An Example a Day, 2023-2024

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Abstract

For a manifesto on what this document is about, please see the initial version.

Contents

10/1/2023 *Metaplectic representations

Mostly from the wikipedia and nLab pages for metaplectic group/representation and Heisenberg group.

The story begins with the representation theory of the Heisenberg group, which can be defined as the matrix subgroup

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset Mat_{3\times 3}(R)$$

for any commutative ring R. Higher dimensional Heisenberg groups can be defined as

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & I_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \right\} \subset M_{n+2\times n+2}(R)$$

For a chosen parameter $\hbar > 0$, H_{2n+1} has a unitary representation, $\Pi_\hbar$, on $L^2(\mathbb{R}^n) \equiv \mathcal{H}$ by the formula

$$\left[\begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & I_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \cdot \psi \right](x) := e^{i\hbar c} e^{i\mathbf{b}\cdot x} \psi(x + \hbar \mathbf{a})$$

By varying the parameters a, b, c we can translate within position space and momentum space and vary the overall phase of the state (I'm not sure I understand in what way this representation is motivated by position and momentum operators, but anyway this is certainly *a* representation).

Theorem (Stone-Von Neumann): *Every (adjective) unitary irrep of H with non-trivial central action is equivalent to $\Pi_\hbar$ for some $\hbar$.*

The Heisenberg group is a (one dimensional) central extension of $\mathbb{R}^{2n}$:

$$0 \to \mathbb{R} \to H_{2n+1} \to \mathbb{R}^{2n} \to 0$$

given by inclusion of $t \hookrightarrow \{\mathbf{a} = \mathbf{b} = 0\} \subset H_{2n+1}$ and projection onto coordinates $(\mathbf{a}, \mathbf{b})$. In general, if G is a central extension of H,

$$0 \to K \to G \to H \to 0$$

for $K \hookrightarrow Z(G)$, then given a linear representation of G, we can always consider the quotient representation $H \cong G/K \to GL(V)$. In general, this is not well defined unless the action of K on V is trivial, which is obviously not the case for arbitrary G and K. However in the case of a central extension, by Schur lemma it is the case that K must act by scalar multiplication, since it is included into the center. Thus $G/K \to PGL(V)$ is a valid projective representation.

In this example it means that linear representations of the Heisenberg group can be viewed as projective representations of $\mathbb{R}^{2n}$ and the action of the center is by $e^{i\hbar c}$.

We can also consider the Heisenberg group of a symplectic VS, (V, ω) and $H(V)$, constructed in roughly the same way: Let $\{e_\alpha, f^\alpha\}$ be a Darboux basis for the symplectic VS and define $H(V)$ as $V \times \mathbb{R}$, so that elements have the form $(\bar{p}, \bar{q}, u)$, admitting the faithful matrix representation (meaning multiplication in $H(V)$ can also be realized as matrix multiplication)

$$\begin{pmatrix} 1 & \vec{p} & u \\ 0 & I_n & \vec{q} \\ 0 & 0 & 1 \end{pmatrix}$$

There is a similar Stone-von Neumann statement in this setting, i.e. there exists a unique irrep of $H(V, \omega)$ for each $\hbar$ on $\mathcal{H} = L^2(V)$. Uniqueness implies that for any other representation $\rho' : H(V, \omega) \to U(\mathcal{H})$, there exists a unitary transformation $\psi_{\rho'} \in U(\mathcal{H})$ such that

$$\rho' = \psi_{\rho'} \circ \rho \circ (\psi_{\rho'})^{-1}$$

It follows that this $\psi_{\rho'}$ is unique up to multiplication by norm 1 constant: If $\phi_{\rho'} \in U(\mathcal{H})$ is another such conjugation automorphism, then

$$\rho' = \psi_{\rho'} \circ \rho \circ (\psi_{\rho'})^{-1} = \phi_{\rho'} \circ \rho \circ (\phi_{\rho'})^{-1}$$
$$\Rightarrow \varphi^{-1}\psi \circ \rho \circ \psi^{-1}\varphi = \rho$$

Thus $\psi\phi^{-1}$ is an intertwiner for the representation ρ to itself. ρ is an irrep, so by Schur Lemma, $\psi\phi^{-1} = \lambda Id$, so the conjugating automorphism ψ is projectively unique.

If $F \in Aut(H(V, \omega))$ acts as the identity (stronger than just preserving) on $Z(H(V, \omega))$[1], then $\rho \circ F$ is another irreducible representation $\rho \circ F : H(V, \omega) \to U(\mathcal{H})$, so by Stone-von Neumann,

$$\rho \circ F = Ad_{\psi_F} \rho, \qquad \psi_F \in U(\mathcal{H})$$

Because ψ_F is only defined up to non-zero constant, this establishes a morphism

$$Symp(V, \omega) \to PU(\mathcal{H})$$

$$F \mapsto [\psi_F]$$

So the symplectic group on (V, ω) has a projective unitary representation on $\mathcal{H}$. By the correspondence discussed above, this unitary representation on $\mathcal{H}$ by $Symp(V, \omega)$ corresponds to a linear representation of a central extension of $Symp(V, \omega)$.
FINISH This central extension must be a double cover[2], and we define the metaplectic group to be this double cover, and we have constructed the (linear) Weil representation of it.

[1] Is this required for this part? I know that restricting to only automorphisms which act as identity on the center means the automorphism group is just the symplectic group, which is what we desire to construct the metaplectic group, but is it necessary for this part?

[2] WHY

10/3/2023 Ordinary quantization of classical field theory

From Dan Freed's "5 Lectures in Supersymmetry".

A classical field theory consists of the data:
i) A spacetime manifold M.
ii) A set of fields, $\mathcal{F} \equiv C^\infty(M, W)$, where W is some target manifold of the theory.
iii) A set of equations of motion (conditions on the fields)
iv) A set of solutions to those equations of motion, $\mathcal{M} \subset \mathcal{F}$ and a symplectic struture on $\mathcal{M}$.

Example: For a free particle in $\mathbb{R}^n$, the spacetime manifold is just one dimension of time: $M = \mathbb{R}$ and the target is the $W = \mathbb{R}^n$ in which the particle lives, so that the fields, $C^\infty(\mathbb{R}, \mathbb{R}^m)$ are paths in space. A free particle experiences no force, so the equations of motion are just Newton's laws, $\frac{d^2x(t)}{dt^2} \equiv \ddot{x}(t) = 0$. The subspace of solutions, $\mathcal{M} \subset \mathcal{F}$, is the maps with constant velocity, $\{\dot{\psi} = 0\} \cong W \oplus W \equiv TW$, making the solution space a symplectic vector space. On $W \oplus W$, the symplectic form is

$$\omega\big((v, w), (v', w')\big) = m\big(\langle w, v'\rangle - \langle w', v\rangle\big)$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $W = \mathbb{R}^n$. From this formulation it is easy to see that each W living in $\mathcal{M}$ is a Lagrangian subspace.

To quantize a classical field theory is to associate to the symplectic solution space a projective Hilbert space in a "functorial manner", that is, sending symplectomorphisms to unitary automorphisms. To do so, we note that the symplectic group of our symplectic vector space[3] of solutions has a projective metaplectic representation $Symp(\mathcal{M}, \omega) \to PU(\mathcal{H})$ where $\mathcal{H}$ the Hilbert space of the target $L^2(\mathbb{R}^n)$. So the associated quantum Hilbert space is $\mathbb{P}L^2(\mathbb{R}^n)$, and the metaplectic representation realizes the functoriality[4].

10/9/2023 Generating Lagrangian submanifolds with graphs (symplectomorphisms and one-forms)

The statement was briefly mentioned in lecture 1 of Lev Rozansky's online mini course posted to the YT channel Informal Mathematical Physics seminar, but this is pretty standard symplectic geometry.

As a warm-up, let's prove that the graph of a symplectomorphism is a Lagrangian submanifold.

Proposition: *Let (M, ω) be symplectic and $\psi : M \to M$ be a diffeomorphism. Then ψ is a symplectomorphism iff $\Gamma_\psi \subset M \times M$ is a Lagrangian submanifold*

[3] I suppose this only works if the differential equation is nice enough to ensure the solution set is indeed a vector space, but I don't know enough about DEs to say more.

[4] I'm really not sure this is right.

Proof: The product symplectic form on $M \times M$ is $(-\omega, \omega)$. To check whether Γ_ψ is Lagrangian, we have to evaluate $(-\omega, \omega)|_{T\Gamma_\psi}$. For a point $(p, \psi(p)) \in M \times M$, we have

$$T(M \times M)_{(p,\psi(p))} = T_pM \times T_{\psi(p)}M$$

Let Ψ be the map $M \to M \times M$ sending $x \mapsto (x, \psi(x))$. Note that Ψ is a diffeomorphism from M onto $\Gamma_\psi \subset M \times M$, so that $d\Psi : TM \to T\Gamma_\psi$ is an isomorphism. Thus $T\Gamma_\psi$ is the image of the differential $d\Psi = d(Id_X, \psi) = (Id_{TM}, d\psi)$. The image of the differential of Ψ is exactly the graph of the differential of ψ:

$$\Gamma_{d\psi} = \{(v, d\psi(v)) \mid v \in T_pM, d\psi(v) \in T_{\psi(p)}M\} = T\Gamma_\psi$$

In fact this statement is really obvious from a calculus perspective. Then Γ_ψ is Lagrangian iff

$$\begin{aligned}
&(-\omega, \omega)|_{T\Gamma_\psi} = 0 \\
\iff &\forall (\xi, d\psi(\xi)), (\eta, d\psi(\eta)) \in \Gamma_{d\psi} \times \Gamma_{d\psi},\ -\omega(\xi, \eta) + \omega(d\psi(\xi), d\psi(\eta)) = 0 \\
\iff &-\omega(\xi, \eta) + \psi^*\omega(\xi, \eta) = 0 \\
\iff &\psi \text{ is a symplectomorphism}
\end{aligned}$$

□

In the setting above, W is called the "generating function" for the Lagrangian submanifold, Γ_{dW}.

