An Example a Day 2023

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Abstract

I tend to enjoy concrete calculations/examples less than reading and proving big abstract theorems, or even just working with examples with high generality. That is a nice way of saying I am quite lazy, and probably afraid of working with too much detail. This document is an attempt to fix that. I will update this document roughly every day, and push the change online roughly monthly. But I will also give myself some grace to miss days. After all, I'm not a mathematical machine. And I don't think I want to be one even if I could be. The idea for this document was inspired (stolen) from Tom Gannon, a (at the time, now he is an assistant adjunct professor at UCLA) grad student I met at UT, though I doubt he remembers me. He has a "What I learned today" document on his website. I've tweaked the focus to concrete computations/examples/calculations according to what I need to practice more to become a better mathematician, but that will just be a focus, not a strict rule. I also decided to start this because I see a lot of those "advice for early grad students" type documents which usually indicate that they wish they wrote more when they were young, so this is part of my attempt to do that. I also have a very strong tendency, nay, pathology, to forget everything I've done before, so here is my avenue to remember. The main audience for this document is my future self, but I publish it in case anyone else may also get some value from it. I feel like I have written that line before, but I can't remember where from. This document is probably riddled with mistakes. If you find one, please email me! There are some entries that I know how to finish but haven't gotten around to yet, usually indicated with $\boxed{\text{FINISH}}$ at the end. There are others that I don't know how to finish, indicated with boxed text briefly describing the problem (but usually it just means I don't know what to do). If you know (or think you may know) how to finish one of these, please email me! All unfinished entries (of both kinds) are indicated in the ToC with an asterisk at the start.

Contents

1/23/2023 $U(1)$ is a symplectic action on $\mathbb{C}^n$:

We recall that a symplectic action is an action by a Lie group G on some symplectic manifold (X, ω) such that for every $g \in G$, $\rho(g)$ is a symplectomorphism on (X, ω), i.e. preserves the symplectic form under pullback. The simplest symplectic manifold is $\mathbb{C}^n$ with standard symplectic form

$$\omega = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$$

The simplest action is that of $U(1)$ on X acting by

$$e^{i\theta}(z_1, \ldots, z_n) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n)$$

To check this is symplectic, we compute

$$\begin{aligned} \varphi^* \omega &\equiv \varphi^* \left(\frac{i}{2} \sum_{n=1}^{n} dz_i \wedge d\bar{z}_i \right) \\ &= \frac{i}{2} \sum_{i=1}^{n} d(\varphi^* z_i) \wedge d(\varphi^* \bar{z}_i) \end{aligned}$$

We compute

$$\begin{aligned} (\varphi^* z_i)(z_1, \ldots, z_n) &= (z_i \circ \varphi)(z_1, \ldots, z_n) \\ &= z_i(e^{i\theta} z_1, \ldots, e^{i\theta} z_n) = e^{i\theta} z_i \\ (\varphi^* \bar{z}_i)(z_1, \ldots, z_n) &= \cdots = e^{-i\theta} \bar{z}_i \end{aligned}$$

So that

$$\varphi^* \omega = \frac{i}{2} \sum_{i=1}^{n} e^{i\theta} dz_i \wedge e^{-i\theta} d\bar{z}_i = \omega$$

As desired. So the action $U(1) \curvearrowright \mathbb{C}^n$ is a symplectic action. We should note that $\big(\mathbb{C}^n \setminus 0/\big)U(1) = \mathbb{C}P^n$, but I don't know if this calculation sheds any light on that.

1/25/2023 S^1 is a symplectic action on S^2

Next we would like to consider the other standard action: $S^1 \curvearrowright S^2$ by rotation. Note it is sort of a miracle that this action can even be considered, whether or not it ends up being symplectic: most spheres are not Lie groups, but S^1 is (of course $S^1 = U(1)$). Most even dimensional spheres are not symplectic manifolds, but S^2 is (the only such). However from a different perspective, T^n are all Lie groups, of which $S^1 = T^1$, and $S^2 = \mathbb{C}P^2$, all of which are symplectic manifolds with the Fubini-Study metric.

Anyway, back to the point: First we must understand the symplectic structure of S^2. We note that in dimension 2, any symplectic form is necessarily a volume form. In this case, S^2 has the area form $\omega_p(v, w) := \langle p, v \times w \rangle$, for $v, w \in T_p S^2$. One can easily check that this in fact a volume form. We'd like to write this out in terms of differentials so that we can easily compute pullbacks. We can view points of S^2 as triples (x, y, z) with norm 1. Then in the standard basis, we have $\omega(\partial_x, \partial_y) = z$, $\omega(\partial_y, \partial_z) = x$, $\omega(\partial_z, \partial_x) = y$. Note the sign switch which comes from the cross product. Then the differential form can only be

$$\omega = zdx \wedge dy + xdy \wedge dz + ydz \wedge dx$$

Since we want to consider the action of rotation, it is not ideal to have ω written in terms of cartesian coordinates: Using the map

$$F : \mathbb{R}^3_{r,\theta,h} \to S^2_{x,y,z}$$

$$(r, \theta, h) \mapsto (r \cos\theta, r \sin\theta, h)$$

We can pull back the form ω to write it in cylindrical coordinates. I did this on the chalk board, trust me. The key is to remember that $r \equiv 1$ when looking at S^2, thus $dr \equiv 0$. You get

$$F^*\omega = d\theta \wedge dh$$

Now we can look at the action of rotation by $\theta_0 \in S^1 \cong [0, 2\pi]$ around the h-axis. This sends $(\theta, h) \mapsto (\theta + \theta_0, h)$. Clearly, this sends

$$d\theta \wedge dh \mapsto d(\theta + \theta_0) \wedge dh = d\theta \wedge dh + d\theta_0 \wedge dh = d\theta \wedge dh + 0$$

where $d\theta_0 = 0$ because θ_0 is a constant. One should verify that this is actually how the pullback works, though. It is quick but it must be done to be completely precise. In words, it is because the pullback splits over wedge product and commutes with the differential and reduces to composition on functions. Thus we have verified that this action is a symplectomorphism.

As an intuitive argument[1], we can observe that in dimension 2, "symplectic form preserving" is the same as "volume preserving". But rotation by θ is obviously volume preserving.

1/26/2023 Also Hamiltonian

To check that it is Hamiltonian, we must describe a moment map $\mu : S^2 \to \mathfrak{g}^*$. In our case, $G = S^1$, so $Lie(G) \equiv \mathfrak{g} \cong \mathbb{R} \Rightarrow \mathfrak{g}^* \cong \mathbb{R}$. I just realized I used θ for the coordinate on S^1 and S^2 in the previous entry. We will now change the notation so that the angular coordinate in cylindrical coordinates on S^2 is ϕ. The obvious candidates for the moment

[1] Actually I'm not sure how intuitive this is, in the pejorative sense. It might constitute a complete proof. I prefer to do the calculation just to be completely sure.

map are the coordinate projections, ϕ or h. But this map must make the diagram

$$\begin{tikzcd}
 & \mathfrak{X}(S^2) \arrow[dr] & \\
\mathfrak{g} \cong \mathbb{R} = \langle \partial_\theta \rangle \arrow[ur] \arrow[dr] & & \Omega^1(S^2) \\
 & C^\infty(S^2) \arrow[ur] &
\end{tikzcd}$$

commute. Let's try h: To go through the bottom maps:

$$\begin{tikzcd}
c\partial_\theta \arrow[dr] & & d(-) \\
 & \Big((\phi, h) \mapsto \mu(\phi, h)(c\partial_\theta)\Big) \arrow[ur] &
\end{tikzcd}$$

where above we interpreted the height function as a map into $\mathbb{R}$. We are free to do this, of course, since $\mathfrak{g}^* \cong \mathbb{R}$. But then it is not clear how to pair a real number with a vector field and get a number. To recover this notion, we must reinterpret the moment map literally, as sending $(\phi, h) \mapsto hd_\theta$, where of course d_θ is the single element in the dual basis to $\mathfrak{g}$. Now we can simplify

$$\begin{tikzcd}
c\partial_\theta \arrow[dr] & & d(-) \\
 & \Big((\phi, h) \mapsto (hd_\theta)(c\partial_\theta) = hc\Big) \arrow[ur] &
\end{tikzcd}$$

Thus we can compute d:

$$\begin{tikzcd}
c\partial_\theta \arrow[dr] & & cdh \\
 & \Big((\theta, h) \mapsto (hd_\theta)(c\partial_\theta) = hc\Big) \arrow[ur] &
\end{tikzcd}$$

Now we must track through the top arrows. It is clear intuitively that ∂_θ must be sent to ∂_ϕ, since we are looking at the infinitesimal action of rotation by θ, which points in the direction of the angular coordinate on S^2, which is ∂_ϕ. We can also see from the definition:

$$\begin{aligned}
\rho_*(\partial_\theta) &\equiv \frac{d}{dt}\Big|_{t=0} exp(t\partial_\theta)\cdot(\phi,h) \\
&= \frac{d}{dt}\Big|_{t=0} t\theta\cdot(\phi,h) \\
&= \frac{d}{dt}\Big|_{t=0} (\phi+t\theta,h) \\
&\equiv \partial_\phi \\
&\Rightarrow \rho_*(c\partial_\theta) = c\partial_\phi
\end{aligned}$$

Then compute (we amend the notation from our previous calculation)(and using the embedding of exterior algebra into alternating tensor algebra):

$$F^*\omega = d\phi \wedge dh$$
$$F^*\omega(c\partial_\phi) = d\phi(c\partial_\phi)\otimes dh - d\phi\otimes dh(c\partial_\phi) = cdh$$

So we have confirmed that the height function is a moment map.

