What's in a scheme?

Abstract

Schemes are one of the fundamental objects of algebraic geometry. However, they have a reputation of being abstract and therefore difficult to understand as well as being difficult to motivate. In this talk, we will go on a journey from manifolds to varieties and ultimately to schemes, highlighting the natural connection and progression of ideas which led to the development of schemes. After highlighting some basic notions of schemes, I want to talk about the phenomenology of schemes, paying particular attention to illuminating examples.

Overview of talk

The structure of the talk will be roughly following the flow of the following diagram:

$$\textbf{Manifolds} \longrightarrow \textbf{Varieties} \longrightarrow \textbf{Schemes}.$$

I want to highlight the continuity and flow of ideas that led to the development of schemes. My plan then once we get to schemes then is to start off with some basic theory, peppered with examples. Then we will move on to discuss the phenomenology of schemes. I want to finish the talk by discussing the ultimate, modern definition of what a variety is.

Manifolds

We start off with a review of the basic concept of manifolds. The basic conceit of manifolds is that we want to glue together topological spaces that locally look like subsets of affine space $\mathbb{R}^n$ or $\mathbb{C}^n$. That is, we have some charts $(U_\alpha, \varphi_\alpha)$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ (or $\mathbb{C}^n$) $U_\alpha \to \varphi(U_\alpha)$ homeomorphisms, and we glue together these charts via transition maps $\tau_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$: $\varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\beta \cap U_\beta)$ that are diffeomorphisms (or homeomorphisms if you're weird).

So the idea that we get from manifolds is gluing: how do we glue things that are locally flat space together and how do make sure things are compatible on this space, i.e. how do we define functions on this space? The idea of gluing is exactly what allows us to describe and work with (smooth) functions on this space.

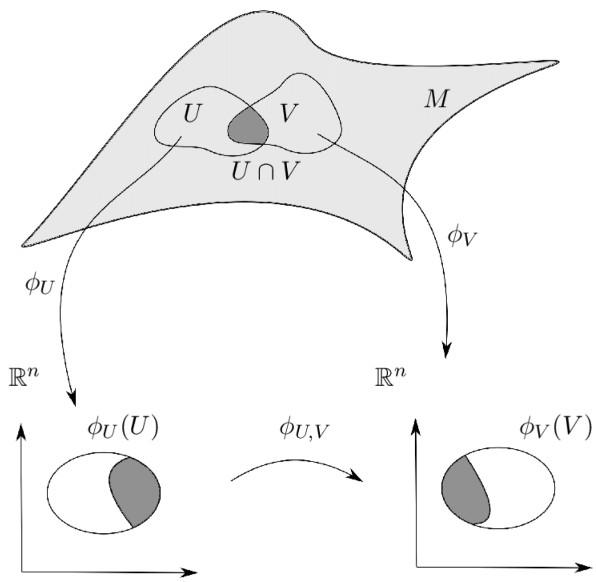


Figure 1: Charts on a manifold

Varieties

Next, I want to move on to the world of varieties, and from here things get more algebraic. Classically speaking, a variety is a **quasiprojective variety**, that is, a closed subset of an open subset in projective space (under the Zariski topology). Some details of these varieties are spelled out below.

Definition 1. Affine n-space $\mathbb{A}^n = k^n$. $\mathbb{A}^n$ is endowed with the **Zariski topology** where the closed sets are exactly

$$Z(S) = \{a \in \mathbb{A}^n \mid f(a) = 0, \forall f \in S\}$$

where $S \subset k[x_1, \ldots, x_n]$.

An **affine variety** X is then just a closed subset of $\mathbb{A}^n$ under the Zariski topology.

Of course, in algebraic geometry, we realized that it was better to not just work in affine space, but rather projective space as well. Working in projective space, we get the concepts of projective and quasiprojective varieties.

Definition 2. $\mathbb{P}^n = k^{n+1} \setminus 0/x \sim \lambda x$. $\mathbb{P}^n$ has projective coordinates $[z_0, \ldots, z_n]$. Given a homogeneous polynomial $F \in k[x_0, \ldots, x_n]$, we note that the locus

$$\{[z_0, \ldots, z_n] \in \mathbb{P}^n \mid F(z_0, \ldots, z_n) = 0\}$$

is well defined because $F(\lambda z_0, \ldots, \lambda z_n) = \lambda^k F(z_0, \ldots, z_n) = \lambda^k \cdot 0 = 0$.