Example: If $X = T^*\mathbb{C} \cong \mathbb{C}^2$, and $W_1 = \frac{x^{n+1}}{n+1}$ as a map $\mathbb{C} \to \mathbb{C}$, with differential $dW_1 = x^n : \mathbb{C} \to \mathbb{C}^2$, then $\Gamma_{dW_1} \subset \mathbb{C}^2$ is a Lagrangian submanifold. Similarly taking $W_2 = 0$, we have dW_2 as a Lagrangian submanifold. In the case of $n = 2$, that picture looks like the graph $y = x^2$ and $y = 0$, and the intersection is the "fat point" $(x, y) = (0, 0)$. It is a fat point because it is non-reduced as a scheme.

So we know that the graph of an exact form is a Lagrangian submanifold. One may ask if this is a necessary and sufficient condition:
If M is a cotangent bundle, $M = T^*X$, with the canonical symplectic form, $\omega = -d\lambda$,

Proposition: *If $\alpha : X \to M = T^*X$ is a one-form, then $\alpha^*(\lambda) = \alpha$.*

Proof: Note that this equation makes sense as λ is a one-form over T^*X and α is a one-form over X. To compute the pullback, we choose local coordinates $(q_1, \ldots, q_n)$ around $p \in X$ and coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ around $\alpha(p) \in T^*X$. Then if $\hat{\alpha}$ is the coordinate representation of α, ie $\hat{\alpha} = (q_1, q_2, \ldots, q_n, \alpha^1, \alpha^2, \ldots, \alpha^n)$ (recall the function on the points must be the identity function, since α must be a section),

$$\begin{aligned}\alpha^*(\lambda)(\partial_{q_j}) &= \lambda(\alpha_*(\partial_{q_j}))\\ &= \lambda\left(\partial_{q_j}\hat{\alpha}^i\partial_{q_i} + \partial_{q_j}\hat{\alpha}^i\partial_{p_i}\right)\\ &= \lambda(\partial_{q_j} + \partial_{q_j}\alpha^i\partial_{p_i})\\ &= \sum(p_k dq_k)(\partial_{q_j}) + \cancel{\sum(p_k dq_k)(\partial_{q_j}\alpha^i\partial_{p_i})}\\ &= \sum p_k(\sigma(p))dq_k(\partial_{q_j})\\ &= p_j(\sigma(p)) = \alpha^j(p)\\ &= \alpha(\partial_{q_j})\end{aligned}$$

□

Throughout we have suppressed the point at which vector fields are evaluated, but it is clear from context.

Then we know that the graph $\Gamma_\alpha \subset X \times T^*X$ is Lagrangian iff Γ_α is half dimensional and $\alpha^*(d\lambda) = 0 \iff d(\alpha^*\lambda) = 0 \iff d\alpha = 0$, so the graph of α is Lagrangian iff α is closed as a one-form (half-dimensional comes for free).

*10/14/2023 Hamiltonian reduction as critical locus

If X^s is a symplectic variety with Hamiltonian G action and moment map μ, then we can consider its Hamiltonian reduction $\mu^{-1}(0) /\!\!/ G$. We may also consider the function $W : X^s \times \mathfrak{g} \to \mathfrak{g}$, defined by $(x, X) \mapsto Tr(\mu(x)X)$. The critical locus of W is isomorphic to the zero set of μ, so one may think of Hamiltonian reduction as a quotient on the critical locus of W.

Example: Let X^s be the symplectic variety $Rep(Q, n)$ where Q is the quiver of a single vertex and two edges. This variety is acted on by $GL(n)$, where n is the framing of that vertex. Then $Rep(Q, n) \cong End(\mathbb{C}^n)^2$. The moment map $\mu : End(\mathbb{C}^n)^2 \to \mathfrak{gl}_n(\mathbb{C})$ is

$$(A, B) \mapsto AB - BA$$

So the zero level set is the "commuting variety", $\{(A, B) \in End(\mathbb{C}^n)^2 \mid [A, B] = 0\}$. The function $W : End(\mathbb{C}^n)^2 \times \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$ is

$$((A, B), X) \mapsto Tr\big((AB - BA)X\big)$$

To compute the differential, we choose a path in the space $\big((A+tA', B+tB'), X+tV\big)$ and compute

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} Tr\big(((A+tA')(B+tB')-(B+tB')(A+tA'))(X+tV)\big)\\
&= Tr\big((AB-BA)X\big)\cdot \frac{d}{dt}\Big|_{t=0}\big(((A+tA')(B+tB')-(B+tB')(A+tA'))(X+tV)\big)\\
&= Tr\big((AB-BA)X\big)\cdot\big(AB'X+A'BX-BA'X-B'AX+ABV-BAV\big)\\
&= Tr(\big([A,B]X\big)\cdot\big((AB'-B'A+A'B-BA')X+[A,B]V\big)\\
&= Tr(\big([A,B]X\big)\cdot\big(([A,B']+[A',B])X+[A,B]V\big)
\end{aligned}$$

which is equal to 0 for all A', B', V iff $AB-BA=0$ and $X=0$ (I'm not convinced this is true, but it is what is supposed to be true. Am I missing something? Eg, what if $Tr((AB-BA)X)$ is 0.). Thus the critical locus is isomorphic to the zero set of μ under projection.

$\boxed{\text{ADDRESS PARENTHETICAL.}}$

10/15/2023 First Koszul complex

From wikipedia page for Koszul complex, Eisenbud, and Tiger Cheng helped me work out some of the details in person. We omit all instances of the phrase "co", as in cohomology, cochain complex, codifferential, etc.

Let R be a mmutative ring, $x \in R$ and M an R-module. Then $\cdot x : M \to M$ is a morphism of R-modules and trivially extends to a chain complex

$$0 \to M \xrightarrow{\cdot x} M \to 0$$

This is the Koszul complex of $x \in R$. The homology of this chain complex is

$$H_0 = ker(\cdot x) = Ann_M(x), \qquad H_1 = M/xM$$

which encodes important data about multiplication by x.

For a pair $(x, y) \in R^2$ we define the sequence of free R-modules and morphisms

$$0 \to R \to R^2 \to R \to 0$$

Where the first nontrivial arrow is the matrix $\begin{pmatrix} y \\ x \end{pmatrix}$ and the second arrow is the matrix $\begin{pmatrix} -x & y \end{pmatrix}$. This is a chain complex, because

$$r \mapsto (ry, rx) \mapsto -(ry)x + (rx)y = 0$$

This is the Koszul complex of a pair (x, y). The Koszul homology in this case is

$$H_0 = ker\begin{pmatrix} y \\ x \end{pmatrix} = \{r \in R \mid rx = ry = 0\} = Ann_R(x) \cap Ann_R(y)$$

in particular if x or y is a nonzerodivisor, then $H_0 = 0$.

$$H^1 = ker \begin{pmatrix} -x & y \end{pmatrix} \Big/ im \begin{pmatrix} y \\ x \end{pmatrix}$$

If $(a,b) \in ker$, then $-ax + by = 0 \iff b \in (x : y)$. For the rest of this entry, assume x is a nonzerodivisor. Then a is uniquely determined by b: If $(a,b), (a',b) \in ker$ then

$$ax = by = ax' \Rightarrow a = a'$$

thus in this case, $ker \cong (x : y)$, via the projection onto the coordinate b. The image of the left map is all elements of the form (ry, rx). Under the above isomorphism, $\{(ry, rx)\} \cong (x)$, so that

$$H^1 \cong (x : y)/(x)$$

If $[r] \in H^1$, then there exists $b \in R$ such that

$$ry = bx = 0$$

which implies $r = 0$ exactly if y is a nonzerodivisor in $R/(x)$. In other words, if (x, y) is a regular sequence in R, then $H^1(K(x, y)) = 0$. This is a general phenomenon.

Example: Let $R = k[x, y, z]/(x-1)z$, and consider the Koszul complex $K(x, (x-1)y)$. The sequence $(x, (x-1)y)$ is a regular sequence since x is a nonzerodivisor in R and $R/(x) = k[z, y]/(z)$, so y is a nonzerodivisor in $R/(x)$, thus

$$H^1(K(x, (x-1)y)) = 0$$

However if we consider the reversed sequence $(x-1)y, x$, this is not regular because $(x-1)y$ is a zero divisor in R, thus $H^1(K((x-1)y, x)$ may be nonzero.

10/20/2023 Residues at infinity

I need to remember how residues work for the next entry.

Let f be analytic apart from finitely many points, z_i, and let C be a circle at the origin of radius R such that all $z_i \subset int(C)$. The <u>residue at infinity</u> of f is the integral

$$Res(f, \infty) := -\frac{1}{2\pi i} \int_C f(z) dz$$

by the residue theorem, this implies

$$Res(f, \infty) = -\sum Res(f)$$

Theorem:

$$Res(f, \infty) = -Res\left(\frac{1}{w^2} f(1/w), 0\right)$$

This amounts to choosing the other affine chart on $\mathbb{P}^1$.

Example: Let $f(z) = (5z-2)/z(z-1)$. Then

$$\int_{|z|=2} f(z)dz = 2\pi i\Big(Res(f,0) + Res(f,1)\Big) = 2\pi i(2+3) = 10\pi i$$

$$\Rightarrow Res(f,\infty) = -5$$

We may also compute

$$\begin{aligned} Res(f,\infty) &= -Res\left(\frac{1}{w^2}f(1/w), 0\right) \\ &= -Res\left(\frac{(2w-5)}{w(w-1)}, 0\right) \\ &= -5 \end{aligned}$$

As desired.

10/23/2023 Equivariant Localization as Residues at infinity

Here is an interesting way to think about equivariant localization, as read in "Integration over homogenous spaces for classical Lie groups using iterated residues at infinity" by Magdalena Zielenkiewicz.