1/28/2023: *The canonical symplectic action

If a Lie group G acts on a manifold X, there is an induced action on T^*X by pulling back. Explicitly, if $\alpha = (p, \alpha_p) \in T^*X$, we can define a new one-form $g\cdot\alpha \in T^*X$ above $g\cdot p$: for any $v = (g\cdot p, v_{g\cdot p}) \in TX$

$$(g\cdot\alpha)(v) := \alpha_p\big((\rho_{g^{-1}})_*(v_{g\cdot p}\big) = \big((\rho_{g^{-1}})^*\alpha_p\big)(v_{g\cdot p})$$

where $\rho_g : X \to X$ is the diffeomorphism of g acting on X. T^*X is canonically an exact symplectic manifold with primitive one-form given by, for $v = (p, v_p) \in T(T^*M)$ with $p \in T^*M$ and $v_p \in T_p(T^*M)$,

$$\lambda(p, v_p) := p\cdot\pi_*(v_p)$$

and symplectic form given by $\omega = d\lambda$. In coordinates, T^*X has the form $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ where q_i are coordinates on the base space and $p_i \equiv dq_i$ are the fiber coordinates. We now need to distinguish between two different differentials: There is $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$, and $d' : \Omega^\bullet(T^*M) \to \Omega^\bullet(T^*M)$. We have[2]

$$\lambda = \sum_i p_i d' q_i$$

We must also be careful with notation here since we are dealing with an induced action by pullbacks. If $\rho_g : X \to X$ is the diffeomorphism associated to the action of g on X, then $(\rho_{g^{-1}})^* : T^*X \to T^*X$ is the induced diffeomorphism on cotangent bundle. Denote this by Φ_g. Then we must show that

$$\big((\rho_{g^{-1}})^*\big)^*\omega \equiv (\Phi_g)^*\omega = \omega$$

[2] I need to work this out too at some point

Note that Φ_g^* preserves ω if it preserves λ, by naturality of pullbacks wrt differentials. So it suffices to compute

$$\Phi_g^*(\lambda) = \Phi_g^*\left(\sum_i p_i d' q_i\right)$$
$$= \sum_i \Phi_g^* p_i \wedge d'\left(\Phi_g^* q_i\right)$$

p_i and q_i are functions on the manifold T^*U, so pullback acts by precomposition:

$$= \sum_i (p_i \circ \Phi_g) d\left(q_i \circ \Phi_g\right)$$

But this is exactly the definition of λ in the "domain" cotangent bundle, equipped with its local coordinates! If we choose a chart U, φ around $\Phi_g(p)$, then there is a chart $(\Phi_g^{-1}(U), \varphi \circ \Phi_g)$ around p. So at the level of cotangent bundle, T^*U has coordinates $(q_i \circ \Phi_g, d(q_i \circ \Phi_g)) = (q_i \circ \Phi_g, p_i \circ d\Phi_g)$, while $T^*\Phi_g(U)$ has coordinates (q_i, p_i). In the codomain coordinates, then, $\lambda = \sum_i p_i d' q_i$, since we chose our initial charts around this point. In the domain coordinates, $\lambda = \sum_i (p_i \circ d\Phi_g) d'(q_i \circ \Phi_g)$. So we have shown the action is symplectic. Seems like we are done, but notice the second λ has a $d\Phi_g$ while the first λ has a Φ_g. I do not know how to reconcile this. One of my friends raised some concerns about the use of the chain rule to find the coordinates around T^*U, which I concede may not be the correct application of the chain rule (for example, I'm not even sure that expression type checks), but we are unsure what the correct thing to do is. $\boxed{\text{ANYONE KNOW WHY THIS DIDNT WORK?}}$

As a final note, from our suggestive notation of Φ_g, we did not reference the group action at all, so really any diffeomorphism of any manifold induces a symplectic action on the cotangent bundle. I knew this from the beginning, and utilizing this fact would have simplified my life/the notation greatly, but the point is to suffer at least a little bit.

1/31/2023 *The canonical Hamiltonian action

The above described action is also Hamiltonian, which we defined previously. To do this, we must show the existence of a moment map $\mu : M \to \mathfrak{g}^*$, which satisfies the moment map condition:

$$d(\langle \mu(-), a\rangle) = \omega(X_a, -)$$

for every $a \in \mathfrak{g}$, where X_a is the vector field generated by the infinitesimal action of a on X. It can be viewed as the image of a through the map $d\rho_e : \mathfrak{g} \to \mathfrak{X}(M)$.

We define the Hamiltonian function[3] $H_a \in C^\infty(T^*M)$ as

$$(x, \alpha) \mapsto \langle \alpha, X_a(x)\rangle \equiv \langle \mu(x, \alpha), a\rangle$$

[3]Here we define the element $\mu(x, \alpha)$ as an element of $\mathfrak{g}^*$ by describing how it acts on an element of $\mathfrak{g}$.

Note that by definition, $X_a(x) \in T_xX$ and $\alpha \in T_x^*X$, so this is the natural pairing of a one-form with a vector field.

Because we showed the action on T^*X is symplectic, we know

$$\mathcal{L}_{\widetilde{X}_a}\lambda = 0$$

where $\widetilde{X}_a$ is the vector field generated by the infinitesimal action of $a \in \mathfrak{g}$ on T^*M:

$$\widetilde{X}_a(q,p) = \frac{d}{dt}\Big|_{t=0} \exp(tX) \cdot (q,p)$$

(In the particular case of the canonical symplectic action, we showed that the tautological one-form was preserved, so we can write the above equation directly, but it is also true in general for a symplectic action on an exact symplectic manifold because $[\mathcal{L}, d] = 0$ by Cartan formula)

$$d(i_{\widetilde{X}_a}(\lambda)) + i_{\widetilde{X}_a}(d\lambda) = 0$$

$$d(\langle \lambda, \widetilde{X}_a \rangle) = -\omega(X_a, -)$$

So we are close to fulfilling the moment map condition. We just have to check what $\langle \lambda, \widetilde{X}_a \rangle$ is. By definition of pairing, it sends a point $(x, \alpha) \in T^*M$ to

$$(x,\alpha) \mapsto \langle \lambda_{(x,\alpha)}, \widetilde{X}_a(x,\alpha) \rangle$$

By definition of the tautological one-form, $\lambda_{(x,\alpha)} = \alpha \circ d\pi_{(x,\alpha)}$,so

$$\langle \lambda_{(x,\alpha)}, \widetilde{X}_a(x,\alpha) \rangle = \alpha\big(d\pi_{(x,\alpha)}\widetilde{X}_a(x,\alpha)\big)$$

if there is any justice in the universe, this right hand side should be equal to

$$\alpha(X_a(x))$$

$\boxed{\text{IDK HOW TO SHOW IT THO}}$

2/1/2023 The canonical Hamiltonian reduction

Now that we have written down the moment map, we can compute the Hamiltonian/symplectic/Marsden-Weinstein-Meyer reduction/quotient. There are many names, I will choose Hamiltonian reduction. This is possible due to the theorem cited for example in CdS:

Theorem (Marsden-Weinstein-Meyer): *Let (M, ω, G, μ) be a Hamiltonian G-action for G compact Lie group. Let $i : \mu^{-1}(0) \to M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*
i) The orbit space $M_{red} := \mu^{-1}(0)/G$ is a manifold,
ii) $\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle, and
iii) there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$.

$\square$

So in our case, we look at $\mu^{-1}(0) \subset T^*M$. The vanishing moment map condition says that

$$\langle \alpha, X_a(x) \rangle = 0$$

for $(x, \alpha) \in T^*M$. Because X_a generates the infinitesimal action of G on M, we can think of the above condition as saying that the fiber coordinates, α, are *orthogonal* to the G-orbit on M. Then

$$T^*(M/G) \cong (T^*M)/G$$

2/2/2023 The Cartan subgalgebra of $\mathfrak{sl}(2, \mathbb{C})$

Definition: A Cartan subalgebra of a finite dimensional[4] Lie algebra is a maximal abelian subalgebra consisting of semisimple elements.

Definition: The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is a three-dimensional Lie algebra consisting of generators e, f, h satisfying

$$[e, f] = h, \quad [h, f] = -2f, \quad [h, e] = 2e$$

Of course there is a particular representation

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad , F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

but we need not refer to it. We do need to refer to the adjoint representation, however. This is given by, if we order the basis as e, f, h:

$$e \mapsto E \equiv \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f \mapsto F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \quad h \mapsto H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We see ad_e and ad_f are not diagonalizable, while ad_h is (in fact, diagonal). Unfortunately this does not immediately imply that the Cartan subalgebra is given by $\langle h \rangle$. It's clear that e and f cannot be contained, since they are not semi-simple, but not clear why, for example, $e + f$ could not be contained in it. We could compute its adjoint rep matrix and see whether it's semi-simple or not. In fact, I just checked and it is diagonalizable. So this along with the (obvious) statement that the adjoint map is linear shows that $e + f$ is semi-simple.

Instead I think we can make the following argument: If there were anything else besides the span of h in the algebra, then it would have the form $\lambda e + \gamma f$. Because Cartan is

[4]There is a more general defn for any Lie algebra, which is typically shown to be equivalent to this defintion in the f.d. case

abelian,

$$[h, \lambda e + \gamma f] = 0$$
$$\lambda[h, e] + \gamma[h, f] = 0$$
$$2\lambda e - 2\gamma f = 0$$

But e, f are linearly independent which implies $\lambda = \gamma = 0$. So $\langle h \rangle$ is the Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$.

2/6/2023 Homology of $\mathbb{R}P^n$:

We can equip $\mathbb{R}P^n$ with the CW structure of one cell in each dimension, up to n. The attaching map is provided by the universal cover $S^n = \partial D^{n+1} \to \mathbb{R}P^n$. On the level of chains, the induced map $C_n \to C_{n-1}$ is multiplication by 2 if k is even, and multiplication by 0 if k is odd. In particular, we now must break into cases depending on whether n is odd or even. So we are looking at the chain complex:

$$0 \to \cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to \ldots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \to 0$$

and the $\mathbb{Z}$ to the leftmost is in index n, which means the leftmost map $\mathbb{Z} \to \mathbb{Z}$ is the 0 map if n is odd and $\cdot 2$ is n is even. For now, let's look integer coefficients. In both cases, we have

$$H_0(\mathbb{R}P^n; \mathbb{Z}) = \frac{ker(0)}{im(0)} = \mathbb{Z}/0 = \mathbb{Z}$$

We have (assuming $n > 1$),

$$H_1(\mathbb{R}P^n; \mathbb{Z}) = \frac{ker(0)}{im(\cdot 2)} = \mathbb{Z}/2\mathbb{Z}$$

and

$$H_2(\mathbb{R}P^n; \mathbb{Z}) = \frac{ker(\cdot 2)}{im(0)} = 0/0 = 0$$

and we will see that the odd terms will all be $\mathbb{Z}/2\mathbb{Z}$ and the even terms will all be 0 until we arrive at n. Then if n is odd, we will have

$$H_n(\mathbb{R}P^n; \mathbb{Z}) = \frac{ker(0)}{im(0)} = \mathbb{Z}$$

and if n is even, then

$$H_n(\mathbb{R}P^n; \mathbb{Z}) = \frac{ker(\cdot 2)}{im(0)} = 0/0 = 0$$

In particular, we note that the odd dimension $\mathbb{R}P^n$'s are orientable. For example, $\mathbb{R}P^1 = S^1$, while the evens are non-orientable. One can play this game by altering the coefficients or by swapping the coefficients to $\mathbb{R}$ or $\mathbb{C}$ or maybe something more exotic. For example, when choosing coefficients in $\mathbb{Z}_2$, the map $\cdot 2$ now has a kernel, so the answers will change.