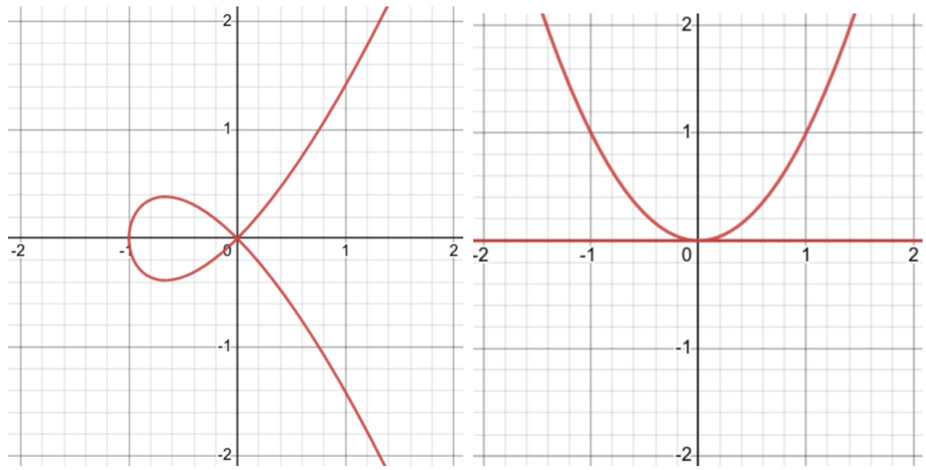


Figure 2: Some affine varieties in $\mathbb{A}^2$

A closed subset $X \subset \mathbb{P}^n$ is defined as the vanishing of a finite number of homogeneous polynomials. A **projective variety** X is then just defined to be a closed subset of $\mathbb{P}^n$. A **quasiprojective variety** X is defined to be a closed subset of an open subset in $\mathbb{P}^n$.

Note that affine varieties are still quasiprojective because we can pick an affine subset $U \subset \mathbb{P}^n$ as the set $\{[z_0, \dots, z_n] \mid z_0 \neq 0\} = \{[1, x_1, x_2, \dots, x_n]\} \simeq \mathbb{A}^n$, this is an open set in projective space, and then an affine variety would be a closed set in an open subset of $\mathbb{P}^n$ and is thus quasiprojective. In a similar vein to this example, we can see that we actually glued together affine varieties together to form a quasiprojective variety much like how a manifold is glued together from affine varieties.

Coordinate rings

Now I will review some basic notions surrounding coordinate rings. These coordinate rings will be the first step to getting to schemes—in the world of schemes, coordinate rings will be built right into the theory in the form of sheaves and affine schemes.

Recall that every affine variety $X \hookrightarrow \mathbb{A}^n$ is associated to a radical ideal $\sqrt{I}$. We define the **coordinate ring** of the affine variety X to be $k[X] = k[x_1, \dots, x_n]/\sqrt{I}$. For any open set $U \subset X$, we define

$$k[U] = \left\{ \frac{f}{g} \;\middle|\; f, g \in k[X], g|_U \neq 0 \right\}.$$

For a quasiprojective variety, we do something similiar, but we restrict to homogeneous polynomials (since homogeneous polynomials is what we care about in projective space).

Aside: varieties in the sense of FAC

Now before we move on to the story of schemes, I want to say that there is an intermediate generalization of varieties by Serre. Notice how in the above picture, the definition of variety is not intrinsic. We defined them as lying in some projective space $\mathbb{P}^n$, which means that we require a choice of coordinates. This is very much unlike the situation with manifolds which are defined without such reference to coordinates. Serre upgraded the classical definition of varieties by adding in the use of sheaves which allows us to glue together affine varieties more intrinsically (I will not go into detail about what Serre did).

Schemes

Now that we have that long-winded story about the classical picture down. We can now move on to schemes. Just like how we started with affine varieties, we will start off our discussion of schemes with affine schemes (which will look very similar to those who know commutative algebra).