We just do the case of projective space for now. Let $\phi(R)$ be some characteristic class of the tautological bundle over $X = \mathbb{P}^n$. Then by Berline-Vergne localization formula (I think we now consider this a specific case of the general phenomena known as Atiyah-Bott localization)

$$\int_X \phi(R) = \sum_{i=0}^{n} \frac{V(t_i)}{\prod_{j\neq i}(t_j - t_i)}$$

the denominator is given by the euler class of the fixed point, and V is some polynomial representing that class. V only depends on t_i because the $\phi(R)$ is specifically a characteristic class of the tautological bundle. Note that for a fixed i,

$$\begin{aligned} &-Res\left(\frac{V(z)}{\prod_{j=0}^{n}(t_j - z)}, z = t_i\right) \\ &= -\lim_{z\to t_i}(z-t_i)\frac{V(z)}{\prod_{j=0}^{n}(t_j - z)} \\ &= \lim_{z\to t_i}\frac{V(z)}{\prod_{j\neq i}(t_j - z)} \\ &= \frac{V(t_i)}{\prod_{j\neq i}(t_j - t_i)} \end{aligned}$$

Therefore the summation over all such i gives

$$\int_X \phi(R) = \sum \frac{V(z)}{\prod_{j\neq i}(t_j - t_i)} = -\sum Res\left(\frac{V(z)}{\prod_{j=0}^{n}(t_j - z)}, z = t_i\right)$$
$$= Res\left(\frac{V(z)}{\prod_{j=0}^{n}(t_j - z)}, z = \infty\right)$$

by the residue theorem.

10/29/2023 Matrix Derivatives

Richard Rimanyi taught me this.

Consider the function $F(A, B, C) = Tr(ABC)$. Then

$$\frac{\partial F}{\partial B_{\alpha\beta}} = \frac{\partial}{\partial B_{\alpha\beta}}\left(\sum_i (ABC)_{ii}\right)$$
$$= \frac{\partial}{\partial B_{\alpha\beta}}\left(\sum_{i,j,k} A_{ij}B_{jk}C_{ki}\right)$$
$$= 0 + \cdots + 0 + \sum_i A_{i\alpha}C_{\beta i} + 0 + \cdots + 0$$
$$\equiv (CA)_{\beta\alpha}$$
$$\Rightarrow \frac{\partial F}{\partial B} = A^T C^T$$

A more elegant solution from "user357269" on stack exchange is to observe that the above expression doesn't really make sense. When we take a partial derivative with respect to a variable B, we are really considering the function of a single variable,

$$F(B) = Tr(ABC)$$

for some fixed, arbitrary A, C. The trace is a linear map in this single variable, so it is its own differential, so what we have written is something like

$$F(B) = Tr(ABC) = A^T C^T$$

which is false. Consider the Frobenius pairing $\langle A, B\rangle = Tr(A^T B)$, which is always non-degenerate. Then we are saying that $Tr(ABC) = \langle A^T C^T, B\rangle$, which is obviously true.

One might ask why we are so interested in taking the derivative with respect to the middle matrix and not the others. The answer is that it allows one to compute all other possible derivatives:

$$\frac{\partial Tr(ABC)}{\partial A} = \frac{\partial Tr\big(Id \cdot A \cdot (BC)\big)}{\partial A} = ID^T(BC)^T = C^T B^T$$

and so on.

11/10/2023 Affine Grassmannian for GL_1

This example was mentioned in a conference talk from Gurbir Dhillon.

The most concrete construction of the affine Grassmannian is as a coset space: For a reductive group G, consider the arc and loop[5] functors $\mathcal{O}, F$ where $G_{\mathcal{O}} = G(\mathbb{C}[[t]])$ and $G_F = G(\mathbb{C}((t)))$. The affine Grassmannian is the coset space $Gr_G = G_F/G_{\mathcal{O}}$. In the case $G = GL_1$, this means the affine Grassmannian is

$$Gr_{GL_1} = \mathbb{C}((t))^\times/\mathbb{C}[[t]]^\times$$

To be a unit the Laurent series ring is just to be non-zero. To be a unit in a power series ring (when the coefficients are a field, i.e. no zero divisors), is just to have non-zero constant term. For any formal Laurent series, we can factor it as

$$\sum_{i=i_0}^{\infty} a_i t^i = t^{i_0} g(t)$$

where $g(t)$ is a power series with non-zero constant term (otherwise just shift the i_0 until it is non-zero). In the quotient, then $g(t)$ dies while t^{i_0} does not, as its constant term is zero, so an element of the quotient is characterized only by its starting index:

$$Gr_{GL_1} \cong \mathbb{Z}$$

I think swapping $\mathbb{C}$ with any field shouldn't be a problem, but replacing it with a ring makes this problem more complicated.

11/20/2023 Canonical basis functions on algebraic tori

I learned this example from Sean Keel's talk "Mirror Symmetry Made Easy", available on Youtube.

If $U = (\mathbb{C}^\times)^r$ is a torus, we can consider the characters, that is morphisms of algebraic groups $U \to \mathbb{C}^\times$. All such must be of the form

$$f(t_1, \ldots, t_r) = t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r}$$

for $a_i \in \mathbb{Z}$. In other words, the characters are the monomials living in $\Gamma(U, \mathcal{O}_U) \cong k[t_1^\pm, \ldots, t_n^\pm]$, and they form a basis of this ring. Similarly, the monomials in $\Gamma(\mathbb{C}^r, \mathcal{O}_{\mathbb{C}^r})$ also form a basis of the coordinate ring. However, again in the case of $\mathbb{C}$ ($r = 1$), while the basis $1, T, T^2, \ldots$ is a basis of $k[T]$, so is $1, (T-3), (T-3)^2, \ldots$, and this basis is just as good. We do not have a canonical basis in this case, because it depended on our choice of basis for $\mathbb{C}$. However the characters for the torus are actually canonical: they can be identified without reference to any choice.

[5] What does this have to do with arc and loop spaces?

Lemma: *$f \in \Gamma(U, \mathcal{O}_U)$ is invertible iff f is (proportional to) a character.*

This is basically just saying that the only functions with no zeros or poles on the torus is monomials (these have no zeros because we removed 0).

So the coordinate ring of the torus has a canonical basis given by characters. Generalizing this phenomena to some classes of CY varieties is the goal of Sean Keel's talk.

11/23/2023 Fun with theta functions

I should try to learn about elliptic things soon for research so here's some of the background info. Mostly from the wiki page on theta functions, also drawing from Paul Aspinwall's unpublished (at the time of writing) textbook on string theory.

For $z, \tau \in \mathbb{C},\ Im(\tau) > 0$, we define the Jacobi Theta function

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z)$$

First of all this is a convergent sum on any compact subset[6] of $\mathbb{C} \times \mathbb{H}$: if we write $z = x + iy$ and $\tau = u + iv$ with $v > 0$, then

$$\begin{aligned} \left|e^{\pi i n^2 \tau} e^{2\pi i n z}\right| &= \left|e^{i(\pi n^2 + 2\pi n x) - (2\pi n y + \pi n^2 v)}\right| \\ &= e^{-\pi n(2y + nv)} \end{aligned}$$

which is less than one as long as $2y + nv \geq 0$. v is always positive, and n will eventually become positive, and the magnitude of y is bounded because we consider z valued in a compact set, so the terms in this series will eventually start to die exponentially fast.

Observe

$$\begin{aligned} \vartheta(z+1;\tau) &= \sum_{n=-\infty}^{\infty} \exp\big(\pi i n^2 \tau + 2\pi i n(z+1)\big) \\ &= \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z + \cancel{2\pi i n}) = \vartheta(z;\tau) \end{aligned}$$

so ϑ is 1-periodic in z, and

$$\begin{aligned} \vartheta(z+\tau;\tau) &= \sum_{n=-\infty}^{\infty} \exp\big(\pi i n^2 \tau + 2\pi i n(z+\tau)\big) \\ &= \sum_{n=-\infty}^{\infty} \exp\big(\pi i n^2 \tau + 2\pi i n z + 2\pi i n \tau\big) \end{aligned}$$

[6]Wikipedia doesn't mention compact subset, but the only proofs I could (easily) find were for compact subsets.

by completing the square

$$= \sum_{n=-\infty}^{\infty} \exp\bigl(\pi i(n+1)^2\tau - \pi i\tau + 2\pi inz\bigr)$$

$$= \sum_{n=-\infty}^{\infty} \exp\bigl(\pi i(n+1)^2\tau - \pi i\tau + 2\pi i(n+1)z - 2\pi iz\bigr)$$

$$= \exp(-\pi i\tau - 2\pi iz)\vartheta(z;\tau)$$

So that ϑ is quasi τ-periodic in z. Indeed it could not actually be τ-periodic in z, since it was already 1-periodic. This would imply that, for fixed τ, ϑ would descend to a holomorphic function on the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, which is compact, therefore ϑ must be constant by Liouville theorem.

In τ, we have

$$\vartheta(z;\tau+1) = \sum_{n=-\infty}^{\infty} \exp\bigl(\pi in^2(\tau+1) + 2\pi inz\bigr)$$

$$= \sum_{n=-\infty}^{\infty} \exp\bigl(\pi in^2\tau + \pi in^2 + 2\pi inz\bigr)$$

If n is an integer, then $e^{\pi in}$ depends only on the parity of n: If n is even then its value is 1, if n is odd then its value is -1. Therefore in the summation over "exp", we may always replace any individual n term with n^2 and vice versa, because $n \equiv n^2 \mod 2$.

$$= \sum_{n=-\infty}^{\infty} \exp\bigl(\pi in^2\tau + \pi in + 2\pi inz\bigr)$$

$$= \sum_{n=-\infty}^{\infty} \exp\left(\pi in^2\tau + \pi in + 2\pi in\left(z+\frac{1}{2}\right)\right)$$

$$= \vartheta(z+\frac{1}{2};\tau)$$

So translating by one in τ translates by a half in z. Inspired by this translation we may also define the other theta functions

$$\theta_1(z;\tau) = e^{-\pi i(z+1/2+\tau/4)}\vartheta(z+\frac{1}{2}\tau+\frac{1}{2};z)$$
$$\theta_2(z;\tau) = e^{\pi i(z+\tau/4)}\vartheta(z+\frac{1}{2}\tau;\tau)$$
$$\theta_3 = \vartheta(z;\tau)$$
$$\theta_4 = \vartheta(z+\frac{1}{2};\tau) = \vartheta(z;\tau+1)$$

If we introduce the nome[7], $q = \exp(\pi i \tau)$, then the original definition of the Jacobi theta function can be rewritten

$$\vartheta(z;\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2\pi n z)$$

(the sin terms cancel because they are odd).