2/7/2023 Classical invariant theory

Consider the action of $\mathbb{Z}_2$ on $\mathbb{C}$ sending $z \mapsto -z$. This induces the action on $\mathbb{C}[x,y]$ sending $x, y \mapsto -x, -y$. We want to compute the fixed locus

$$\mathbb{C}[x,y]^{\mathbb{Z}_2}$$

Obviously any monomials with odd total degree will not be invariant, thus any polynomials containing even one monomial of odd total degree will not be invariant. Thus the fixed locus is just polynomials with each term having even total degree. This is generated by the lowest even dimensional monomials x^2, y^2, xy:

$$\mathbb{C}[x,y]^{\mathbb{Z}_2} = \mathbb{C}[x^2, y^2, xy] \cong \frac{\mathbb{C}[a,b,c]}{(ac-b^2)}$$

2/8/2023 Determinantal variety

Consider 3 x 3 matrices over $\mathbb{C}$. Define $Y_2 = \{M \in Mat_{3\times 3}(\mathbb{C}) \mid rk\ M \leq 2\}$ as a set. Enumerate the entries of the matrix as 1,...,9 from top left to bottom right. Then $Y_2 \subset \mathbb{C}^9$. The condition to belong to Y_2 is exactly that M does not have full rank, which is given exactly by having determinant 0 in this case. So

$$Y_2 = Z(det(M))$$

$$= Z\big(z_{11}(z_{22}z_{33} - z_{23}z_{32}) - z_{12}(z_{21}z_{33} - z_{31}z_{23}) + z_{13}(z_{21}z_{32} - z_{31}z_{22})\big)$$

So Y_2 is an affine variety in $\mathbb{C}^9$. Y_1 is defined similarly, but now the condition is more than just having determinant 0. Generally, the condition for a matrix to have rank $\leq r$ is given by the vanishing of all $(r+1) \times r+1$ minors. So

$$Y_1 = Z(M_{1,1}, M_{1,2}, \ldots, M_{3,3})$$

where $M_{i,j}$ is the (i,j) minor, i.e. the determinant of the 2 x 2 matrix obtained from M by deleting the ith row and jth column. These are all polynomials in the entries of the matrix, i.e. the coordinates of $\mathbb{C}^9$, so Y_1 is also an affine variety. From the definitions, it is clear that $Y_1 \subset Y_2$. But writing out the ideal of Y_1 and Y_2 as we did above shows this as well, since we can write

$$Y_2 = Z\big(z_{11}(M_{1,1}) - z_{12}(M_{1,2}) + z_{13}(M_{1,3})\big)$$

so that clearly

$$Z(M_{1,1}, \ldots, M_{3,3}) \subset Z\big(z_{11}(M_{1,1}) - z_{12}(M_{1,2}) + z_{13}(M_{1,3})\big)$$

The ideal $(M_{1,1}, \ldots, M_{3,3}) \subset k[\mathbb{C}^9] = k[x_1, \ldots, x_9]$ is called the <u>determinantal ideal</u>, perhaps corresponding to rank 1 inside dimension 3.

The 2 x 2 minors $M_{i,j}$ are homogeneous polynomials by observation, so we may also consider Y_1 as a projective variety. The determinant of a 3 x 3 matrix can be given by, for example, $z_{11}(M_{1,1}) - z_{12}(M_{1,2}) + z_{13}(M_{1,3})$ as we displayed above. Thus the 3 x 3 minors are homogeneous of degree 3 (in matrices of dimension 3 there is only one such minor but if we increase the dimension then there will be more). Using induction in this way, we can see that the determinantal ideal is always a homogeneous ideal, thus Y_r considered inside of $Mat_{m,n}$ is a projective (or affine) variety, given by the zero set of the determinantal ideal.

2/10/2023 *Second fundamental theorem of invariant theory for general linear group

Let V, W be f.d. complex vector spaces. $Hom(V, W)$ is an affine variety, for example choosing a basis yields $Hom(V, W) = Mat_{n,m}(\mathbb{C}) \cong \mathbb{C}^{nm}$ for n, m the dimensions of V and W. $GL(V) \times GL(W) \curvearrowright Hom(V, W)$ by conjugation where appropriate, thus inducing an action on polynomials $GL(V) \times GL(W) \curvearrowright \mathbb{C}[Hom(V, W)] \cong \mathbb{C}[x_1, \dots, x_{nm}]$. This induces an action

$$GL(V) \times GL(W) \curvearrowright Spec\big(\mathbb{C}[Hom(V, W)]\big)$$

elementwise. Let $I_{r+1} \in Spec(\mathbb{C}[Hom(V, W)])$ be a determinantal ideal corresponding to rank $(r+1) \times (r+1)$ minors[5].

For concreteness let's consider $n = m = 3$ and $r = 1$ as in the previous example. In this example, I_{1+1} is generated by the 9 minors $M_{i,j}$ $i, j \in \{1, 2, 3\}$. For example, $M_{1,1} = z_{22}z_{33} - z_{23}z_{32} \in I_2$. In our case, $GL(V) = GL(W)$, so we consider conjugation by $A \in GL_3(\mathbb{C})$. We may compute (here we replace z_{ij} with A_{ij}, the elementary matrix ij, as this is exactly the isomorphism $M_{k,n}(\mathbb{C}) \cong \mathbb{C}^{nm}$)

$$\begin{aligned}(A, A) \cdot M_{1,1} &\equiv AA_{22}A^{-1} \cdot AA_{33}A^{-1} - AA_{23}A^{-1} \cdot AA_{32}A^{-1} \\ &= AM_{1,1}A^{-1}\end{aligned}$$

$\boxed{\text{I WANT TO CHECK THAT THE DETERMINANTAL IDEALS ARE INVARIANT}}$

2/9/2023 The naive definition of equivariant cohomology is not well defined

The naive definition being $H^*_G(X) = H^*(X/G)$. In fact, what we're really showing is that this is not the right quotient to take, independent of taking cohomology. Let $X = \mathbb{R}$ and $G = \mathbb{Z}$ acting on $\mathbb{R}$ by translation. There is a G-equivariant homotopy equivalence $X \to pt$, but no G-equivariant homotopy equivalence $X/G \to Y/G$, since $S^1 \not\simeq pt$. In other words, this naive definition would not be homotopy invariant. One fixes this by introducing the homotopy quotient, then taking ordinary cohomology.

To fix this, we end up considering the homotopy quotient. In this case, we consider

[5] Actually I don't know why this is prime, but I do know that it is.

$E\mathbb{Z} \times_{\mathbb{Z}} \mathbb{R}$ and $E\mathbb{Z} \times_{\mathbb{Z}} pt$. $E\mathbb{Z}$ is a contractible space which $\mathbb{Z}$ acts on freely. I'm pretty sure we can just take $\mathbb{R}$ and translation as a model. Then we are looking at spaces $\mathbb{R}^2/(r_1, zr_2) \sim (zr_1, r_2)$ and $\mathbb{R}/\mathbb{Z} \cong S^1$. Uhh are these homotopy equivalent?

2/10/2023 Equivariant cohomology of a point

In a sense, this is the most important equivariant cohomology calculation. Let G act on pt. By definition,

$$H_G^*(pt) \equiv H^*(EG \times_G pt) = H^*(EG/G) = H^*(BG)$$

We are frequently considering torus actions, so let's consider something like $G = S^1 \cong \mathbb{C}^*$. Then

$$H_{S^1}^*(pt) = H^*(BS^1)$$

Let's show that that $BS^1 = \mathbb{C}P^\infty$. Why? First we need to identify EG, since $BG = EG/G$. ES^1 is a space, unique up to homotopy equivalence, which is contractible and acted on freely by S^1. If you think about it, S^1 naturally acts on the spheres S^n freely by rotation, but none of these are contractible (e.g. $\pi_k(S^k) = \mathbb{Z}$)) until you consider S^∞, which we may think of as the union of all the n-spheres. So we may choose a representative (I think sophisticated people usually use the word model, but to me that feels like stolen valor) of EG as S^∞. Then

$$BS^1 = S^\infty/S^1 \equiv \mathbb{C}P^\infty$$

It is known that $H^*(\mathbb{C}P^\infty) = \mathbb{Q}[u]$ (or whatever coefficients we are taking) where u is a hyperplane generator[6], i.e. lives in degree 2. Thus

$$H_{S^1}^*(pt) \cong \mathbb{Q}[u]$$

2/12/2023 The GIT quotient of $GL_n(k)$ acting on $End(k^n)$

Let's consider $M = Hom(V, W) = k^{nm}$ again. This has an action by $GL(V) \times GL(W)$ by conjugation where appropriate. We restrict to the case $V = W$, since I'm not sure this example makes sense otherwise. (we may also need to restrict k to be something tame like $\mathbb{R}$ or $\mathbb{C}$ but I don't see a reason to.) Then

$$M /\!\!/ GL_n(k) \equiv Spec\big(k[M]^{GL_n(M)}\big)$$

Let's identify $k[M]^{GL_n(k)}$. This is $GL_n(k)$-invariant (under precomposition) polynomials in $n \times n$ matrices. We know that diagonalizable matrices are dense in the set of invertible matrices (THIS REQUIRES k TO BE ALGEBRAICALLY CLOSED), so it suffices to identify the values on these:

$$k[M] = k[\text{diagonalizable matrices} \subset M]$$

[6] As a heuristic calculation, we can consider the ordinary cohomology of $\mathbb{C}P^n$ as $\mathbb{Q}[u]/(u^n)$, where u has degree 2, and taking the limit as $n \to \infty$.

But if we are looking at $GL_n(k)$ invariant polynomials (again acting by precomposition then conjugation), it suffices to look at diagonal matrices:

$$k[M]^{GL_n(k)} = k[\text{diagonalizable matrices} \subset M]^{GL_n(k)} = k[\text{diagonal matrices} \subset M]^{GL_n(k)}$$

$$= k[\lambda_1, \ldots, \lambda_n]^{GL_n(k)}$$

But $GL_n(k)$ acts by permutation on the diagonal entries:

$$= k[\lambda_1, \ldots, \lambda_n]^{S_n}$$

So the GIT quotient is

$$M /\!\!/ GL_n(k) = Spec(k[\lambda_1, \ldots, \lambda_n]^{S_n})$$

By the fundamental theorem of symmetric polynomials,

$$= Spec(k[e_1(\lambda_1, \ldots, \lambda_n), \ldots, e_n(\lambda_1, \ldots, \lambda_n)])$$

As in the standard story, we have the prime ideals consisting of kernels of evaluation morphisms. But now what we are really plugging in is not points of k^n, but equivalence classes of k^n/S^n. For example, we may consider $ev_{\vec{a}} : k[e_1(\vec{\lambda}), \ldots, e_n(\vec{\lambda})] \to k$, for $\vec{a} \in k^n$, then $ker(ev_{\vec{a}})$ is a prime ideal. But it is exactly equal to $ker(ev_{\sigma(\vec{a})})$ for any $\sigma \in S_n$, since $e_i(\vec{a}) = 0 \iff e_i(\sigma(\vec{a})) = 0$. So the set of prime ideals is in one to one correspondence with

$$\cong k^n/S_n \cong k^n$$

2/14/2023 Affine GIT moduli space for double quiver

Let Q be the quiver with one vertex and one edge and let the dimension vector be $\vec{v} = (n)$. Then $Rep(Q, \vec{v}) = End(\mathbb{C}^n) = Mat_{n \times n}(\mathbb{C})$. This vector space is acted on by $GL_n(\mathbb{C})$, and the affine GIT quotient is

$$R_0(\vec{v}) \equiv R(Q, \vec{v}) /\!\!/ PGL(\vec{v})$$

(PGL because we act by conjugation, so the overall scale will pull out and cancel).

$$\equiv Spec(\mathbb{C}[Rep(Q, \vec{v})]^{PGL(\vec{v})}) = Spec(\mathbb{C}[z_1, \ldots, z_n]^{PGL(n)}) = k^n$$

by the same argument as above. We can also consider the doubled quiver, $Q^\sharp$ with one vertex and two edges. Then $Rep(Q^\sharp, n) = T^*(Rep(Q, n))$ is acted on by $GL_n(\mathbb{C})$, with moment map

$$\mu : R(Q^\sharp, n) \to \mathfrak{gl}_n(\mathbb{C})$$

$$z \mapsto [z_{\bar{h}}, z_h]$$

since there is one vertex and two edges whose target is that vertex, and the sign shift comes from taking the orientation function $\epsilon = \epsilon_\Omega$. Thus

$$\mu^{-1}(0) = \{(X, Y) \in End(\mathbb{C}^n) \mid [X, Y] = 0\}$$

We can take the affine GIT quotient:

$$\mu^{-1}(0) /\!\!/ G \equiv Spec(\mathbb{C}[\mu^{-1}(0)]^{PGL(n)})$$

Commuting matrices are simultaneously diagonalizable, so we can make the same argument as in the above example:

$$= Spec(\mathbb{C}[\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n]^{S_n})$$
$$= \mathbb{C}^{2n}/S_n$$

which is singular. For example, taking $n = 2$, we have

$$= Spec(\mathbb{C}[\lambda_1, \lambda_2, \mu_1, \mu_2]^{\mathbb{Z}_2})$$
$$\cong \mathbb{C}^2 \times Spec\left(\frac{\mathbb{C}[a, b, c]}{(ac - b^2)}\right)$$

where the second term we encountered in 2/7/2023 Classical invariant theory. We immediately recognize the term on the right as singular, for example by computing its Jacobian (gradient). So this affine quotient is often singular.