Definition 3. For a commutative ring A, $\operatorname{Spec} A = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal in } A\}$. We can give $\operatorname{Spec} A$ with the **Zariski topology** in a similar way to what we did with $\mathbb{A}^n$: we define the closed subsets of $\operatorname{Spec} A$ to be sets

$$V(I) = \{\mathfrak{p} \mid I \subset \mathfrak{p}\}.$$

On $\operatorname{Spec} A$, we can define a sheaf $\mathcal{O}_A$ (yes, I am abusing notation) in the following the manner: For an open set $U \subset \operatorname{Spec} A$, $\mathcal{O}(U)$ is defined to be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

subject to the conditions that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ and for all $\mathfrak{p} \in U$ there is a neighborhood V of $\mathfrak{p}$ and $a, f \in A$ such that for all $\mathfrak{q} \in V$ $f \notin \mathfrak{q}$ and

$$s(\mathfrak{q}) = \frac{a}{f} \in A_{\mathfrak{q}}.$$

Now that I've given the stuffy definition that no one understands, let me tell you how real people in the real world thinks of the sheaf $\mathcal{O}_A$:

Theorem 1. Let A, $\operatorname{Spec} A$, $\mathcal{O} = \mathcal{O}_A$ be as above.

i) $\mathcal{O}_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.

ii) Define $D(f)$ to be the complement of the set $V((f))$, i.e. $D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$. Then $\mathcal{O}(D(f)) \simeq A_f$.

iii) $\Gamma(\operatorname{Spec} A, \mathcal{O}) \simeq A$.

Definition 4. An **affine scheme** is the pair $(\operatorname{Spec} A, \mathcal{O}_A)$ for some commutative ring A.

It's now a good time here to pause and look at some examples. These examples will show you that already affine scheme captures all of the affine varieties.

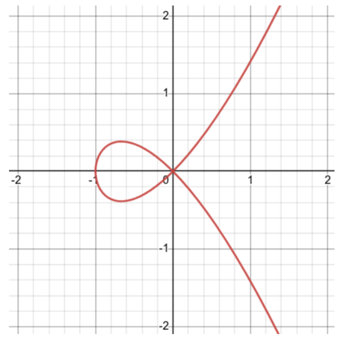


Figure 3: $\operatorname{Spec} k[x_1, \ldots, x_n]/(y^2 - x^3 - x^2)$

Example 1. $\operatorname{Spec} k[x] = \mathbb{A}^1$. $\operatorname{Spec} k[x_1, \ldots, x_n] = \mathbb{A}^n$. (For the experts, yes I am abusing notation.)

So I believe this example requires a little convincing. The classic Nullstellensatz tells us that the closed points (i.e. maximal ideals) of $\operatorname{Spec} k[x_1, \ldots, x_n]$ consists of ideals $(x_1 - a_1, \ldots, x_n - a_n)$ for all $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$, so the closed points correspond to the classic picture. But now we have certain extra points, namely those prime ideals which are not maximal, e.g. (0), $(y - x^2)$, $(y^2 = x^3 + x^2)$, etc. These points are the spooky points known as **generic points**. More precisely, (0) is the **generic point** of $\mathbb{A}^n$, and for each nonmaximal prime $\mathfrak{p}$, it corresponds to an irreducible subvariety X of $\mathbb{A}^n$, so the point $\mathfrak{p}$ is the generic point of X. The closure of these generic points are the varieties that they represent.

Example 2. Every affine variety X is an affine scheme. For the first example, say you have $Z(f) \subset \mathbb{A}^n$. This affine variety can be identified with the scheme

$$\operatorname{Spec}(k[x_1, \ldots, x_n]/(f)).$$

For each radical ideal $\sqrt{I} \subset A$, the variety that it corresponds to is identified with the scheme

$$\operatorname{Spec}(k[x_1, \ldots, x_n]/\sqrt{I}).$$

Example 3. $\operatorname{Spec} k[x_1, \ldots, x_n]/(y^2 - x^3 - x^2)$ is a scheme.

General schemes

Knowing what an affine scheme is will now allow us to define what a general scheme is. They are going to be basically spaces that are glued together from affine schemes.

Before we define what a scheme is, it is easier if we first define locally ringed spaces and maps between them.

Definition 5. A **locally ringed space** consists of the data $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that the stalk $\mathcal{O}_{X,x}$ is a local ring.

A **morphism of locally ringed spaces** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of the data $(f, f^{\#})$ where $f : X \to Y$ is a map of topological spaces and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves such that it respects the local structure, that is:

$$f^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a map of local rings (i.e. it sends max ideals to max ideals).

Definition 6. A **scheme** is then a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}$ such that each $(U_i, \mathcal{O}_X \mid_{U_i})$ is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_A)$ for some A as locally ringed spaces.

So the slogan here is that schemes are glued together from affine schemes. Speaking of gluing, our first example will be an example of gluing.