11/20/2023 Free and Forget are adjoint functors

If $F : C \to D$ and $G : D \to C$ are functors, we say they are adjoints if for all $X \in C$ and $Y \in D$, we have a set bijection

$$hom_D(FX, Y) \cong hom_C(X, GY)$$

which is natural in X and Y, ie there is a natural isomorphism of functors from D to Set, $hom_C(G-, X) \cong hom_D(-, FX)$ and a natural isomorphism of functors from C to Set $hom_C(-, GY) \cong hom_D(F-, Y)$.

Example: We have two categories Grp and Set, and the functors $F = Free : Set \to Grp$ and $G = Forget : Grp \to Set$, where Free sends a set to the free group on that set and forget sends a group to its underlying set. If G is a group and X is a set, we have to show a bijection

$$hom_{Grp}(Free(X), G) \cong hom_{Set}(X, Forget(G))$$

Given such an X and G, define a set function

$$hom_{Grp}(Free(X), G)) \to hom_{Set}(X, Forget(G))$$

by observing that for f in the LHS, we can define the set function $\tilde{f}$ by sending an element in X to its image under f. That is to say, for $x \in X$, there is a generator in $Free(X)$ which is also labelled x, so define $\tilde{f}(x) := f(x)$. This is injective because if two group homomorphisms from $Free(X)$ to G map to the same set function, then they must must coincide on the generators of $Free(X)$. Because they are group homomorphisms, that means they coincide on all of $Free(X)$, so they are the same group homomorphism. It is surjective because if you have a set function $X \to Forget(G)$ then you can just define a group homomorphism $Free(X) \to G$ by defining the images of the generators by the given set function, then extending to all of $Free(X)$ by imposing that it be a group homomorphism. We decline to check naturality.

The universal property of free groups is essentially defined so that this relation holds by definition, i.e. it puts set functions in bijection with group homomorphisms.

[7] What an unpleasant name.

11/30/2023 Representability of schemes

We will sometimes blur distinctions between (thing) and (co-thing).

For a scheme X, we have a functor $H(-,X) : AffSch \to Set$ given by sending $Y \mapsto Hom_{AffSch}(Y,X)$, and sending morphisms to their composite. Such a functor is called the functor of points for the scheme X.

The Yoneda lemma states that the Yoneda embedding $Sch \to [Sch^{op}, Set]$ is fully faithful, i.e. furnishes an isomorphism of categories $Sch \cong \{h_S \mid S \in Sch\}$, where h_S is the typical hom-functor. However a stronger statement is true: A scheme X is determined by its functor of points, i.e. the "functor of points" functor is also a fully faithful functor $Sch \to [AffSch^{op}, Set]$. Intuitively, Yoneda lemma tells us that in order to understand a scheme, X, we need to understand all scheme morphisms into X, while the functor of points approach says that to understand a scheme, it suffices to understand only *affine* scheme morphisms into X, which is desirable because it means we need to understand a much simpler class of morphisms.

Example: $X = \mathbb{A}^r = Spec(\mathbb{Z}[x_1, \dots, x_n])$. If we interpret the functor of points as eating commutative rings instead of affine schemes (which we are allowed to do because the categories are isomorphic), then the functor of points is $Hom(\mathbb{Z}[x_1, \dots, x_r], -) : CommRng \to Set$. It sends an affine scheme $Spec(A)$ to the hom-set $Hom(\mathbb{Z}[x_1, \dots, x_r], A)$. Any morphism is determined by the image of $x_1, \dots, x_n$, and so this hom-set is equal to the set of r-element subsets of A. Then the functor of points for $\mathbb{A}^r$ is the functor sending a ring A to the set of r-element subsets of A, as a set. When $r = 1$, this is just the forgetful functor.

The pedagogically precise statement is to first consider the functor $CommRng \to Set$ which sends a ring to the set of r element subsets, $\underline{\mathbb{A}}^r$. The claim is that this functor is representable, and further is represented by the ring $\mathbb{Z}[x_1, \dots, x_r]$, furnishing a natural transformation of functors $CommRng \to Set$, $\underline{\mathbb{A}}^r \simeq Hom_{CommRng}(-, \mathbb{Z}[x_1, \dots, x_r]) \cong Hom_{AffSch}(-, Spec(\mathbb{Z}[x_1, \dots, x_r])$. Thus we say the "classical affine space" $\mathbb{A}^r = Spec(\mathbb{Z}[x_1, \dots, x_r])$ represents the functor of points $\underline{\mathbb{A}}^r$.

Example: $\mathbb{G}_m = Spec(\mathbb{Z}[x, x^{-1}])$. The functor of points is the functor $A \mapsto Hom(A, \mathbb{Z}[x, x^{-1}]) \cong A^\times$. Ring morphisms send units to units so this indeed defines a functor. Therefore the functor is the "units" functor. There is a similar "pedagogically correct" modification as in the previous example.

Example: $\mathbb{P}^n$.

12/2/2023 *Functor of points of the line with two origins

$\boxed{\text{TODO}}$

12/6/2023 $2 \times 3 = 4 + 2$

I learned this in Paul Aspinwall's Quantum mechanics and string theory course at Duke a year(ish) ago.

Recall that because $SU(2)$ is simply connected, there is a bijection between its group representations and the Lie algebra representations of its lie algebra, $\mathfrak{su}(2) = Lie(SU(2))$, so we may blur the distinction between these two. If we complexify, $\mathfrak{su}_{\mathbb{C}}(2) \cong \mathfrak{sl}(2;\mathbb{C})$ is generated by 3 elements, H, X, Y satisfying the commutation relations $[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = H$. Given an $\mathfrak{su}_{\mathbb{C}}(2)$ representation, we refer to the eigenvalues of the operator H as the weights of the representation.

Theorem: *For each non-negative integer m, there is a unique irrep with highest weight m. Each irrep is equivalent to one of these. The representation with highest weight m has dimension $m + 1$ with weights $-m, -m + 2, -m + 4, \ldots, m - 4, m - 2, m$, each having multiplicity one.*

Using physicists notation, let $\mathbf{n}$ denote the irrep of dimension n. To describe the representation, we only need to say how the 3 generators act on the weight spaces. H of course acts by scaling and X and Y are known as ladder operators, i.e. X sends the weight α eigenspace to the weight $\alpha + 2$ eigenspace, and Y sends the weight α eigenspace to the weight $\alpha - 2$ eigenspace. These facts are immediate from the commutation relations.

Given any two irreps $\mathbf{a}$, $\mathbf{b}$, we can tensor to get the representation $\mathbf{a} \otimes \mathbf{b}$, which will usually be reducible [8], so we may ask how to decompose it into a direct sum of irreducibles.

The key to analyze how to break up the reps into irreps is to examine the spectrum of H on both sides using the observation: If v_1, v_2 are eigenvectors of H with eigenvalues λ_1, λ_2 in some representations, $\mathbf{n}_1$, $\mathbf{n}_2$, then in the tensor product, $v_1 \otimes v_2$ is an eigenvector with eigenvalue $\lambda_1 + \lambda_2$. Therefore if X is the spectrum of H in the representation $\mathbf{n}_1$ and Y is the spectrum of H in the representation $\mathbf{n}_2$, then all the pairwise sums of eigenvalues, $X + Y$, will be contained in the spectrum of H in the representation $\mathbf{n}_1 \otimes \mathbf{n}_2$, but this is also the right number of eigenvalues, so $X + Y$ is the spectrum of H in the tensor product. Similar analysis shows that the spectrum of H in the direct sum corresponds to $X \sqcup Y$. Clearly from the theorem, the spectrum of H determines the representation.

Example: $\mathbf{2} \otimes \mathbf{3} \cong \mathbf{2} \oplus \mathbf{4}$. The weights for $\mathbf{2}$ are ± 1 and the weights for $\mathbf{3}$ are $-2, 0, 2$. For posterity we also note that the eigenvalues for $\mathbf{4}$ are $-3, -1, 1, 3$. Then the spectrum of H in the tensor product is

$$-3, -1, -1, 1, 1, 3$$

which is the spectrum of $\mathbf{2}$ disjoint unioned with the spectrum of $\mathbf{4}$, so this representation must be $\mathbf{2} \oplus \mathbf{4}$.

[8] I'm pretty sure it will always be reducible, but I don't care to investigate that right now.

Example: $\mathbf{2} \otimes \mathbf{n} \cong (\mathbf{n}+1) \oplus (\mathbf{n}-1)$. The eigenvalues of H in $\mathbf{2}$ are ± 1. The eigenvalues of H in $\mathbf{n}$ are $-n+1, -n+3, \ldots, n-3, n-1$. So the eigenvalues of H in $\mathbf{2}\otimes\mathbf{n}$ are

$$\Big\{-n, -n+2, -n+2, \ldots, n-2, n-2, n\Big\}$$

which are also the eigenvalues of H in $(\mathbf{n-1}) \oplus (\mathbf{n+1})$. Thus

$$\mathbf{2} \otimes \mathbf{n} = (\mathbf{n-1}) \oplus (\mathbf{n+1})$$

12/7/2023 Fusion-categorical interpretation of $2 \times 3 = 4 + 2$

To interpret the above, the category $FdRep(\mathfrak{su}_{\mathbb{C}}(2))$ is (almost) a fusion category: All objects have duals, it is $\mathbb{C}$-linear, semisimple (all direct sums exist and each object is a direct sum of simple objects), monoidal, and endomorphisms of a 1-dimensional vector space is just $\mathbb{C}$. This category does fail one criteria, which is that it has infinitely many isomorphism classes of simple objects, since we have an irreducible representation for every natural number, but let's ignore that.

Fusion categories are categories where one can nicely "fuse" (tensor) objects in the category. The conditions in the definition allow one to write down things like

$$V \otimes V' \cong \sum_{i=0}^{N} c_i V_i$$

where V_i are the simple objects in the category. The c_i's are called structure constants. (in our case the sum is a priori infinite, due to the infinitely many simple objects, but in practice the sums are always finite since, for example, we know the c_i for V_i whose dimensions are higher than the product of dimensions will all be 0. This is the reasoning for disregarding missing one of the conditions. Experts may take exception to that.). In the case of $\mathfrak{sl}_n$, the structure constants are provided by the Littlewood-Richardson rule.