2/17/2023 Twisted GIT moduli space for a framed quiver

Let Q be a type A quiver of length ℓ.

$$Q = \underbrace{\bullet \to \bullet \to \cdots \to \bullet}_{\ell}$$

And consider a framing, $\vec{W}$, which is 0 at all but the final vertex, where we place a vector space of dimension r:

$$Q_{Fr} = \begin{array}{ccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 & & 0 & & \cdots & & \mathbb{C}^r \end{array}$$

which we will abbreviate as

$$Q_{Fr} = \begin{array}{ccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet \\ & & & & & & \downarrow \\ & & & & & & \mathbb{C}^r \end{array}$$

We abbreviate $\vec{w} \equiv (0, \dots, 0, r)$ as just r, so that

$$Rep(Q_{Fr}, \vec{v}, \vec{w}) = Rep(Q_{Fr}, \vec{v}, r)$$

From general theory this is given by

$$Rep(Q_{Fr}, \vec{v}) \bigoplus Hom(V_\ell, \mathbb{C}^r)$$

so we denote such an element as (x, j). Clearly this space carries an action of $GL(\vec{v})$, so we can compute the twisted GIT quotient, for a choice of character χ_θ:

$$R_\theta(\vec{v}, r) \equiv Rep(Q_{Fr}, \vec{v}, r) /\!\!/_{\chi_\theta} GL(\vec{v})$$

Also from general theory, this space on the right is isomorphic to equivalence classes of semi-stable orbits, so we need to identify these.

Theorem[7]: *Let Q be a quiver. Let $\theta \in \mathbb{Z}^I, \theta > 0$, and let χ_θ be the corresponding character of $GL(\vec{v})$. Then*
i) $(x, j) \in Rep(Q_{Fr}, \vec{v}, \vec{w})$ is $\chi-$semistable iff the following condition holds:
For any subrepresentation[8] $V' \subset V$, $V' \subset Ker j \Rightarrow V' = 0$.
ii) Any χ-semistable element is automatically stable.

□

Suppose we have a representation (x, j):

$$\begin{array}{ccccccc} V_1 & \xrightarrow{x_1} & V_2 & \xrightarrow{x_2} & \cdots & \xrightarrow{x_{\ell-1}} & V_\ell \\ & & & & & & \downarrow{\scriptstyle j} \\ & & & & & & \mathbb{C}^r \end{array}$$

and we want to determine what conditions there are on x and j in order for the representation to correspond to a semistable orbit. Let's look at $\ell = 2$ first.

$$\begin{array}{ccc} V_1 & \xrightarrow{x} & V_2 \\ & & \downarrow{\scriptstyle j} \\ & & \mathbb{C}^r \end{array}$$

Then we ask what conditions will guarantee that given any $V_i' \subset V_i$ such that $x|_{V_1'} \subset V_2'$ (this is the definition of a subrepresentation), $V_2' \subset ker\ j \Rightarrow V_1' = V_2' = 0$.

Suppose (x, j) is semistable and consider $v \in V_1$, such that $(j \circ x)(v) = 0$. Then we can define a subrepresentation

$$\begin{array}{ccc} span(v) & \xrightarrow{x} & span(x(v)) \\ & & \downarrow{\scriptstyle j} \\ & & \mathbb{C}^r \end{array}$$

which is contained in $ker\ j$, since $j(x(v)) = 0$. Thus by the above theorem, $span(v) = span(x(v)) = 0 \Rightarrow v = 0$. Thus $j \circ x$ is injective. In fact, we may also consider the subrepresentation $V_1' = 0$, with V_2' arbitrary. Then if $V_2' \subset ker\ j$, $V_2' = 0$. In other words, the only

[7]Kirillov 10.22

[8]Subrepresentation here refers to only the x, a subrepresentation of $x \in Rep(Q, \vec{v})$.

subspace of $ker\ j$ is the 0 subspace, so that $ker\ j = 0$. So $j \circ x$ is injective and j is injective.

So for general ℓ, if (x, j) is χ-semistable, then we can consider an element of $ker\ j \circ x_{\ell-1} \circ \cdots \circ x_a$, for any a, including $a = 0$. We consider the subrepresentation

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] & span(v) \arrow[r] & span(x_a(v)) \arrow[r] & \cdots \arrow[r, "x"] & span(x_{\ell-1}(x_{\ell-2}(\dots(x_a(v))\dots) \arrow[d, "j"] \\ & & & & & W^r \end{tikzcd}$$

so that $span(x_{\ell-1}(x_{\ell-2}(\dots(x_a(v))\dots) \subset ker\ j \Rightarrow span(v) = \cdots = 0 \Rightarrow v = 0$, and $j \circ x_{\ell-1} \circ \cdots \circ x_a$ is injective for any a. So this is the set of semi-stable orbits:

$$Rep(Q, \vec{v}, r)^{ss} = \{(x, j) \mid j \circ x_{\ell-1} \circ \cdots \circ x_a \text{ is injective } \forall\ a\}$$

Note in particular that x_a is always injective in such a case, for every a. Thus the dimensions must be increasing in order for this to be non-empty.

But any such element determines a (generically partial) flag in $\mathbb{C}^r$: For every vertex, we have an injective map $j \circ \cdots \circ x_a : V_a = \mathbb{C}^{\vec{v}_a} \to \mathbb{C}^r$, and inclusions of successive spaces. Thus

$$R_\theta(\vec{v}, \vec{w}) \cong \mathcal{F}(\vec{v}_1, \dots, \vec{v}_\ell, r)$$

In particular, this is empty unless the dimension vector is increasing.

2/20/2023 *Same as above but a different framing

Let's repeat the above with a different framing: Let Q be a type A quiver of length ℓ, and instead of $\vec{w} = (0, \dots, 0, r)$, let's take $\vec{w} = (0, \dots, 0, r_1, r_2)$:

$$\begin{tikzcd} V_1 \arrow[r, "x_1"] & V_2 \arrow[r, "x_2"] & \cdots \arrow[r, "x_{\ell-1}"] \arrow[d, "j_1"] & V_\ell \arrow[d, "j_2"] \\ & & \mathbb{C}^{r_1} & \mathbb{C}^{r_2} \end{tikzcd}$$

Again to determine the stability condition, let $\ell = 3$. So a representation, (x, j) has the form

$$\begin{tikzcd} V_1 \arrow[r, "x_1"] & V_2 \arrow[r, "x_2"] \arrow[d, "j_1"] & V_3 \arrow[d, "j_2"] \\ & \mathbb{C}^{r_1} & \mathbb{C}^{r_2} \end{tikzcd}$$

Assume (x, j) is semistable and consider $v \in V_1$ such that $(j_1 \circ x_1)(v) = (j_2 \circ x_2 \circ x_1)(v) = 0$, i.e. $x \in ker(j_1 \circ x_1) \cap ker(j_2 \circ x_2 \circ x_1)$. Then we can consider the subrepresentation

$$\begin{tikzcd} span(v) \arrow[r, "x_1"] & span(x_1(v)) \arrow[r, "x_2"] \arrow[d, "j_1"] & span(x_2(x_1(v))) \arrow[d, "j_2"] \\ & \mathbb{C}^{r_1} & \mathbb{C}^{r_2} \end{tikzcd}$$

which is contained in $ker\ j$ by construction, so $v = 0$. In this case, neither $ker(j_1 \circ x_1)$ nor $ker(j_2 \circ x_2 \circ x_1)$ need to be trivial, but the intersection must be. In other words, we have maps

$$j_1 \circ x_1 : V_1 \to \mathbb{C}^{r_1}, \quad j_2 \circ x_2 \circ x_1 : V_1 \to \mathbb{C}^{r_2}$$

which are not necessarily injective. But it does define an injective map

$$(j_1 \circ x_1, j_2 \circ x_2, x_1) : V_1 \to \mathbb{C}^{r_1} \oplus \mathbb{C}^{r_2}$$

since the kernel is exactly equal to the intersection of the kernels (note you cannot take $V_1 \oplus V_1 \to \mathbb{C}^{r_1} \oplus \mathbb{C}^{r_2}$, since the kernel is then given by the direct sum, which properly contains the intersection, thus provides no information.).

By the same argument as the previous example, j_2 is injective, but we don't know that j_1 is injective. We can also make an identical argument to show that $ker(j_1) \cap ker(j_2 \circ x_2) = 0$. Then the semistable points are identified as

$$\left\{ (x_1, x_2, j_1, j_2) \;\middle|\; \begin{array}{l} ker(j_1) \cap ker(j_2 \circ x_2) = 0 \\ ker(j_1 \circ x_1) \cap ker(j_2 \circ x_2 \circ x_1) = 0 \\ ker(x_1) = ker(j_2) = 0 \\ ker(j_1) \cap ker(x_2) = 0 \end{array} \right\}$$

which could be written in more generalization-friendly notation as

$$\left\{ (x_1, x_2, j_1, j_2) \;\middle|\; \begin{array}{l} ker(j_1 \circ x_{\ell-2} \circ \cdots \circ x_a) \cap ker(j_2 \circ x_{\ell-1} \cdots \circ x_a) = 0 \\ ker(x_1) = ker(j_2) = 0 \\ ker(j_1) \cap ker(x_2) = 0 \end{array} \right\}$$

where $a \in \{1, \ldots, \ell - 1\}$, which in our case would be just $\{1, 2\}$, and we understand that if a exceeds $\ell - 2$ for the left term or $\ell - 1$ for the right term, then no x_a term appears in the kernel. In the single-framed example, it was obvious that all the x_a's were injective, because the compositions $(j \circ x_{\ell-1} \circ \cdots \circ x_a)$ were injective, which always implies the first map is injective. In this case we don't have that, so we have to argue directly: If $v \in ker(x_1)$, then we can define the subrepresentation consisting of its span and the 0 spaces at V_2' and V_3'. This is a subrep contained in $ker\ j$ so $v = 0$. This cannot be applied to x_2 unless the chosen v also belongs to $ker\ j_1$, hence the final condition.