Example 4. On schemes, there is a well-defined procedure that we can use to glue two schemes together by some open set. So let k be a field, and look at two copies of $\mathbb{A}^1$, i.e. let $X_1 = X_2 = \mathbb{A}^1$ be two schemes. Let $U_1 = U_2 = \mathbb{A}^1 \setminus 0$ (here I am abusing notation, and 0 is actually the point (x)). We can actually glue X_1 and X_2 together by identifying U_1 and U_2 together to get the line with two origins.

We can check here clearly that this is a scheme because the complement of either origin point is isomorphic to $\mathbb{A}^1$, but the whole thing is not an affine scheme!

Projective space

So we have some analogs of affine space and we know what a general scheme is, but we still need to reproduce the phenomenon of projective space. To do this, we're going to need the Proj construction.

Definition 7. Let $S = \bigoplus_d S_d$ be a graded ring and let $S_+ = \bigoplus_{d>0} S_d$. $\operatorname{Proj} S$ is going to be the set of homogeneous prime ideals $\mathfrak{p}$ in S that do not contain all of S_+. Similar to $\operatorname{Spec} A$, $\operatorname{Proj} S$ has the Zariski topology defined by letting the closed sets be exactly

$$V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{a} \subset \mathfrak{p}\}$$

for $\mathfrak{a}$ a homogeneous ideal in S.

We can define a sheaf of rings $\mathcal{O}$ in an exactly analogous way to how we defined the sheaf on affine schemes. An analogous theorem about how $\mathcal{O}$ works also hold, but for sake of brevity, I will omit it.

Definition 8. If A is a ring, the **projective n-space** over A is defined to be the scheme

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n].$$

In particular, if k is an algebraically closed field, then $\mathbb{P}^n_k$ is going to be a scheme whose closed points corresponds precisely to the classical projective n-space.

Aside: an important notion

So one of the great insights of Grothendieck was that we should really do algebraic geometry in the relative setting, and that we should really consider schemes are being over some base scheme S. I will explain this.

Definition 9. Let S be a fixed scheme. A **scheme over** S is a scheme X together with a map $X \to S$. If X and Y are schemes over S, then a map $X \to Y$ of schemes over S (also known as S-morphism) is a morphism $f : X \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

commutes. The category of schemes over S is denoted $\mathfrak{Sch}(S)$. If A is a ring, then we will abuse notation and denote $\mathfrak{Sch}(A)$ to be the category of schemes over $\operatorname{Spec} A$.

Note that the category $\mathfrak{Sch}(\mathbb{Z})$ is actually the category of all schemes.

Proposition 1. Let k be an algebraically closed field. There is a natural fully faithful functor F from the category of varieties over k, $\mathfrak{Var}(k)$ to the the category of schemes over k, $\mathfrak{Sch}(k)$. For any variety V, the set of closed points of $F(V)$ is going to be homeomorphic to V as topological spaces, and the sheaf of regular functions on V is going to be isomorphic to the restriction of the structure sheaf of $F(V)$ restricted to its closed points.

What the above proposition says is that we can regard every variety as a scheme, and indeed we can recover the variety itself by just restricting our attention to the closed points.

Some phenomenology

The slogan to keep in mind for this section is that schemes are in some way dual to rings. So ideas from rings are going to have an associated idea in schemes. We start off with one of the most important notions between rings and schemes, that of noetherian rings/schemes.

Definition 10. A scheme X is **locally noetherian** if it can be covered by open affine subsets $\operatorname{Spec} A_i$ such that each A_i is a noetherian ring. X is **noetherian** if in addition to being locally noetherian, it is also quasi-compact (or compact for normies).

Clearly we have that every (quasi-projective) variety is a noetherian scheme. In fact, I have it on good authority that basically every scheme that people care about in their day to day is noetherian or at least locally noetherian.

Exercise 1. Try to guess the definition of what it means for a scheme to be

- **integral**,

- **reduced**,
- **normal**,
- **irreducible**,
- **connected**.

Definition 11. A morphism of schemes $f : X \to Y$ is **locally of finite type** if there is a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$ where each A_{ij} is a finitely generated B_i algebra. f is of **finite type** if in addition each $f^{-1}(V_i)$ can be covered by a finite number of the U_{ij}'s.

Most of the time, we want to just think of schemes are over some field k (by which I really mean of course $\operatorname{Spec} k$). I then think of this finite type definition as saying that the scheme is like some kind of finite dimensional manifold. So you can think of this as some kind of technical defintion that allows us to actually do geometry.