The conditions on a fusion category also imply that the Grothendieck ring is generated by (isomorphism classes of) the irreps $\mathbf{n}$, and the way to multiply is given by the Little-Richardson coefficients, and addition is direct sum. The rigidity condition implies that the Grothendieck ring is a fusion ring. In this context, the Littlewood-Richardson rule is known as a fusion rule.

The Clebsch Gordon coefficients from quantum mechanics are another example of a fusion rule.

12/9/2023 Alternating tensor representations of $SO(m)$.

From Knapp's "Representation Theory of Semisimple Groups".

Let $\mathbb{R}^m$ have the standard basis $e_1, \ldots, e_m$. $SO(m)$ acts on this by matrix multiplication. By imposing i-linearity, $SO(m)$ acts on $\mathbb{C}^m$. Suppose $m = 2n$. Then a basis for the Cartan subalgebra can be given by $H_1, \dots, H_n$ where H_i has the 2 x 2 block $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the ith diagonal 2 x 2 block (of which there are n), and 0 elsewhere. Then for example, $e_1 + ie_2 \in \mathbb{C}^m$ is acted on by H_1 with weight $-i$ and $e_1 - ie_2$ is acted on by $+i$, therefore they are both weight vectors with weights w_1, w_2, where $w_1(H_1) = i$ and $w_2(H_1) = -i$, while $w_1(H_i) = w_2(H_i) = 0$ for $i \neq 1$. Continuing in this manner, there is a pair of weight vectors for each H_i, $e_i \pm ie_{i+1}$, with weights w_i and w_{i+1}, satisfying $w_i(H_i) = w_{i+1}(H_{i+1}) = \pm i$, $w_i(H_j) = 0$ for $i \neq j, j+1$. If m is odd, then there are still $H_1, \dots, H_n$ basis vectors for the Cartan, described in the same way, but with an extra row and column of all 0's appended to the right and bottom. There is thus an additional weight, 0, corresponding to the weight vector e_{2n+1}.

There is an induced action on $\bigwedge^k \mathbb{C}^m$ for all $k \leq m$ by Leibniz rule. For example, if we consider a simple 2-vector, and v_i are weight vectors of weights w_i, then

$$\begin{aligned}
H_1(v_1 \wedge v_2) &= H_1(v_1 \otimes v_2 - v_2 \otimes v_1) \\
&= H_1(v_1 \otimes v_2) - H_1(v_2 \otimes v_1) \\
&= H_1(v_1) \otimes v_2 + v_1 \otimes H_1(v_2) - H_1(v_2) \otimes v_1 - v_2 \otimes H_1(v_1) \\
&= w_1(H_1)v_1 \otimes v_2 + v_1 \otimes w_2(H_1)v_2 - w_2(H_1)v_2 \otimes v_1 - v_2 \otimes w_1(H_1)v_1 \\
&\equiv w_1(H_1)v_1 \wedge v_2 + v_1 \wedge w_2(H_1)v_2 \\
&= (w_1(H_1) - w_2(H_1))v_1 \wedge v_2
\end{aligned}$$

So that in general, the weights of $\bigwedge^k \mathbb{C}^m$ are all possible sums of $\pm w_i$'s in increasing order of size k, and the highest weight is the sum over the first k w_i's, with positive signs.

Theorem: *Each $\bigwedge^k \mathbb{C}^{2n}$ representation of $SO(2n)$ is irreducible, for $k < n$. When $k = n$, this representation is reducible.*

Theorem: *Each $\bigwedge^k \mathbb{C}^{2n+1}$ representation of $SO(2n+1)$ is irreducible, for $k \leq n$.*

Once we pass the halfway point, we are looking at the same representation, since $\bigwedge^k \mathbb{C}^m \cong \bigwedge^{m-k} \mathbb{C}^m$. So all the exterior reps of $SO(2n+1)$ are irreducible, while $SO(2n)$ has one half-dimensional representation which splits as two irreps. Viewed as representations of $\mathfrak{so}(m)$, these are all fundamental representations, that is reps whose highest weight is a fundamental weight, (not a sum of any other weights), and these cover almost all of the fundamental representations.

12/9/2023 Spin representations

The notation in this section is a complete mess and I will not fix it.

If V is a vector space (here we will only consider over $\mathbb{C}$ and $\mathbb{R}$) with non-degenerate quadratic form Q, Spin(V, Q) is the unique double cover of $SO(V, Q)$. As a result, any representation of $SO(V, Q)$ induces a representation of Spin(V, Q). In particular, the irreps above of $SO(V, Q)$ induce irreps of Spin(V, Q). As Spin(V, Q) is a covering of $SO(V, Q)$, they are locally homeomorphic and thus have the same lie algebra, $\mathfrak{so}(V, Q)$. Therefore first we study the Lie algebra spin representations. These are representations of the Lie algebra $\mathfrak{so}(V,Q)$, but as $SO(V,Q)$ is not simply connected, they do not necessarily integrate to representations of $SO(V,Q)$. In the case of spin reps, they do not, and instead have to pass to the universal cover.

Up to isomorphism, we may consider $V = \mathbb{C}^n$ and Q to be the standard quadratic form, inducing the standard symmetric bilinear form $\langle -,-\rangle$ (dot product), so that $\mathfrak{so}(V,Q) \cong \mathfrak{so}(n,\mathbb{C})$, the Lie algebra of skew symmetric complex matrices. If $m = 2n$, choose a splitting $V = W \oplus W^*$ of maximal isotropic subspaces of V. When $m = 2n+1$, choose a splitting $V = W \oplus W^* \oplus U$, where U is a one-dimensional subspace, orthogonal to $W \oplus W^*$. In general, we can always construct a basis of V so that $\langle e_i, e_j\rangle = 0$ unless $j = n+i$, in which case you get 1. In such a setting, we can let the first n basis vectors be a basis of W and the final n be a basis of W^*. When m is odd, there will be an extra vector left over, and let that span U. Because $\langle -,-\rangle$ is non-degenerate, it induces a perfect pairing $W \times W^* \to \mathbb{R}$, so W^* is canonically linear dual to W. Let $S = \bigwedge^\bullet W$, and for every $v \in V$, split v uniquely into $v = w + w^*$ if n is even and $v = w + w^* + u$ if n is odd. For every $\psi \in S$, define[9]

$$v \cdot \psi := \sqrt{2}(w \wedge \psi + \iota(w^*)\psi)$$

and if n is odd, then define the action of u by identity if $\psi \in \bigwedge^{\text{even}} W$ and by multiplication by -1 if $\psi \in \bigwedge^{\text{odd}} W$. w^* acts on ψ by expanding it as an alternating tensor and using Leibniz rule: In the case $\psi = v_1 \wedge \cdots \wedge v_k$, then

$$\iota(w^*)\psi = \sum_{i=1}^{k} (-1)^i Q(w_i, w^*) w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_k$$

where $\widehat{w_i}$ indicates the omission of the ith vector, so $\iota(w^*)$ is a degree -1 map on the exterior algebra. We can check that the Clifford relation, $v \otimes v = Q(v)1$ is respected:

$$\begin{aligned}(v\otimes v)\psi \equiv v\cdot(v\cdot\psi) &= \sqrt{2}\big(w \wedge (\sqrt{2}(w\wedge\psi + \iota(w^*)\psi)) + \iota(w^*)(\sqrt{2}(w\wedge\psi + \iota(w^*)\psi)\big)\\ &= 2\big(w\wedge\iota(w^*)\psi + \iota(w^*)(w\wedge\psi) + \iota(w^*)\iota(w^*)\psi\big)\\ &= 2\big(w\wedge\iota(w^*)\psi + \iota(w^*)w\wedge\psi + w\wedge\iota(w^*)\psi + \iota(w^*)\iota(w^*)\psi\big)\end{aligned}$$

Of these, only one term remains:

$$= 2(\iota(w^*)w\wedge\psi) = 2\langle w, w^*\rangle\psi$$

By polarization,

$$= 2\left(\frac{1}{2}Q(w+w^*) - \cancel{Q(w)} - \cancel{Q(w^*)}\right)\psi = Q(v)\psi$$

[9] This is known as the "geometric product" or "Clifford product".

The cancellations occur because W, W^* are Lagrangian subspaces. This shows[10] that $v \cdot v$ acts as $Q(v)$, so S is a $Cl(V, Q)$ module, where $Cl(V, Q)$ is the Clifford algebra of V, Q.

Now we give an alternate construction of the Spin group inside of the Clifford algebra, and show that it coincides with the abstract double cover definition. Thus the Clifford module S will become a spin module by restriction. The Clifford algebra, $Cl(V, Q)$, inherits a $\mathbb{Z}_2$-grading from the tensor algebra (but not the $\mathbb{Z}$-grading: that is destroyed by the Clifford relation. Instead it has a $\mathbb{Z}$-filtration, since we are allowed to decrease the degree.). The Pin group, $\mathrm{Pin}_n(\mathbb{C})$, is the subgroup of the Clifford algebra consisting of products of vectors of norm ± 1. The Spin group, $\mathrm{Spin}_n(\mathbb{C})$, is the even (wrt the $\mathbb{Z}_2$-grading) part of the Pin group.