IMPORTANT ASIDE: when writing down subrepresentations to determine the semistability conditions, it is important to note that you are allowed to write down things like

$$\cdots \to 0 \to span(v) \to \ldots$$

But you are not allowed to write

$$span(v) \to 0 \to \ldots$$

unless you specifically choose v in the kernel of the assigned map. If not, then you have to continue the sequence as

$$span(v) \to span(x_1(v)) \to \ldots$$

END ASIDE:

We want to get some kind of flag variety out of this. We have these three conditions on kernels of maps:

$$\begin{cases} a=1 & ker(j_1 \circ x_1) \cap ker(j_2 \circ x_2 \circ x_1) = 0 \\ a=2 & ker(j_1) \cap ker(j_2 \circ x_2) = 0 \\ & ker(x_1) = ker(j_2) = 0 \end{cases}$$

As mentioned above, then we get injective maps

$$\begin{cases} a=1 & (j_1 \circ x_1, j_2 \circ x_2 \circ x_1) : V_1 \hookrightarrow \mathbb{C}^{r_2} \oplus \mathbb{C}^{r_2} \\ a=2 & (j_1, j_2 \circ x_2) : V_2 \hookrightarrow \mathbb{C}^{r_1} \oplus \mathbb{C}^{r_2} \\ & \end{cases}$$

But we get NO SUCH MAP from V_3:

$\boxed{\text{IS THERE ANY WAY TO PROCEED? ANSWER MAY BE NO.}}$

2/21/2023 Twisted GIT moduli space for a framed, doubled quiver (The Big Boy)

Now we consider the full quiver variety. Let A be a type A quiver of length ℓ and consider the framing

$$Q_{Fr} = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[d] \\ & & & \mathbb{C}^r \end{tikzcd}$$

Now consider the double

$$\overline{Q_{Fr}} = \begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[d, shift left] \\ & & & \mathbb{C}^r \arrow[u, shift left] \end{tikzcd}$$

Kirillov denotes $\overline{Q_{Fr}}$ as $Q^\sharp$ and we will do the same. We need to upgrade the previous theorem cited about semistability in the context of framed representations to now account for this doubling.

Definition: Let $\theta \in \mathbb{R}^I$. Then a W-framed representation V of $Q^\sharp$ is <u>θ-semistable</u> if for any $Q^\sharp$ representation $V' \subset V$, we have

$$V' \subset ker\, j \Rightarrow \theta \cdot \mathbf{dim}\, V' \leq 0 \tag{0.0.1}$$

$$V' \supset im\, i \Rightarrow \theta \cdot \mathbf{dim}\, V' \leq \theta \cdot \mathbf{dim}\, V \tag{0.0.2}$$

and $\underline{\theta\text{-stable}}$ if the inequalities are strict.

Theorem[9]**:** *Let $\theta \in \mathbb{Z}^I$ and χ_θ the corresponding character of $GL(\vec{v})$. Then an element $m = (z, i, j) \in R(Q^\sharp, \vec{v}, \vec{w})$ is χ_θ-semistable (in the sense of the general theory of twisted GIT quotients) (respectively stable) iff the corresponding W-framed representation V is θ-semistable (in the sense of the definition above) (respectively stable).*

Rmk: Generically, semistable implies stable. See Kirillov for details.

Rmk: Note if we choose $\theta > 0$, then condition 1 is the same as for framed varieties, as above and condition 2 is trivially satisfied always.

Now we want to determine $R^{ss}(Q^\sharp, \vec{v}, (0, \ldots, 0, r))$ for the above. By the above remark, we make the same argument to conclude that every composition $j \circ x_{\ell-1} \circ \cdots \circ x_a$ is injective, and because condition 2 is trivial, this is the only requirement.

As before, this implies R^{ss} is empty unless $\vec{v}_1 \leq \vec{v}_2 \leq \cdots \leq \vec{v}_\ell$. But note that WLOG we can assume the inequalities are strict: If there are equalities, then we have an arrow like $x_a : \mathbb{C}^n \to \mathbb{C}^n$, which can clearly be removed from the quiver without altering the quiver variety (up to isomorphism) in this case. One may object to say that only seems true if the provided map is an isomorphism. That is true, but x_a is guaranteed to be an isomorphism in this case because we showed each x_a is injective, and it is a self-map.

Therefore as before, each element in R^{ss} defines a partial flag in $\mathbb{C}^r$. The moment map condition implies

$$\Big(-y_1 x_1, -y_2 x_2 + y_1 x_1, \ldots, x_{\ell-1} y_{\ell-1} - ij \Big) = (0, \ldots, 0)$$

Componentwise, this means

$$y_1 x_1 = 0, \quad x_1 y_2 = y_2 x_2, \quad \ldots, \quad x_{\ell-1} y_{\ell-1} = ij$$

$$\Rightarrow y_1|_{im(x_1) \cong V_1} = 0, \quad y_2|_{im(x_2) \cong V_2} = x_1 y_1, \quad \ldots, \quad i|_{im(j) \cong V_\ell} = x_{\ell-1} y_{\ell-1}$$

We can regard i as a map $\mathbb{C}^r \to \mathbb{C}^r$, since V_ℓ embeds into $\mathbb{C}^r$. By the embeddings $j \circ x_{\ell-1} \circ \cdots \circ x_a : V_a \to \mathbb{C}^r$, we can consider $i|_{V_k}$ for every k. Because each space includes into every space after it (going left to right), we can write

$$i|_{V_k} = \bigg(\Big((i|_{V_\ell})|_{V_{\ell-1}} \Big) \Big|_{V_{\ell-2}} \cdots \bigg) \bigg|_{V_k}$$

in English: We have a map on $\mathbb{C}^r$, and we can restrict it to V_k immediately, or we can restrict it to V_ℓ then restrict that to $V_{\ell-1}$ then restrict that to $V_{\ell-2}$ and so on until we reach

[9] Kirillov 10.37

V_k. These result in the same map. Then we can go down the chain of relations we found

$$= \left(\left((x_{\ell-1}y_{\ell-1})\big|_{V_{\ell-1}}\right)\Big|_{V_{\ell-2}} \dots\right)\Bigg|_{V_k}$$

$$= \left((x_{\ell-1}x_{\ell-2}y_{\ell-2})\Big|_{V_{\ell-2}} \dots\right)\Bigg|_{V_k}$$

$$\vdots$$

$$= x_{\ell-1}x_{\ell-2}\dots x_k y_k$$

and the inclusions x_i serve only to embed the space back into $\mathbb{C}^r$, so we can view i as a map from $\mathbb{C}^r$ to itself.

In total, we now have a map $i : \mathbb{C}^r \to \mathbb{C}^r$, whose restrictions to each space V_j are equal to the maps y_{j-1}, and further they send $i(V_j) \subset V_{j-1}$. Setting the moment map condition to 0 allows you to amalgamate all the y maps into a single map of the large space inducing all of the smaller maps. Therefore the quiver variety is the space of semi-stable points (modulo the appropriate equivalence relation) and the moment map condition:

$$\mathcal{M}(Q, \vec{v}, \vec{w}) \cong \left\{ \begin{array}{l} F \in \mathcal{F}(\vec{v}_1, \dots, \vec{v}_\ell, r) \\ i : \mathbb{C}^r \to \mathbb{C}^r, \quad i(V_j) \subset V_{j-1} \end{array} \right\}$$

It reamins to see why this right hand side is isomorphic to the cotangent bundle of a full flag variety. I finished this by talking to Richard Rimanyi in person but I can't be bothered to write it down now. If someone is reading this and wants an explanation just ask me.

2/27/2023 Regular points vs singular points

Consider the action of $k^\times$ on $M = k^2$ by $t \cdot (x_1, x_2) = (tx_1, t^{-1}x_2)$. Recall $x \in M$ is regular if its orbit is closed and its stabilizer is trivial. Denote the set of such as M^{reg}. This set is G-invariant, so we can ask what are the regular G-orbits, ie those orbits which are represented by a regular point. This set is $(M /\!\!/ G)^{reg}$. Now we calculate

$$M /\!\!/ G \equiv Spec(k[k^2]^{k^\times}) = Spec(k[x_1, x_2]^{k^\times})$$

But $k[x_1, x_2]^{k^\times} \cong k[x_1 x_2] \cong k[\epsilon]$ for some indeterminate ϵ:

$$\cong Spec(k[\epsilon]) \cong k$$

so the affine quotient is nonsingular, and we want to determine $(M /\!\!/ G)^{reg}$. From general theory, any regular point is non-singular, but the converse does not hold: The regular points in this case are those of $M /\!\!/ G$ which have trivial stabilizer and closed orbit. 0 is a fixed point of G, so it is not regular. Note that every orbit in $M /\!\!/ G$ is closed, from general theory, so in here we need only check stabilizers. Thus $(M /\!\!/ G)^{reg} = k \setminus 0$, and we see that nonsingular does not imply regular.

2/30/2023 *Baby's first Springer resolution

Consider the action of $k^\times$ on k^2 by scaling. The set theoretic quotient $k^2/k^\times$ is given by $\{0, \mathbb{P}^1\}$, basically by definition. The affine quotient is

$$\begin{aligned} k^2 /\!\!/ k^\times &\equiv Spec(k[k^2]^{k^\times}) \\ &= Spec(k[x,y]^{k^\times}) \\ &= Spec(k) = pt \end{aligned}$$

Note that $T^*(k^2) \cong k^2 \oplus (k^2)^* \cong Hom(k, k^2) \oplus Hom(k^2, k) \equiv \{(i,j)\}$. From general theory, because $k^\times$ acts on k^2, it induces a Hamiltonian action on $T^*(k^2)$. Because k^2 is a VS, its cotangent component is just the dual vector space, where $k^\times$ now acts by inverse. So the action on $T^*(k^2)$ is given by

$$\lambda(i,j) = (\lambda i, \lambda^{-1} j)$$

One can guess what the moment map to this action is, but it also follows from the most general formula of an induced Hamiltonian action on a cotangent bundle. To guess it: The moment map is from

$$\mu : T^*(k^2) \to \mathfrak{gl}(1)^* \cong k$$

there is only one way to combine $i : k^2 \to k$ and $j : k \to k^2$ to get a one by one matrix (number), which is

$$\mu(i,j) = ij$$

Then

$$\mu^{-1}(0) = \{(i,j) \mid ij = 0\}$$

In accordance with the general theory of Hamiltonian reduction, this is an affine space, and admits an action[10] by $G = k^\times$, so we may consider the affine GIT quotient:

$$\mu^{-1}(0) /\!\!/ k^\times$$

Whatever this is, we may define a map $\mu^{-1}(0) \to Mat_{2\times 2}(k)$ sending $(i,j) \mapsto ji$. This is clearly $k^\times$-invariant[11], so it descends to a morphism

$$\mu^{-1}(0) /\!\!/ k^\times \to Mat_{2\times 2}(k)$$

The image of this map is:

$$im = \{A \in Mat_{2\times 2}(k) \mid Tr\ A = det\ A = 0\} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a^2 + bc = 0 \right\} \equiv \mathcal{N}_{\mathfrak{sl}_2}$$

To justify the first equality, we must be careful: i and j are not square matrices, so you cannot, for example, distribute the determinant over the product to say $det(ji) = det(j)det(i) =$

[10]Generally, I think the Hamiltonian condition (the 1-form equation) guarantees that the fibers of the moment map are G-invariant.

[11]That would work if we were taking an ordinary quotient, but we are taking a GIT quotient, so a priori, probably more justification is needed here.

