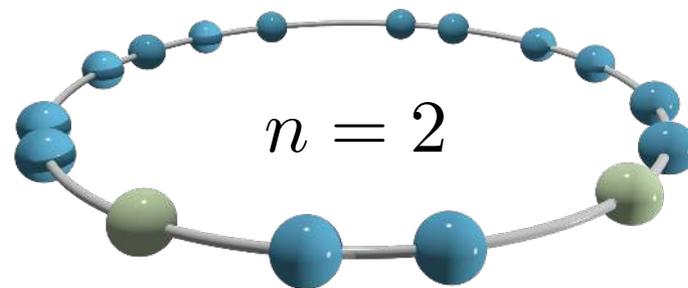
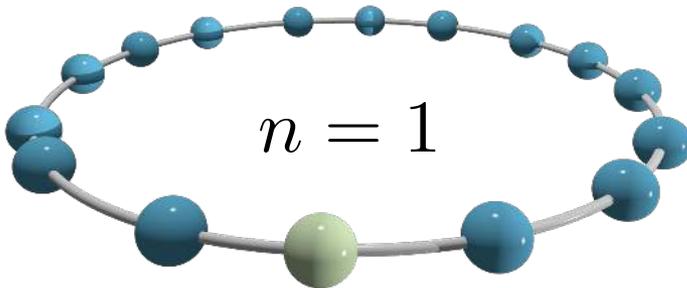
Interacting bosons

$$S(n) \sim \frac{n}{K} \ln N \quad \text{bosonic Luttinger liquids}$$



Luttinger parameter

C. M. Herdman and A. Del Maestro, Phys. Rev. B **91**, 184507

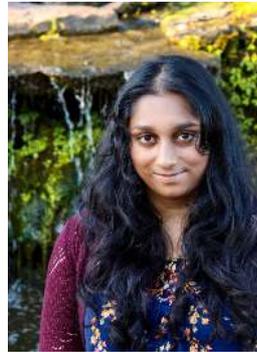


# CONCLUSIONS

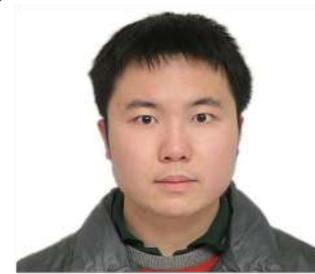
- **Unary coding for bosonic states is as efficient as the conventional binary encoding for fermionic states.**
- **Time complexity is reduction by  $O(L)$ .**
- **In the Bose–Hubbard model, restricting the bosonic occupation numbers to a few bosons per lattice site could lead to large relative errors in the accessible entanglement.**
- **Similar behavior is expected in the case of particle bipartition.**

# Thank You!

## Del Maestro Group



Emanuel Casiano-Diaz Harini Radhakrishnan



Tong Shen



Brenda Rubenstein



BROWN



Bernd Rosenow



Matthias\_Thamm



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# Thank You!

## Del Maestro Group

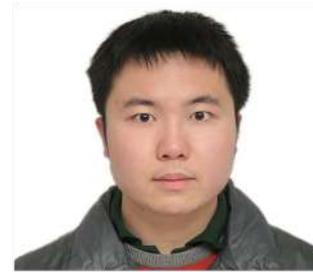


Emanuel Casiano-Diaz Harini Radhakrishnan



Poster sessions

One-particle entanglement for one dimensional spinless fermions after an interaction quantum quench



Tong Shen



Brenda Rubenstein



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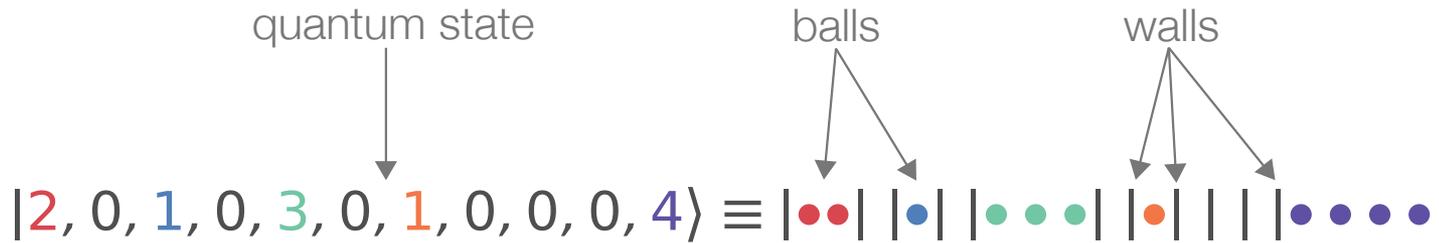


Matthias\_Thamm



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# Thank You!



## BALLS & WALLS

$$I = 2541296 \equiv 1001101100011011110000$$

base 10 integer

machine readable binary number



Code:

[https://github.com/DelMaestroGroup/Bose-Hubbard\\_diagonalize](https://github.com/DelMaestroGroup/Bose-Hubbard_diagonalize)



Paper: <https://github.com/DelMaestroGroup/papers-code-UnaryBosonicBasis>

hbarghat@utk.edu