Example 5. Every quasiprojective variety is of finite type over k.

Example 6. A simple example of a scheme not of finite type is simply

$$\operatorname{Spec} k[\{y_\alpha\}_{\alpha \in A}]$$

for some infinite set A. This is kind of a silly example.

If $x \in X$ is a point of a variety with local ring $\mathcal{O}_x$, then

$$\operatorname{Spec} \mathcal{O}_x$$

is an integral noetherian scheme which is not in general of finite type over k.

An interesting phenomenon of schemes is that a closed set of a scheme can have many different subscheme structures.

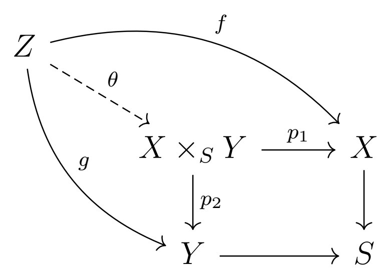
Example 7. Let $A = k[x, y]$, then $\operatorname{Spec} A = \mathbb{A}^2$ is the affine plane over k. The ideal $\mathfrak{a} = (xy)$ gives a reducible subscheme that's the union of the x and y-axes. The ideal $\mathfrak{a} = (x^2)$ gives a subscheme structure to the y-axis with nilpotents. The ideal $\mathfrak{a} = (x^2, xy)$ gives another subscheme structure on the y-axis, but this one only has nilpotents at the origin. This nilpotent is then an **embedded point** for this subscheme.

The next construction is going to be another example of the general theme of 'rings and schemes are dual to each other.' Recall that for rings we have this notion of coproducts (I'm sorry for the overly categorical language). More specifically, if we have algebras over a common ring R, that is we have morphism $R \to A$ and $R \to B$, we can then define the tensor product $A \otimes_R B$, and for general rings R and S we can simply view them as algebras over $\mathbb{Z}$ and so we can define $R \otimes S = R \otimes_{\mathbb{Z}} S$. Now schemes are dual to rings, so we can dualize the construction above: coproduct now becomes a product, that is, we're going to talk about product of schemes.

Definition 12. Let X and Y be schemes over a base scheme S. The **fibered product** of X and Y along S, denoted $X \times_S Y$ is a scheme, together with morphisms $p_1 X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ which makes the following diagram commute.

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p_1} & X \\ \big\downarrow{\scriptstyle p_2} & & \big\downarrow \\ Y & \longrightarrow & S \end{array}$$

We ask further that $X \times_S Y$ be universal. That is, given morphisms $f : Z \to X$ and $g : Z \to Y$ of S-schemes, we as that there be a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$. That is we want morphism θ such that the following diagram commutes.



If we're just talking about schemes without reference to any base schemes, then, as in the case of rings, we're just going to take our base scheme to be $\operatorname{Spec} \mathbb{Z}$.

Now the real way to think about this construction is to first think about the affine case. In the case that $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $S = \operatorname{Spec} R$, we just have that

$$X \times_S Y = \operatorname{Spec} A \otimes_R B.$$

For general schemes the construction is something similar, except that we're now going to have to glue together on the affine patches. The details are horrible. If the base scheme $S = \operatorname{Spec} R$ for some ring R, then often times, instead of writing $X \times_{\operatorname{Spec} R} Y$, we will often just write $X \times_R Y$.

Example 8. A simple example is

$$\begin{aligned} \mathbb{A}^n \times_k \mathbb{A}^m &= \operatorname{Spec}(k[x_1, \dots, x_n]) \times_k \operatorname{Spec}(k[y_1, \dots, y_m]) \\ &= \operatorname{Spec}(k[x_1, \dots, x_n] \otimes_k k[y_1, \dots, y_m]) \\ &= \operatorname{Spec} k[x_1, \dots, x_{n+m}] = \mathbb{A}^{n+m} \end{aligned}$$

as expected.

Here is a GIT-y example.