$0 * 0$, nor naively apply the cyclic property of the trace. The cyclic property of trace, luckily, works as long as the matrix products make sense, so we can use that one directly to conclude $Tr(ji) = 0$. For the determinant, if $j = \begin{pmatrix} a \\ b \end{pmatrix}$, $i = \begin{pmatrix} c & d \end{pmatrix}$, then $ij = 0 \Rightarrow ac + bd = 0$.

$$ji = j \otimes i = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$$

$$det(ji) = (ac)(bd) - (bc)(ad) = 0$$

(actually we didn't even need $ij = 0$). This shows $im \subset Q$.

To see the other containment, we would need to decompose any given 2 x 2 matrix with trace = det = 0 into a tensor product ji, then show that $ij = 0$. I haven't done this, let's take it for granted for now. You just need to mess around with some equations for 2 x 2 matrices. I know the point of this document was to *not* avoid doing the nitty gritty details, but this example has dragged on long enough already and it will certainly drag much further.

This "shows" that the morphism described is surjective onto $\mathcal{N}_{\mathfrak{sl}_2}$. We must also show that this is injective, but this is clear since the outer product ji is 0 iff i, j are 0.

For the second equality, consider the characteristic polynomial, $c_A(\lambda) = det(A - \lambda I)$. In particular, $c_A(0) = det(A)$, and for a 2 x 2 matrix, the linear term is minus the trace of A. Thus the characteristic polynomial of any $A \in Q$ is $c_A(\lambda) = \lambda^2 - 0 + 0$. By Cayley-Hamilton, then $A^2 = 0$, so A is nilpotent. But the RHS is exactly the form of any 2 x 2 nilpotent matrix.

So $\mu^{-1}(0) /\!\!/ k^\times \cong Q$, which is clearly singular. We encountered this example before.

The affine GIT quotient being singular is common. Now we examine the twisted GIT quotient:

Per the general theory, we need to check that $(\mu^{-1}(0))^{reg}$ is non-empty and that the twisted GIT quotient is nonsingular. This will show that the twisted GIT quotient is a resolution of singularities of the affine GIT quotient. $(\mu^{-1}(0))^{reg}$ is the set of (i, j) with closed orbit and trivial stabilizer. By the definition of the G-action, (i, j) has trivial stabilizer unless both $i, j = 0$, so we have to throw out 0, and closed orbits are those which survive in the affine GIT quotient, so the regular points are exactly given by $Q \setminus 0$. In particular, it is non-empty, and we can sanity check the fact that the regular locus is non-singular, though this was already guaranteed by the general theory (regular implies non-singular, but not necessarily the other way around).

Now we check that the twisted quotient is nonsingular. Let χ be the identity character[12] on $k^\times$. Then $(i, j) \in \mu^{-1}(0)$ is semi-stable iff j is injective iff $j \neq 0$, and semi-stability

[12] I suppose this is called projective quotient, and sees more use in pure algebraic geometry. To my knowledge, twisted GIT quotients mostly appear only in the context of quiver varieties.

is equivalent to stability in this case. Then

$$M /\!\!/_\chi k^\times = M^{ss} /\!\!/ k^\times = M^s / k^\times$$

$$= \{(i,j) \mid j \neq 0\} = \{(V,i) \mid V \subset k^2,\ dim\ V = 1, i : k^2 \to V,\ i|_V = 0\}$$

This is clearly identified as $T^*\mathbb{P}^1$, thus establishing $T^*\mathbb{P}^1$ as a resolution of singularities of the nilpotent cone $\mathcal{N}_{\mathfrak{sl}_2}$.

3/1/2023 The Hall algebra of a single vertex

For any $q = p^r$ with p prime, we can define the $\mathbb{F}_q$ Hall algebra, $H(Q, \mathbb{F}_q)$ as an algebra over $\mathbb{C}$ (or any ring) with basis generated by isomorphism classes of $\mathbb{F}_q$ representations of Q and multiplication defined by

$$[M_1] * [M_2] := \sum_{[L]} F^L_{M_1 M_2} [L]$$

where

$$F^L_{M_1 M_2} = \frac{X^L_{M_1 M_2}}{|Aut_Q(M_1) \times Aut_Q(M_2)}$$

where $X^L_{M_1 M_2}$ is the number of SES's

$$0 \to M_1 \to L \to M_2 \to 0$$

Let Q be the quiver with one vertex and no edges. Then the representations of Q are just vector spaces (and no morphisms). So the basis elements are indexed by $\mathbb{N}$ up to isomorphism, $[nS]$, where S is the one-dimensional vector space. To determine multiplication:

$$[nS][mS] := \sum_{[L] \in H_q(\bullet, \mathbb{F}_q)} F^L_{M_1 M_2}[L] = \sum_{j=1}^{\infty} F^{jS}_{nSmS} jS$$

Recall $F^{jS}_{nSmS} = \frac{X^{jS}_{nSmS}}{|Aut_Q(nS) \times Aut_Q(mS)|}$ where X^{jS}_{nSmS} is defined as the number of pairs (f, g) with

$$0 \to mS \xrightarrow{f} jS \xrightarrow{g} nS \to 0$$

an exact sequence, i.e. counts the number of extensions of nS by mS. Because we are in the setting of vector spaces (over $\mathbb{F}_q$, but the statement still holds), short exact sequences always split, thus the only such extension is given by the direct sum

$$jS \cong mS \oplus nS$$

In particular, $j = m + n$. So up to isomorphism, we have

$$[nS][mS] = F^{(n+m)S}_{nSmS}(n+m)S$$

and it only remains to compute its coefficient. The number of SES's is exactly equivalent to, up to isomorphism, a choice of m dimensional subspace of $\mathbb{C}^{n+m}$. (After you have made such a choice, the map g must be projection onto the remaining coordinates, so g contains no information in this case). In other words,

$$|F_{nSmS}^{(n+m)S}| = |Gr_{\mathbb{F}_q}(n, n+m)| = \frac{(q^{n+m}-1)(q^{n+m}-q)\cdots(q^{n+m}-q^{m-1})}{(q^m-1)(q^m-q)\cdots(q^m-q^{m-1})} = \binom{n+m}{m}_q$$

So

$$[nS][mS] = \binom{n+m}{m}_q [(n+m)S]$$

3/11/2023 A projective resolution of a quiver representation

Let Q be a type A quiver of length ℓ but with arrows pointing to the left. Define a family of representations of Q, $S(i) \in Rep(Q)$, as having a single copy of k at the ith vertex and 0 at all others, so that all the linear maps must also be 0. Also define $P(n) \in Rep(Q)$ as the representation with k placed at every vertex $1, 2, \ldots, n$, for $n \in \{1, 2, \ldots, \ell\}$, and identity morphism for all arrows that can possibly be non-zero. Note that there is a natural transformation $P(1) \to P(2)$: We have to first give a map

$$P(1)(\bullet_i) \to P(2)(\bullet_i)$$

for every i. This is clear: If $i = 1$, then

$$P(1)(\bullet_1) = k \to P(2)(\bullet_1) = k$$

so we are free to choose the identity map. For $i = 2$, then $P(1)(\bullet_2) = 0$, so we are forced to choose 0, and similarly for all $i > 2$. This defines the components of the natural transformation, and we also have to check the naturality condition: Diagrammatically, this is given by

$$\begin{tikzcd}
P(2) & k & k \arrow[l, "id"'] & 0 \arrow[l, "0"'] & \cdots \arrow[l, "0"'] & 0 \arrow[l, "0"'] \\
P(1) \arrow[u] & k \arrow[u, "id"] & 0 \arrow[l, "0"] \arrow[u, "0"] & 0 \arrow[l, "0"] \arrow[u, "0"] & \cdots \arrow[l, "0"] & 0 \arrow[l, "0"] \arrow[u, "0"]
\end{tikzcd}$$

So all squares commute. Notice that if the arrows went the other way, as in the classical type A quiver, then the first square would not commute. Instead I think we would get a morphism $P(2) \to P(1)$, so these are really the same statements, just different "chirality".

In the same fashion, we have a morphism

$$P(n-1) \to P(n)$$

for every n, and thus a sequence of maps

$$P(1) \to P(2) \to \cdots \to P(n)$$

It is easy to check that each arrow defines a natural transformation. Finally, there is a map

$$P(i) \to S(i)$$

$$\begin{tikzcd}
S(i) & 0 & 0 \arrow[l, "0"'] & \dots \arrow[l, "0"'] & k \arrow[l, "0"'] & 0 \arrow[l, "0"'] & \dots \arrow[l, "0"'] & 0 \arrow[l, "0"'] \\
P(i) \arrow[u] & k \arrow[u, "0"] & k \arrow[l, "id"] \arrow[u, "0"] & \dots \arrow[l, "id"] \arrow[u, "0"] & k \arrow[l, "id"] \arrow[u, "id"] & 0 \arrow[l, "0"] \arrow[u, "0"] & \dots \arrow[l, "0"] & 0 \arrow[l, "0"] \arrow[u, "0"]
\end{tikzcd}$$

Observe that in each component[13], the map $P(i) \to S(i)$ is surjective, while the map $P(n-1) \to P(n)$ is injective. So we can conjecture an exact sequence of the form

$$0 \to P(i-1) \xrightarrow{f} P(i) \xrightarrow{g} S(i) \to 0$$

Diagrammatically:

$$\begin{tikzcd}
S(i) & 0 & 0 \arrow[l, "0"'] & \dots \arrow[l, "0"'] & k \arrow[l, "0"'] & 0 \arrow[l, "0"'] & \dots \arrow[l, "0"'] & 0 \arrow[l, "0"'] \\
P(i) \arrow[u, "g"] & k \arrow[u, "0"] & k \arrow[l, "id"] \arrow[u, "0"] & \dots \arrow[l, "id"] \arrow[u, "0"] & k \arrow[l, "id"] \arrow[u, "id"] & 0 \arrow[l, "0"] \arrow[u, "0"] & \dots \arrow[l, "0"] & 0 \arrow[l, "0"] \arrow[u, "0"] \\
P(i-1) \arrow[u, "f"] & k \arrow[u, "id"] & \dots \arrow[l, "id"] & k \arrow[l, "id"] \arrow[u, "id"] & 0 \arrow[l, "0"] \arrow[u, "0"] & 0 \arrow[l, "0"] \arrow[u, "0"] & \dots \arrow[l, "0"] & 0 \arrow[l, "0"] \arrow[u]
\end{tikzcd}$$

and it remains to check exactness at the middle index. As we claimed, it suffices to check this componentwise in f and g. We check at the component $\bullet_j$:

$$0 \to P(i-1)(\bullet_j) \xrightarrow{f_{\bullet_j}} P(i-1)(\bullet_j) \xrightarrow{g_{\bullet_j}} S(i)(\bullet_j) \to 0$$

This is a sequence in *Vect*. Note that $g_{\bullet_j}$ (the jth vertical map in the top), by definition, only has kernel if $j < i$. If $j > i$, then the domain is 0, thus there can be no non-zero kernel. If $j = i$, then the map is the identity. If $j < i$, then the map is $0 : k \to 0$, thus has kernel k. However for all $j < i$, $ker\ g_{\bullet_j} = k = im\ f_{\bullet_j}$, since $f_{\bullet_j} = id$ for $j < i$. Therefore the sequence is exact componentwise, so

$$0 \to P(i-1) \to P(i) \to S(i) \to 0$$

[13]We are implicitly using a statement here: A morphism of functors (nat trans) is injective/surjective iff it is injective/surjective in each component. Certainly this is not true in general. In fact, if the target category doesn't have a 0 object, this doesn't even make sense. However for us, it reduces nicely: A sequence of functors is exact iff the component sequence is exact for each component. Thus it reduces to a linear algebra statement of checking exactness of vector spaces.

is exact. So it remains to show that $P(i)$ is a projective object of $Rep(Q)$.