The units (under tensor product) of $Cl(V, Q)$, U, act on $Cl(V, Q)$: For $x \in U$, define $y \mapsto \alpha(x) \cdot y \cdot x^{-1}$ where α is the main involution. Here product means tensor product. The Lipschitz group, Γ, is the subgroup of the group of units which preserves the set of vectors under this action, therefore $\Gamma \curvearrowright V$ by construction. When $r \in \Gamma$ belongs to V, we have the familiar reflection formula

$$\begin{aligned}
\alpha(r) \cdot v \cdot r^{-1} &= \frac{1}{Q(r)}(-r \cdot v \cdot r) \\
&= \frac{1}{Q(r)} - r(rv + 2\langle v, r\rangle) \\
&= \frac{1}{Q(r)}(r \cdot r) \cdot v - \frac{2\langle v, r\rangle r}{Q(r)} \\
&= v - \frac{2\langle v, r\rangle}{Q(r)} r
\end{aligned}$$

where the second equality is equivalent to the defining clifford relation apart from characteristic 2. This action is orthogonal wrt Q, so we have a morphism

$$\Gamma \to O(V, Q)$$

which is surjective (theorem), inducing the exact sequence

$$1 \to \mathbb{C} \to \Gamma \to O(V, Q) \to 1$$

and also

$$1 \to \mathbb{C}^\times \to \Gamma^\times \to SO(V, Q) \to 1$$

The Pin group sits inside Γ, so by restriction, we have a map

$$\mathrm{Pin}_n(\mathbb{C}) \to O(V, Q)$$

[10] I haven't seen anywhere online covering this topic mention what is technically happening here (even I could be more precise here by spelling out what "action" means in each context). Maybe it is totally obvious, but it confused me for about an hour. We define the action of V on S. Then we consider an induced action of TV, the tensor algebra on V, on S. It is defined by repeated action, so $v \otimes w \cdot \psi := v \cdot (w \cdot \psi)$. This is a not-often-mentioned intermediate step that makes all of these equations actually make sense. Then we show that under this action by TV, any term of the form $v \otimes v - Q(v)\mathbf{1} \in TV$ acts trivially, so this action descends to the quotient $TV/(v^2 - Q(v)) = Cl(V, Q)$.

which is again surjective (in finite characteristic it is not so, but that is not my business), and whose kernel is now just ± 1:

$$1 \to \{\pm 1\} \to \mathrm{Pin}_n(\mathbb{C}) \to O(V,Q) \to 1$$

$$1 \to \{\pm 1\} \to \mathrm{Spin}_n(\mathbb{C}) \to SO(V,Q) \to 1$$

establishing Spin and Pin as double covers, and thus agreeing with our initial definition of Spin as a double cover.

Because we have identified Spin and Pin as subgroups of the Clifford algebra, the Clifford modules S and S' are also Spin and Pin modules. These modules are called the spin representations, and they also induce Lie algebra representations on $\mathfrak{so}(n,\mathbb{C})$.

To get some explicits formulas and to compute weights, we need to recall the algebra isomorphism[11] $\bigwedge^2 V \cong \mathfrak{so}(n,\mathbb{C}) = \mathfrak{so}(V,Q)$ via the map

$$v \wedge w \mapsto \Big(x \mapsto 2\big(\langle w,x\rangle v - \langle v,x\rangle w\big)\Big)$$

If we present V with the Lagrangian splitting and basis as described at the start of the entry, (again this depends on the parity of m), then the basis of $\mathfrak{so}(m,\mathbb{C})$ of $E_{ij} - E_{ji}$ is sent to $\alpha_i \wedge a_j \in \bigwedge^2 V$. In this case, the Cartan subalgebra is sent to $\alpha_i \wedge a_i$. We can also identify $\bigwedge^2 V \cong Cl^2(V)$, the degree 2 vector space of the Clifford algebra, by the map[12]

$$v \wedge w \mapsto \frac{1}{4}[v,w]$$

Then the action of $\mathfrak{h}$ is by

$$\begin{aligned}
(\alpha_i \wedge a_i) \cdot \psi &= \frac{1}{4}(\alpha_i a_i - a_i \wedge \alpha_i) \cdot \psi \\
&= \frac{1}{4}\alpha_i \cdot (a_i \cdot \psi) - \frac{1}{4}a_i \cdot (\alpha_i \cdot \psi) \\
&= \frac{\sqrt{2}}{4}\alpha_i \cdot (a_i \wedge \psi + 0) - \frac{\sqrt{2}}{4}a_i \cdot (0 + \iota(\alpha_i)\psi) \\
&= \frac{2}{4}\iota(\alpha_i)(a_i \wedge \psi) - \frac{2}{4}a_i \wedge \iota(\alpha_i)\psi \\
&= \frac{1}{2}\psi - a_i \wedge \iota(\alpha_i)\psi
\end{aligned}$$

If $\psi = a_{i_1} \wedge \cdots \wedge a_{i_k} \in \bigwedge^\bullet W$, then if $i \in \{i_1, \ldots, i_k\}$, the second term is equal to ψ, and $a_{i_1} \wedge \cdots \wedge a_{i_k}$ is an eigenvector of eigenvalue -1/2 and if not, then the second term is just 0, so we have an eigenvalue of $\frac{1}{2}$, so all such are weight vectors with weights $(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$.

If n is even, then $S_+ = \bigwedge^{\text{even}} W$ and $S_- = \bigwedge^{\text{odd}} W$ are invariant subspaces. These turn

[11] Let's not get into the algebra structure on $\bigwedge^2 V$ and proof of compatibility.

[12] But the second thing is not a subalgebra, so this must be only a linear identification. That's weird.

out to be irreps, and are called half-spin representations. Their elements are called Weyl spinors, and highest weights are $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2})$.

If n is odd, these representations are all irreducible: $\bigwedge^{\text{even}} W$ and $\bigwedge^{\text{odd}} W$ are not preserved by the action of $u \wedge w$, (recall u always acts by scalar multiplication). The highest weight is $(\frac{1}{2}, \ldots, \frac{1}{2})$. Elements of S are called Dirac spinors.

In both cases, note that $-1 \in \mathrm{Spin}_n(\mathbb{C})$ does not act trivially, so this does not[13] descend to a representation of $SO(n, \mathbb{C})$.

*Spin(3)-reps.

Consider $\mathbb{C}^3$ with the standard quadratic form $Q(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2$. We can choose a basis $\{e, f, u\}$ of $\mathbb{C}^3$ so that $\langle e, f\rangle = \langle u, u\rangle = 1$, $\langle e, e\rangle = \langle f, f\rangle = \langle u, f\rangle = \langle u, e\rangle = 0$. Then $Cl(V, Q)$ is generated by $1, e, f, u$ in their appropriate degrees, with the relations $e^2 = f^2 = 0$, $u^2 = 1$ and graded anti-symmetry. In particular, we can always consider the highest homogeneous degree to be 3, which is a 1-dimensional space.

*12/10/2023 Spin of a particle

From Dan Freed's Five Lectures on Supersymmetry.

In quantum mechanics, the Hilbert space of wavefunctions of a particle in $\mathbb{R}^m$ is modeled by $L^2(\mathbb{R}^m, W)$ where W is some representation of $\mathrm{Spin}(m)$, the double cover of $SO(m)$.

12/15/2023 Positive Grassmannian

From Youtube video "Lauren K. Williams: Cluster algebras and the amplituhedron - definition".

Recall the identification of the Grassmannian $Gr_k(n)$ with rectangular matrices of full rank, quotiented by left multiplication action of $GL(k)$. The identification is afforded by "row span" function into $Gr_k(n)$. Full rank implies the existence of at least one non-zero matrix minor. There are $\begin{pmatrix} n \\ k \end{pmatrix}$ Plucker coordinates for $Gr_k(n)$, corresponding to the choice of possible minors in the rectangular matrix representing a subspace. For example in $Gr_3(\mathbb{R}^5)$, we have a subspace represented by

$$\begin{pmatrix} 2 & 1 & 0 & 4 & 3 \\ -1 & 0 & 3 & 2 & 1 \\ 3 & 2 & 1 & 7 & 5 \end{pmatrix}$$

So the corresponding subspace is the span of $2e_1 + e_2 + 4e_4 + 3e_5$ and so on. The maximal rank of such a matrix is 3, so we examine all 3 x 3 minors. For example, the [123] minor

[13] Does this really suffice as justification? I'm thinking that when we define the action by TV, the degree 0 portion, which is just $\mathbb{C}$, should just act by scalar multiplication, in which case the action of -1 is non-trivial, which it would need to be to factor through the covering projection.

(indicating the columns being chosen) is -2.

In contrast, the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{pmatrix}$$

has rank 2, thus must have all zero 3 x 3 minors, and this is the case.

So we have coordinate functions to $\mathbb{R}^{\binom{n}{k}}$, which are not well-defined after quotient: If we act by left multiplication, the minors can be changed by an overall constant: The minors of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$ are -5,-10,-5, while the minors of $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$ are -10,-20,-10. In fact the minors will all change by a factor of the determinant of the acting matrix[14]. This means we have really found projective coordinates (also because we know that full rank implies the 0 vector is not in the image), these are the Plucker coordinates.

Definition: The positive (or totally non-negative) Grassmannian, $Gr^{\geq 0}(k, n)$, is the subset of points of the ordinary Grassmannian, $Gr(k, n)$, whose Plucker coordinates are all non-negative, up to sign.

For example, the matrix written above belongs to $Gr^{\geq 0}(2, 3)$, since its 3 coordinates have the same sign.

*12/20/2023 Amplituhedron is a polyhedron

Continuation of above entry. Let Z be a $n \times k + m$ matrix with $k + m \leq n$ (so Z is like a vertical triangle), whose $k + m \times k + m$ minors are all positive. Then define

$$\tilde{Z} : Gr^{\geq 0}(k, n) \to Gr(k, k + m)$$

$$C \mapsto [CZ]$$

The fact that Z has maximal minors positive implies that the resulting matrix is full rank, so the map is well defined.

Definition: The Amplituhedron, $\mathcal{A}_{n,k,m}(Z) := \tilde{Z}(Gr^{\geq 0}(k, n))$.

Note this is not interesting if Z is square, because then $\tilde{Z}$ is an isomorphism, and the amplituhedron reduces to the positive Grassmannian.

If $k = 1, m = 2$, then Z will be an $n \times 3$ matrix with positive maximal minors, and $\mathcal{A}_{n,1,2}(Z) \subset Gr(1, 3) = \mathbb{P}^2$ is a polygon in $\mathbb{P}^2$: The map $\tilde{Z}$ is from $Gr^{\geq 0}(1, n) \to Gr(1, 3)$. If we represent $C \in Gr^{\geq 0}(1, n)$ by a matrix $\begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}$ where a_i all have the same sign (because each one is a minor), then $\tilde{Z}(e_i) = Z_i$, the ith row of Z. Because Z has positive

[14] Is there a geometric way to prove that?

maximal minors, the Z_i's must be in convex position.