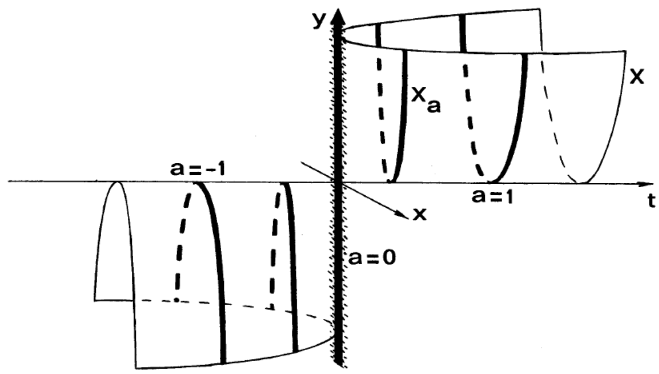


Figure 4: A family of parabolas parameterized by k

Example 9. So we let k be an algebraically closed field as usual. Let

$$X = \operatorname{Spec} k[x, y, t]/(ty - x^2).$$

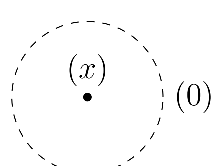
Let $Y = \operatorname{Spec} k[t]$ and $f : X \to Y$ be the morphism defined by the associated ring map $k[t] \to k[x, y, t]/(ty - x^2)$. Now, Y is of course the affine line $\mathbb{A}^1$ and we can identify the closed points of Y with just k. For each $a \in k$, $a \neq 0$, the fiber X_a is the plane curve $ay = x^2$ in $\mathbb{A}^2$, so just a parabola. But something interesting happens in the limit as $a \to 0$. Here, we have that X_0 is the nonreduced scheme given by $x^2 = 0$ in $\mathbb{A}^2$. This is an example of a jump phenomenon in GIT, and our scheme X is a family of parabolas parameterized by k.

An analysis-y example.

Example 10. As we know, neighborhoods in the Zariski topology are humongous, so we don't really get any infinitesimal information around any point, at least in the classical world. However, in the world of schemes, there are actually ways to get around such issues. Consider the scheme

$$\operatorname{Spec} k[[x]].$$

We know that $k[[x]]$ is a local PID with only two prime ideals: (0) and (x). Here (x) is the only maximal ideal while (0) is a nonmaximal prime. We can visualize this scheme as a disk:



The center point of the disk corresponds to the closed point (x), while the rest of the disk part corresponds to the generic point (0). This example is related to other phenomenon such as valuative criteria and Zariski's Main Theorem.

The Zariski topology is famously not Hausdorff, and it has way too many compact sets, so one can be forgiven for thinking that algebraic geometry just carry no analytic information. However, just like the example before, this is really not the case. Algebraic geometry does have an analog for Hausdorff and compact, namely separatedness and properness.

Definition 13. Let $f : X \to Y$ be a morphism of schemes. The **diagonal morphism** is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with both projection maps is the identity map. f is **separated** if Δ is a closed immersion. In this case, we say that X is **separated** over Y. If we just say **separated**, then we mean separated over $\mathbb{Z}$ (or over k if the context allows).

Remark. The line with two origins defined below is not separated.

Example 11. Any variety over an algebraically closed field is separated.

Definition 14. A morphism $f : X \to Y$ is **proper** if it is separated, of finite type and satisfy the following property:

- for any morphism $Y' \to Y$, the corresponding morphism

  $$f' : X \times_Y Y' \to Y'$$

  is a closed map.

The above property is called f being **universally closed**.

Example 12. The affine line $\mathbb{A}^1$ is not proper over k. But, as we know from classical algebraic geometry, $\mathbb{P}^n$ is proper. In fact, any projective variety over a field is proper.

Speaking of projective space, in the scheme world, there is an analogous notion of projectivizing schemes just as in variety world.

Definition 15. If Y is any scheme, then the **projective n-space** over Y, denoted $\mathbb{P}^n_Y$ is the scheme $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Y$.

A morphism $f : X \to Y$ is said to be **projective** if it factors through as a closed immersion $X \to \mathbb{P}^n_Y$ for some n followed by the projection $\mathbb{P}^n_Y \to Y$.

$f : X \to Y$ is said to be **quasiprojective** if instead if it factors into an open immersion $X \to X'$ followed by a projective morphism $X' \to Y$.

With all of this terminology done, I will finish with one last defintion.

Definition 16. A **variety** (or an **abstract variety**) is a separated scheme of finite type over an algebraically closed field k. If it is additionally proper over k, then we say that the variety is **complete**.

Hironaka proved that this new notion of variety is actually strictly larger than the previous notion. After a little digging, I found that abstract varieties that cannot be embedded into project space usually arise out of certain blow ups. This highlights the fact that blow ups can do great violence to varieties.