In summary, we have produced a family of projective resolutions of representations of the "dual" type A quiver (but the dual is artificial. the same argument applies for the ordinary type A quiver).

3/15/2023 The path algebra of a quiver with one vertex and n arrows

The path algebra by definition is an algebra with basis given by the set of paths in Q with multiplication given by concatenation (if the tip and tail match appropriately, else 0) and formal addition. In the case $n = 1$, there are infinitely many paths, corresponding to going once from $\bullet$ to $\bullet$, or twice, or so on. So $k(Q, n = 1) \cong k[x]$, where x^n corresponds to the path from $\bullet$ to $\bullet$ given by taking the loop n times. It is important to note that in this case the path algebra is commutative, since we have only one loop. However if we consider the quiver Q with one vertex and two edges, there is nothing in our theory which says going around one loop and then the other is the same as going around the other loop and then the one. This is a non-commutative algebra. This is exactly the same as the statement that $\pi_1(S^1 \vee \cdots \vee S^1) \cong *\mathbb{Z} \equiv F_n$ except we don't have to do any topology. This reference is important because the way people typically describe this path algebra as the free algebra on n variables. This should be thought of as the analogue of the free group, and should also be thought of exactly as the tensor algebra and as the algebra of non-commutative polynomials.

3/17/2023 Path algebra of a type A quiver

Let Q be a type A quiver of length ℓ. Consider a path in kQ which starts at ith vertex and ends at j. By the construction of the type A quiver, such a path can only arise as taking each successive path $i \to i+1 \to i+2 \to \cdots \to j$, for $i < j$. So for every $i < j$, there is a single path, and there is no path for $i \geq j$. Define a map

$$kQ \to B_\ell(k)$$

the algebra of $n \times n$ upper triangular matrices by sending a path e_{ij} to the elementary matrix E_{ij}. This map defines an isomorphism.

3/20/2023 Representations of $\bullet \to \bullet \leftarrow \bullet$ (the uwu quiver)

Let Q be the above quiver. Let $R = ((x_1, x_2), (V_1, V_0, V_2))$ be a representation of Q. Then by linear algebra, $V_i = ker(x_i) \oplus V_i'$, where V_i' is the orthogonal complement, and is isomorphic to $V_i/ker(x_i)$. Under such a decomposition, we have $x_i = (0, \widetilde{x}_i)$, where $\widetilde{x}_i$ is the restriction to to the complement/induced map on the quotient, depending on how you want to view V' (I suppose technically if we are considering V_i abstract, not $\mathbb{C}^{v_i}$, then these are not inner product spaces, so you can't take complements. Then view it always as the quotient). Obviously $\widetilde{x}_i$ is injective. So the representation R is isomorphic to:

$$R = \Big((0 \oplus \widetilde{x}_1, 0 \oplus \widetilde{x}_2), (ker(x_1) \oplus V_1', 0 \oplus V_0, ker(x_2) \oplus V_2')\Big)$$

Graphically,

$$\begin{tikzcd} ker(x_1) \oplus V_1' \arrow[r, "(0{,}\widetilde{x}_1)"'] & 0 \oplus V_0 & ker(x_2) \oplus V_2' \arrow[l, "(0{,}\widetilde{x}_2)"] \end{tikzcd}$$

which is a direct sum, by definition:

$$\begin{tikzcd} V_1' \arrow[r, "\widetilde{x}_1"'] & V_0 & V_2' \arrow[l, "\widetilde{x}_2"] \end{tikzcd} \bigoplus \begin{tikzcd} ker(x_1) \arrow[r, "0"'] & 0 & ker(x_2) \arrow[l, "0"] \end{tikzcd}$$

Notice the second term can also be written as a direct sum: $\big((0,0),(ker(x_1),0,ker(x_2)\big) = dim\, ker(x_1)S(1) \oplus dim\, ker(x_2)S(2)$, where $S(i)$ is defined as in the 3/11/2023 example. So overall

$$R \cong n_1 S(1) \oplus n_2 S(2) \oplus V'$$

where V' is the representation with all x_i's injective. Therefore a representation of Q can be classified by two natural numbers and a representation V' with two injective maps.

Thus classification of representations of Q up to isomorphism reduces to classification of triples V_0, V_1, V_2 with V_1, V_2 subspaces of V_0. To classify these:

Proposition: *We can always choose a basis of V_0 which reduces to a basis of V_1, V_2 and $V_1 \cap V_2$ upon restriction*:

Proof: If $V_1 \cap V_2$ is non-empty, choose a basis of it. Extend arbitrarily to a basis of V_1 and a basis of V_2, so $B_1 = \{v_1, \dots, v_\ell, v_{\ell+1}, \dots, v_{\ell+k_1}\}$ and $B_2 = \{v_1, \dots, v_\ell, v'_{\ell+1}, \dots, v'_{\ell+k_2}\}$ are bases of V_1 and V_2, with $v_1, \dots, v_\ell \in V_1 \cap V_2$, $v_{\ell+i} \in V_1 \setminus V_2$ and $v'_{\ell+i} \in V_2 \setminus V_1$. Claim $B_1 \cup B_2$ is a linearly independent set. Examine the linear combination

$$\lambda_1 v_1 + \dots \lambda_\ell v_\ell + \lambda_{\ell+1} v_{\ell+1} + \lambda'_{\ell+1} v'_{\ell+1} + \cdots + \lambda_{\ell+k_1} v_{\ell+k_1} + \cdots + \lambda'_{\ell+k_2} v'_{\ell+k_2} = 0$$

This can be rearranged as

$$\lambda_{\ell+1} v_{\ell+1} + \cdots + \lambda_{\ell+k_1} v_{\ell+k_1} = -\lambda_1 v_2 - \cdots - \lambda_\ell v_\ell - \lambda'_{\ell+1} v'_{\ell+1} - \cdots - \lambda'_{\ell+k_2} v'_{\ell+k_2}$$

in English, we put the components from $V_1 \setminus V_2$ on the LHS and the components from $V_1 \cap V_2$ and $V_2 \setminus V_1$ on the right hand side. But this is a contradiction unless some of the coefficients are 0, because the left hand side is an element of $V_1 \setminus V_2$, while the RHS is an element of V_2. WLOG, This implies the "strictly V_2 coefficients" must be 0 (we could also have chosen the strictly V_1 coefficients to 0): $\lambda' \equiv 0$. So we can rearrange the above equation, moving everything back to the left again, as

$$\lambda_1 v_1 + \dots \lambda_\ell v_\ell + \lambda_{\ell+1} v_{\ell+1} + \cdots + \lambda_{\ell+k_1} v_{\ell+k_1} = 0$$

But $\{v_1, \dots, v_\ell, v_{\ell+1}, \dots, v_{\ell+k_1}\}$ is a basis of V_1, so $\lambda \equiv 0$. Thus $B_1 \cup B_2$ is a linearly independent set. It is not a basis of $V_1 \cup V_2$ because $V_1 \cup V_2$ is not a subspace, but that is okay. Then extend $B_1 \cup B_2$ arbitrarily to a basis of V_0.

$\square$

Thus any such representation is a direct sum of the following representations

$$\overset{k}{\bullet} \to \overset{0}{\bullet} \leftarrow \overset{0}{\bullet}, \qquad \overset{0}{\bullet} \to \overset{0}{\bullet} \leftarrow \overset{k}{\bullet}, \qquad \overset{0}{\bullet} \to \overset{k}{\bullet} \leftarrow \overset{0}{\bullet}$$

$$\overset{k}{\bullet} \to \overset{k}{\bullet} \leftarrow \overset{0}{\bullet}, \qquad \overset{0}{\bullet} \to \overset{k}{\bullet} \leftarrow \overset{k}{\bullet}, \qquad \overset{k}{\bullet} \to \overset{k}{\bullet} \leftarrow \overset{k}{\bullet}$$

3/23/2023 The triple sum dimension formula is wrong

One may guess, for U, V, W subspaces of some vector space, that the formula for the dimension of the triple sum is given by

$$dim(U + V + W) = dim(U) + dim(V) + dim(W) - dim(U \cap V)$$

$$-dim(U \cap W) - dim(V \cap W) + dim(U \cap V \cap W)$$

by applying the correct formula for the single sum:

$$dim(U + V) = dim(U) + dim(V) - dim(U \cap V)$$

twice and following your nose. But this is false, as we may take, for example, three lines through the origin in $\mathbb{R}^2$ to be U, V, W. For generic choices, the left hand side is equal to two, since we will certainly span all of $\mathbb{R}^2$, but the intersections are all just the origin, so the correction terms all vanish, yielding 2=3, a contradiction. The error comes when attempting to distribute the sum across the intersection:

$$(U + V) \cap W \stackrel{?}{=} U \cap W + V \cap W$$

in our example, the left hand side is $\mathbb{R}^2 \cap W = W$, while the right hand side is 0.

Fun fact: This is connected to the representation theory of the 3-subspace quiver, D_4. It can be shown (using the rep theory) that this counterexample presented is the only thing that can go wrong in this case. However for 4 subspaces it is worse: this is because the 4-subspace quiver is not Dynkin, it is a tame quiver. 5 subspaces is even worse because the 5-subspace quiver is wild.

3/25/2023 More equivariant cohomology of a point

Let Q be any quiver and consider the vector space of representations for a fixed weight vector, $\vec{v}$, $Rep(Q, \vec{v})$. As a smooth manifold this is contractible, thus has the homotopy type of a point. In particular, we can compute its equivariant cohomology (under the standard action of $GL(\vec{v})$) via our previous example 2/10/2023:

$$H^*_{GL(\vec{v})}(Rep(Q, \vec{v})) \cong H^*(BGL(\vec{v}))$$

where BG is the classifying space of G. We know that the classifying space respects products, so

$$\cong H^*(BGL(\vec{v}_1) \times \cdots \times BGL(\vec{v}_\ell))$$

and by Kunneth formula,

$$\cong \bigotimes H^*(BGL(\vec{v}_i))$$

Now we need to actually find a model of $BGL(n)$. We computed the model of BS^1 in Andrey's seminar: The total space, ES^1, is a topological space on which S^1 acts freely and the space is contractible. One model is S^∞. Then BG is the quotient by the action, $BS^1 = ES^1/S^1 = S^\infty/S^1 \equiv \mathbb{C}P^\infty$. Note this is exactly our example for $n = 1$, since $GL(1, \mathbb{C}) \cong \mathbb{C}^* \cong S^1$. Written more suggestively,

$$BGL(1) \cong \mathbb{C}P^\infty \equiv Gr_1(\mathbb{C}^\infty)$$

Leading us to the naive guess

$$BGL(n) \cong Gr_n(\mathbb{C}^\infty)$$

which is correct. Then we again guess the formula for the ordinary cohomology ring from the base case:

$$H^*(Gr_1(\mathbb{C}^\infty)) \equiv H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[u], \quad |u| = 2$$

Note that we can't just append more variables in degree 2, as this is what happens when you consider products of groups (and thus products of classifying spaces). Instead we must append variables in increasing degrees:

$$H^*(Gr_k(\mathbb{C}^\infty)) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n], \quad |c_i| = 2i$$

thus

$$H^*_{GL(\vec{v})}(Rep(Q, \vec{v})) \cong \mathbb{Z}[c_{11}, c_{12}, \ldots, c_{1\vec{v}_1}, c_{21}, c_{22}, \ldots, c_{2\vec{v}_2}, \ldots, c_{\ell\vec{v}_\ell}]$$

$$|c_{ij}| = 2i$$

where ℓ represents the number of vertices of Q. The c_i's represent chern classes. At each vertex, the polynomial ring can be written in terms of the symmetric chern roots:

$$\cong \mathbb{Z}[\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1\vec{v}_1}, \ldots, \gamma_{\ell,\vec{v}_\ell}]^{\prod_{j=1}^{\ell} S_{v_j}}$$

The nature of this isomorphism may be discussed later.