$\boxed{\text{IM NOT SURE HOW TO SEE THIS}}$

*1/10/2024 Slodowy Slice

Let $\mathfrak{g} = \mathfrak{sl}(3)$, and consider the nilpotent element $e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. By Jacobson-Morosov, we know that e can be included into an $\mathfrak{sl}(2)$-triple, $e, f, h \subset \mathfrak{g}$. Define $S = e + ker[f, -] \subset \mathfrak{g}$, the Slodowy Slice of the pair $(e, \mathfrak{g})$. In this case we can take

$$f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Let

$$X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathfrak{sl}(3)$$

so that $a + e + i = 0$. Then the action of $ad(f)$ is

$$ad(f)(X) = [f, X] = \begin{pmatrix} -c & 0 & 0 \\ -f & 0 & 0 \\ a - i & b & c \end{pmatrix}$$

So any $X \in ker(ad(f))$ has the form

$$X = \begin{pmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & a \end{pmatrix}$$

and the traceless condition implies $e = -2a$, so

$$= \begin{pmatrix} a & 0 & 0 \\ d & -2a & 0 \\ g & h & a \end{pmatrix}$$

Then the coset $S := e + ker(ad(f))$ is all matrices of the form

$$\begin{pmatrix} a & 0 & 1 \\ d & -2a & 0 \\ g & h & a \end{pmatrix}$$

which is our Slodowy slice. In this case, the G-orbit of e is transversal to "slice direction", $ker(ad(f))$: In the language of linear spaces it just means that they sum to the whole space: The G-orbit of e looks like

$$\left[\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, e \right] = \begin{pmatrix} g & h & i - a \\ 0 & 0 & -d \\ 0 & 0 & -g \end{pmatrix}$$

so we may obtain an arbitrary element of $\mathfrak{sl}(3)$ by adding something from $ker(ad(f))$. Further, the orbit of the Slodowy slice covers $\mathfrak{sl}(3)$ **HOW TO SHOW THIS.**

These properties are enjoyed by Slodowy slices in general: They are transversal slices to nilpotent orbits which cover the Lie algebra.

*Slodowy Slice as Hamiltonian reduction

From SURYA RAGHAVENDRAN's notes.

*1/10/2024 Odd theta functions

From Felder, Rimanyi, Varchenko: "Elliptic Dynamical Quantum Groups and Equivariant Elliptic Cohomology".

Define the <u>Jacobi odd theta</u> function[15]

$$\theta(z) = \frac{\sin(\pi z)}{\pi} \prod_{j=1}^{\infty} \frac{(1-q^j w)(1-q^j w^{-1})}{(1-q^j)^2}$$

where $q = e^{2\pi i \tau}$ is the elliptic nome and $w = e^{2\pi i z}$. We can calculate

$$\frac{d}{dz}\theta(z)$$

$$= \cos(\pi z) \prod_{j=1}^{\infty} \frac{(1-q^j w)(1-q^j w^{-1})}{(1-q^j)^2} + \frac{\sin(\pi z)}{\pi} \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^2} \frac{d}{dz}\Big[(1-q^j w)(1-q^j w^{-1})\Big]$$

$$= \cos(\pi z) \prod_{j=1}^{\infty} \frac{(1-q^j w)(1-q^j w^{-1})}{(1-q^j)^2} + \frac{\sin(\pi z)}{\pi} \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^2} \Big[(-2\pi i q^j w)(1-q^j w^{-1}) + (1-q^j w)(2\pi i q^j w^{-1})\Big]$$

$$\Rightarrow \frac{d}{dz}\bigg|_{z=0} \theta(z)$$

$$= 1 \cancel{\prod_{j=1}^{\infty} \frac{(1-q^j)(1-q^j)}{(1-q^j)^2}} + \cancel{\frac{\sin(\pi z)}{\pi} \prod_{j=1}^{\infty} \frac{(-2\pi i q^j)(1-q^j) + (1-q^j)(q^j 2\pi i)}{(1-q^j)^2}} = 1$$

Shifting $z \mapsto z+1$ preserves w and sends $\sin(\pi z) \mapsto -\sin(\pi z)$, so $\theta(z+1) = -\theta(z)$ (as opposed to the theta functions defined in previous entries, which were 1-periodic in z), and

$$\theta(z+\tau) = -e^{-\pi i \tau} e^{-2\pi i z} \theta(z)$$

so it is also τ-quasi-periodic in z.

[15]Idk the exact relation of this defn to the theta functions defined in 11/23/2023. It is very close to the formula which comes after applying the Jacobi triple product formula, but the versions I have seen don't have any denominators, so I'm not sure where this comes from.

Definition (Spaces of theta functions): Let $z \in \mathbb{C}^n, y, \lambda \in \mathbb{C}$ and define $\Theta_k^{\mp}(z, y, \lambda)$ to be the set of entire, symmetric functions $f(t_1, \ldots, t_k)$, such that the meromorphic function

$$g_-(t_1, \ldots, t_k) := \frac{f(t_1, \ldots, t_k)}{\prod_{j=1}^k \prod_{a=1}^n \theta(t_j - z_a)}$$

$$g_+(t_1, \ldots, t_k) := \frac{f(t_1, \ldots, t_k)}{\prod_{j=1}^k \prod_{a=1}^n \theta(t_j - z_a + y)}$$

satisfies

$$g(t_1, \ldots, t_i + r + s\tau, \ldots, t_k) = e^{\pm 2\pi i s(\lambda - ky)} g(t_1, \ldots, t_k)$$

Example: For $n = 1$ (let's just do -), $z \in \mathbb{C}$, so

$$g_- = \frac{f(t_1, \ldots, t_k)}{\prod_{j=1}^k \theta(t_j - z)}$$

Claim the function

$$\varphi_k^-(t; z, y, \lambda) = \prod_{j=1}^k \theta(\lambda - t_j + z - ky)$$

is a solution. This is clear because

$$\begin{aligned} w^{\pm 1}(\lambda - (t_i + r + s\tau) + z - ky) &= w^{\pm 1}(z) w^{\pm 1}(\lambda - (t_i + r + s\tau) - ky) \\ &= w^{\pm 1}(z) w^{\pm 1}(\lambda - ky) w^{\pm 1}(t_i + r + s\tau) \end{aligned}$$

while

$$w^{\pm 1}(t_i + r + s\tau - z) = w^{\pm 1}(t_i + r + s\tau) w^{\mp}(z)$$

so $\boxed{\text{IDK HOW TO SHOW THIS}}$

1/20/2024 Higher dimensional torus orbits on $Gr(k, n)$.

If $T = (\mathbb{C}^\times)^n \curvearrowright Gr(k, n)$ (or one can take cotangent bundle also), with the standard scaling of coordinate vectors, we know the fixed points are indexed by subsets of size k of the set $\{1, \ldots, n\}$, corresponding to the coordinate k-subspaces. Claim 1-dimensional orbits are of the form

$$\left\{ \langle a e_{i_1} + b e_{i_2}, e_{\ell_1}, \ldots, e_{\ell_{k-1}} \rangle \;\middle|\; a, b \in \mathbb{C}^\times \right\}$$

so there are $\binom{n}{2} \cdot \binom{n}{k-1}$ of these, since to specify such an orbit, you first choose the 2 coordinate vectors to be added together, of which you may choose from n, and then choose the remaining $k-1$ coordinate vectors. This orbit is clearly fixed by T since all but the j coordinate vectors spans are preserved, and $a e_{i_j} + b e_{i'_j} \mapsto a' e_{i_j} + b' e_{i'_j}$, so the corresponding subspace still belongs to the orbit. As a variety, this orbit's closure (closure so we can get the points $a = 0, b \neq 0$ and $a \neq 0, b = 0$. I won't say closure anymore throughout this entry.) is a copy of $\mathbb{P}^1$, since scaling a and b by an overall constant doesn't change the subspace. It can be shown that all 1-dimensional orbits are of this form. Following this, we obtain d-dimensional orbits in $Gr(k,n)$, for $d \leq n$, of the form

$$(\star) \qquad \Big\{a_1 e_{i_1} + a_2 e_{i_2} + \cdots + a_{d+1} e_{i_{d+1}}, e_{\ell_1}, \ldots, e_{\ell_{k-1}}\Big\}$$

These are similarly copies of $\mathbb{P}^d$. But there are other higher dimensional orbits, namely for 2-dimensional orbits, we could split up the summation (let's just consider $Gr(2,n)$ to ease the notation):

$$\Big\{ae_i + be'_i, ce_j + de'_j\Big\}$$

This is also preserved under the action of T and is a 2-dimensional variety, but is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, since each individual coordinate is invariant under scaling. So all 1-dimensional orbits are of the form ($\star$), but for $d > 1$, we have these others. I think in general, each d-dimensional orbit yields an integer partitions of d. For example, a 5-dimensional orbit corresponding to $5 = 3+1+1$, representing a bounded orbit isomorphic to $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$, is

$$\Big\{ae_{i_1} + be_{i_2} + ce_{i_3} + de_{i_4}, fe_{j_1} + ge_{j_2}, he_{k_1} + ie_{k_2} \;\Big|\; a,b,c,d,f,g,h,i \in \mathbb{C}^\times\Big\}$$

Obviously this is not the full story though, since this orbit lies in $Gr(3,n)$. If we apply the same idea to $Gr(4,n)$ then we consider the family

$$\Big\{ae_{i_1} + be_{i_2} + ce_{i_3} + de_{i_4}, fe_{j_1} + ge_{j_2}, he_{k_1} + ie_{k_2}, e_{\ell_1} \;\Big|\; a,b,c,d,f,g,h,i \in \mathbb{C}^\times\Big\}$$

Because we look at a 4-dimensional subspace, we have an extra choice of the final coordinate vector, there are n orbits corresponding to the partition $5 = 3+1+1$ (also assuming n is large enough). This is interesting, I should investigate these combinatorics more. I think it would be interesting to have a complete description.

1/22/2024 Quiver Grassmannians

From Youtube video "Martina Lanini (Università di Roma Tor Vergata): GKM-Theory for cyclic quiver Grassmannians".

Let $\hat{A}_n$ be the affine type A quiver. Let $(M, \vec{M})$ be a quiver representation of $\hat{A}$, so M is a $\mathbb{Z}_n$ graded vector space and $\vec{M}$ is an endomorphism of M such that $\vec{M} M_i \subset M_{i+1}$. Such a representation is called <u>nilpotent</u> if $\vec{M}$ is nilpotent. We define the <u>quiver Grassmannian</u> as a certain moduli space depending on a chosen $\hat{A}$-rep, $(M, \vec{M})$ and a dimension vector $d \in \mathbb{Z}^n_{\geq 0}$

$$Gr_d(M)\Big\{(N_i)_{i \in \mathbb{Z}_n} \;\Big|\; N_i \in Gr(d_i, M_i), \quad \vec{M}(N_i) \subset N_{i+1}\Big\}$$

Really this definition can be given for any quiver, but in this entry we only focus on affine type A. In English: at every vertex, choose a subspace of the vector space there of dimension d_i, and make sure you can compose the maps (subrepresentation condition). This is just the variety which parameterizes all d-dimensional subrepresentations of M (hence moduli space). If $d_i > \dim(M_i)$ for any i, then this variety is empty.

Example: Consider $\hat{A}_1$ (Jordan quiver) and choose the representation $M = \mathbb{C}^n$ and $\vec{M} = 0$ and dimension vector d. Then

$$Gr_d(M) = \left\{ V \in Gr(d, \mathbb{C}^n) \;\middle|\; \emptyset \right\} = Gr(d, \mathbb{C}^n)$$

Consider $Q = A_n$, the non-affine type A quiver of length n. Choose $M_i = \mathbb{C}^{n+1}$ for every i and $\vec{M}_i = Id$, and an increasing dimension vector d. Then

$$Gr_d(M) = \left\{ (N_i)_{i \in \mathbb{Z}_n} \;\middle|\; N_i \in Gr(d_i, \mathbb{C}^{n+1}),\ N_i \subset N_{i+1} \right\} = \mathcal{F}l(d, \mathbb{C}^{n+1})$$

We could have also let $Q = \hat{A}_n$ and choose every map to be identity except the map connecting the final vertex to the first, letting that one be 0, to get the same result.

1/25/2024 Rational/Trigonometric/Elliptic

From "$\hbar$-deformed Schubert Calculus in equivariant cohomology, K-theory, and elliptic cohomology" by Richard Rimanyi. I'll be reading more of this this semester.

Define yet another theta function

$$\theta(x) = (x^{1/2} - x^{-1/2}) \prod_{s=1}^{\infty} (1 - q^s x)(1 - q^s/x)$$

on a double cover of $\mathbb{C}$. The <u>trigonometric limit</u> is

$$\lim_{q \to 0} \theta(x) = (x^{1/2} - x^{-1/2})$$

If we change variables

$$x^{1/2} = e^{iy}$$

then

$$\begin{aligned} x^{1/2} - x^{-1/2} &= \cos(y) + i\sin(y) - \cos(-y) - i\sin(-y) \\ &= 2i\sin(y) \end{aligned}$$

hence the name. We often abuse notation and call the new variable x, so we say $\theta(x) \to \sin(x)$ is the trigonometric limit. If we further approximate $x \to 0$, we have $\theta(x) = x$, which is called the <u>rational limit</u>.

These functions can be used to define extraordinary cohomology theories through formal group laws:

$$(\theta(x), \theta(y)) \to \theta(xy), \quad (\sin(x), \sin(y)) \mapsto \sin(x+y), \quad (x, y) \mapsto x + y$$

1/28/2024 $\mathrm{Ell}_T^\bullet(\mathbb{P}^1)$.

From "$\hbar$-deformed Schubert Calculus in equivariant cohomology, K-theory, and elliptic cohomology" by Richard Rimanyi.

For GKM spaces, the equivariant elliptic cohomology is identified via equivariant localization with $|X^T|$-tuples of sections of line bundles on $E^{\mathrm{rk}T}$, for E some elliptic curve, subject to the GKM moment graph constraints. Let $X = \mathbb{P}^1$, so $T = (\mathbb{C}^\times)^2$. Then $\mathrm{Ell}_T^\bullet(\mathbb{P}^1)$ consists of pairs of functions on $E^2 = \left(C^\times/q^{\mathbb{Z}}\right)^2$ (of course these cannot be holomorphic functions), (f_1, f_2), subject to the single moment graph divisibility constraint, $f_1|_{z_1=z_2} = f_2|_{z_1=z_2}$. Let

$$\theta(x) = (x^{1/2} - x^{-1/2}) \prod_{s=1}^{\infty}(1 - q^s x)(1 - q^s/x)$$

as in the entry above. Then

$$(\theta(z_2/z_1), 0), \qquad \left(\theta'(1)\frac{\theta(z_1\mu_2/z_1\mu_1)}{\theta(\mu_2/\mu_1)}, \theta'(1)\frac{\theta(z_1\hbar/z_2)}{\theta(\hbar)}\right)$$

are two elements of the equivariant elliptic cohomology, as they obviously satisfy the divisibility requirement. One can also guess elliptic weight functions which yield these classes upon restriction:

$$\frac{\theta(z_1\hbar\mu_2/t\mu_1)\theta(z_2/t)}{\theta(\hbar\mu_2/\mu_1)}, \qquad \theta'(1)\frac{\theta(z_1\hbar/t)\theta(z_2\mu_2/t\mu_1)}{\theta(\hbar)\theta(\mu_2/\mu_1)}$$

and easily check that their restrictions yield the two pairs above. This fact obviates the need to even check restrictions.

Hm that was not very satisfying though because we didn't actually do any computation, the divisibility is obvious and used nothing about theta functions. Also the formal setup is not exactly clear: what are the line bundles in question?

*1/30/2024 XXX Spin chains

I learned this from Andrey Smirnov.

A(n XXX) <u>spin chain</u> of length n is an element of the vector space

$$\mathcal{H} = \underbrace{\mathbb{C}^2 \times \cdots \times \mathbb{C}^2}_{n \text{ times}}$$

These model length n chains of "spin up" and "spin down" particles living on a line. We call $e_1 \in \mathbb{C}^2$ the "spin up" basis vector and e_2 the "spin down" vector. So we imagine a horizontal line on which we have placed n particles, each of which is some (complex) linear combination of spin up and spin down. Label the ith particle as a_i. We can define a Hamiltonian at every particle as measuring the where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli sigma matrices, so $H_{a_i} \in End(\mathbb{C}^2 \otimes \mathbb{C}^2)$. The Hamiltonian for the whole spin chain is then

$$H := \sum_{i=1}^{n} H_{a_i}$$

For example if we have three particles in the chain, then

2/8/2024 Hopf algebra representations can be canonically tensored

I learned this (partly) from Andrey Smirnov, who discussed this in the case of $U_\hbar(\mathfrak{sl}_2)$, and I figured it probably applies to all Hopf algebras.

For some examples of Hopf algebra computations, see entry 5/28/2023 and its antecedent. In a Hopf algebra we have comultiplication, antipode, and counit k-algebra morphisms $\Delta : U \to U \otimes U$, $S : U \to U$ and $\epsilon : U \to k$ satisfying some compatibility conditions. Suppose we have a finite dimensional algebra representation of U, so a k vector space V with a homomorphism of k-algebras $\varphi : U \to End(V)$, where $End(V)$ is equipped with composition. Then the claim is that $V \otimes V$ is canonically equipped with a U-module structure. In particular we must exhibit a k-algebra map

$$U \to End(V \otimes V) \cong End(V) \otimes End(V)$$

this isomorphism doesn't always hold, but it does in our case. Then we can compose

$$U \xrightarrow{\Delta} U \otimes U \xrightarrow{\varphi\otimes\varphi} End(V) \otimes End(V)$$

which is a composition of k-algebra homomorphisms as desired. So far we did not need any compatability conditions. But to define an action of U on, for example, $V \otimes V \otimes V$, there is a choice to be made: We have to first apply Δ of course, but then there is a choice of which factor to act on in order to obtain something which can act on a 3-tensor:

$$\begin{tikzcd}
 & U \otimes U \arrow[dr, "\Delta\otimes 1"] & \\
U \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & U \otimes U \otimes U \\
 & U \otimes U \arrow[ur, "1\otimes\Delta"'] &
\end{tikzcd}$$

But the coassociativity of comultiplication condition implies that these choices are the same (we have also implicitly identified $(U \otimes U) \otimes U$ with $U \otimes (U \otimes U)$, which is canonical.). In this fashion, we can continually just apply comultiplication to induce representations on $V^{\otimes n}$ (I think technically you would need to show that $1 \otimes \Delta$ makes $U \otimes U$ into a Hopf algebra and so on in order to show this, but let's not get into that.)

2/10/2024 Tensor powers of $U_\hbar(\mathfrak{sl}_2)$

I also learned this from Andrey Smirnov. This entry can be thought of as a continuation of the above. The quantized enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ is an algebra generated by E, F, H subject to the relations

$$[H, E] = 2E \quad [H, F] = -2F, \quad [E, F] = \frac{H - H^{-1}}{\hbar - \hbar^{-1}}$$

There is a 2-dimensional representation, V,

$$E = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad F = \begin{pmatrix} & \\ 1 & \end{pmatrix}, \quad H = \begin{pmatrix} \hbar & \\ & \hbar^{-1} \end{pmatrix}$$

With this assignment, we can calculate

$$H - H^{-1} = \begin{pmatrix} \frac{\hbar^2 - 1}{\hbar} & \\ & \frac{1 - \hbar^2}{\hbar} \end{pmatrix}$$

$$\frac{H - H^{-1}}{\hbar - \hbar^{-1}} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = [E, F]$$

sometimes called the tautological representation. In this case, the Hopf algebra structure deforms to:

$$\Delta(E) = E \otimes 1 + H \otimes E, \quad \Delta(F) = F \otimes H^{-1} + 1 \otimes F, \quad \Delta(H) = H \otimes H$$

I'm not sure how to see this as a deformation of the usual relations though. Anyway, it means that for example, we can calculate the action of, e.g., E on the basis vector $e_1 \otimes e_2 \in V^{\otimes 2}$ as:

$$\begin{aligned} E(e_1 \otimes e_2) \equiv \Delta(E)(e_1 \otimes e_2) &= (E \otimes 1 + H \otimes E)(e_1 \otimes e_2) \\ &= E(e_1) \otimes e_2 + H(e_1) \otimes E(e_2) \\ &= 0 + \hbar e_1 \otimes e_1 \end{aligned}$$

and so on. We have thus constructed an action of $U_\hbar(\mathfrak{sl}_2)$ on the space of XXX spin chains, and the significance of this will be made clear in a subsequent entry.