3/26/2023 nil vs nilpotent ideal

Recall: An ideal of a ring is called nil if every element in it is nilpotent. An ideal N is called nilpotent if there exists some $n > 0$ such that $N^n = 0$, the 0 ideal. Equivalently, if every n-fold product of elements in N is 0.

Of course nilpotent implies nil, since we could take specific products containing only powers of a single element.

It can be shown that if an ideal is finitely generated by nilpotent elements, then the ideal is nilpotent.

If an ideal is (even infinitely) generated by nilpotent elements, then it is nil: This follows immediately because the sum of nilpotent elements is nilpotent (via binomial theorem) and a product of the form ra is also nilpotent, and any element of the ideal is a finite R-linear combination of the generators. But nil does not (in general) imply the ideal is nilpotent. As a counterexample, consider the polynomial ring

$$\mathbb{Z}[x_1, x_2, \dots]/(x_1^2, x_2^3, x_3^4, \dots)$$

and consider the ideal generated by the images of all x_i's through the quotient. This ideal is infinitely generated by nilpotent elements, so it is nil. However I cannot be nilpotent because for any $N > 0$, we can consider the product N-fold product $\overline{x_N}^N \in I$ which does not vanish.

If we assume that I is finitely generated, then this problem of not having a "global/uniform" power causing every element to vanish goes away. It seems like you may not be able to guarantee that the distinct products will vanish, but I believe you can apply the generalized pigeonhole principle to show this. Therefore I conjecture that f.g. + nil should imply nilpotent, but I haven't checked.

4/2/2023 Something in the spectrum which is not an eigenvalue

Consider the shift operator $T : \ell^2 \to \ell^2$ sending

$$(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$$

This clearly has no eigenvalues: If it did, then $x_1 = 0$ which implies $x_2 = 0$ and so on, so $x = 0$, but we do not count 0 as an eigenvalue unless it has a non-zero eigenvector. However claim that $0 \in \sigma(T)$. This is because the operator $(T - 0I) = T$ does not admit bounded inverse: It admits a set theoretic inverse only on the subset of ℓ^2 with $x_1 = 0$, which is obviously not dense, therefore T^{-1} is unbounded[14]. I believe we could technically say that 0 belongs to the approximate point spectrum and the residual spectrum of T, since the range is not dense and the inverse is unbounded. This also shows that the two spectra don't have to be disjoint.

4/3/2023 Gaussian Elimination in an additive category

I learned this example from Luke Conners (I'm going to try to attribute more going forward. I learn so much from talking to people, and that is probably not reflected as much as it should be in this document.). This is partially an exercise in showing that homotopy equivalence of chain complexes is annoying to deal with.

Proposition: *Let $\mathcal{A}$ be an additive category, and let $Ch(\mathcal{A})$ be the additive category of chain complexes with objects in $\mathcal{A}$ and chain maps as morphisms. Let $A_\bullet \in Ch(\mathcal{A})$ which contains a*

[14] If it were defined on a dense subset of ℓ^2, then we could have extended it uniquely to an inverse.

subcomplex isomorphic to the complex

$$A \xrightarrow{\begin{pmatrix} *_A \\ \alpha \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} \varphi & \lambda \\ \mu & \eta \end{pmatrix}} D \oplus E \xrightarrow{\begin{pmatrix} *_D & \epsilon \end{pmatrix}} F$$

where φ is an isomorphism $B \to D$, and $*_A, *_D$ are any maps satisfying the condition $d^2 = 0$. Then there is a homotopy equivalence from $A_\bullet$ to a chain complex, $B_\bullet$, containing the subcomplex

$$A \xrightarrow{\alpha} C \xrightarrow{\eta - \mu\varphi^{-1}\lambda} E \xrightarrow{\epsilon} F$$

and $A_\bullet = B_\bullet$ away from this subcomplex.

Remark: Note this implies the two complexes are isomorphic in $\mathcal{K}(\mathcal{A})$, the homotopy category of $\mathcal{A}$.

Proof: First we have to prove that the sequence on the bottom is a chain complex. For example, we must show that the map $A \to C \to E = 0$, given by

$$a \mapsto (\eta\alpha)(a) - (\mu\varphi^{-1}\lambda\alpha)(a)$$

Because $A_\bullet$ is a chain complex with 4 terms, there are 2 compositions of maps which we know are 0. We use the first composition here, $A \to D \oplus E$, and the second $B \oplus C \to F$ later. Each one comes with two 0 maps in the components. Let's examine $A \to D \oplus E$: The component map $A \to D$ is the sum of $A \to B \to D$ and $A \to C \to D$:

$$(I) \qquad \varphi *_A + \lambda\alpha = 0 \Rightarrow *_A = -\varphi^{-1}\lambda\alpha$$

The component map $A \to E$ is the sum of $A \to B \to E$ and $A \to C \to E$:

$$\mu *_A + \eta\alpha = 0$$

Combining this with the above line:

$$-\mu\varphi^{-1}\lambda\alpha + \eta\alpha = 0$$

as required.

Similarly, we require the map $C \to F$ to be 0, given by

$$c \mapsto \epsilon\eta(c) - \epsilon\mu\varphi^{-1}\lambda(c)$$

The component $B \to F$ is the sum of maps $B \to D \to F$ and $B \to E \to F$:

$$(II) \qquad *_D\, \varphi + \epsilon\mu = 0 \Rightarrow *_D = -\epsilon\mu\varphi^{-1}$$

Similarly the component $C \to F$ is the sum of maps $C \to D \to F$ and $C \to E \to F$:

$$*_D\lambda + \epsilon\eta = 0$$

Combining this with the line above, we have

$$-\epsilon\mu\varphi^{-1}\lambda + \epsilon\eta = 0$$

as required. So we we have shown $B_\bullet$ is a chain complex.

We must define a homotopy equivalence from $A_\bullet$ to $B_\bullet$. First, this requires the data of chain maps $\Psi : A_\bullet \to B_\bullet$, $\Phi : B_\bullet \to A_\bullet$. We define these as the identity away from the subcomplexes in question, and

$$\Psi_A = Id_A, \quad \Psi_{B\oplus C} = \pi_C, \quad \Psi_{D\oplus E} = \begin{pmatrix} -\mu\varphi^{-1} & 1 \end{pmatrix}, \quad \Psi_F = Id_F$$

and

$$\Phi_A = Id_A, \quad \Phi_C = \begin{pmatrix} -\varphi^{-1}\lambda \\ 1 \end{pmatrix}, \quad \Phi_E = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Phi_F = Id_F$$

I include a picture below[15].

$$\begin{tikzcd}[ampersand replacement=\&]
A \arrow[r, "{\begin{pmatrix} \bullet \\ \alpha \end{pmatrix}}"] \arrow[d, shift left, "(1)"] \& B \oplus C \arrow[r, "{\begin{pmatrix} \varphi & \lambda \\ \mu & \nu \end{pmatrix}}"] \arrow[d, shift right, "(0\ 1)"'] \& D \oplus E \arrow[r, "{(\bullet\ \epsilon)}"] \arrow[d, shift right, "{(-\mu\varphi^{-1}\ 1)}"'] \& F \arrow[d, shift left, "(1)"] \\
A \arrow[r, "(\alpha)"] \arrow[u, shift left] \& C \arrow[r, "(\nu - \mu\varphi^{-1}\lambda)"] \arrow[u, shift right, "{\begin{pmatrix} -\varphi^{-1}\lambda \\ 1 \end{pmatrix}}"'] \& E \arrow[r, "(\epsilon)"] \arrow[u, shift right, "{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}"'] \& F \arrow[u, shift left]
\end{tikzcd}$$

Note that $\Psi_{D\oplus E} : D \oplus E \to E$ is not just projection and $\Phi_C : C \to B \oplus C$ is not just inclusion: because φ is invertible, there are natural maps $D \to B \to E$ and $C \to D \to B$, so we must account for that. Now we show these are chain maps. Unfortunately we do need to check all 3 squares, even though the outside maps are identity maps. First we do Ψ. In the first square:

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[d, mapsto] & (*, \alpha(a)) \arrow[d, mapsto] \\
a \arrow[r, mapsto] & \alpha(a)
\end{tikzcd}$$

[15] I could not get Tikz-cd to play nicely with matrix labels on arrows. If interested, it is because the tikz universe uses ampersand to separate spaces in the diagram, while the matrix uses ampersand to separate spaces in the matrix, so ampersand is overloaded. There are tutorials on line for how to get around this problem easily, but I was unable to implement any of these solutions succesfully. Instead this image is from the arxiv paper: "On Khovanov's cobordism theory for $\mathfrak{su}(3)$ knot homology" by Scott Morrison. Note that their ν is my η.

in the middle square,

$$\begin{tikzcd}
(b,c) \arrow[r, mapsto] \arrow[d, mapsto] & \big(\varphi(b)+\lambda(c), \mu(b)+\eta(c)\big) \arrow[d, mapsto] \\
c \arrow[r, mapsto] & \eta(c)-(\mu\varphi^{-1}\lambda)(c) = -\mu\varphi^{-1}\big(\varphi(b)+\lambda(c)\big)+\mu(b)+\eta(c)
\end{tikzcd}$$

the bottom right equality follows from simple algebra. In the final square,

$$\begin{tikzcd}
(d,e) \arrow[r, mapsto] \arrow[d, mapsto] & *_D(d)+\epsilon(e) \arrow[d, mapsto] \\
-\mu\varphi^{-1}(d)+e \arrow[r, mapsto] & \epsilon(-\mu\varphi^{-1}(d))+\epsilon(e) = *_D(d)+\epsilon(e)
\end{tikzcd}$$

The bottom right equality follows directly from (II). So Ψ is a chain map. Similarly, we check Φ. In the first square,

$$\begin{tikzcd}
a \arrow[r, mapsto] & \big(*_A(a), \alpha(a)\big) = \big(-\varphi^{-1}\lambda\alpha(a), \alpha(a)\big) \\
a \arrow[r, mapsto] \arrow[u, mapsto] & \alpha(a) \arrow[u, mapsto]
\end{tikzcd}$$

where the top right equality follows from (I). In the middle square,

$$\begin{tikzcd}
\big(-\varphi^{-1}\lambda(c), c\big) \arrow[r, mapsto] & \big(\varphi(-\varphi^{-1}\lambda(c))+\lambda(c), \mu(-\varphi^{-1}\lambda(c))+\eta(c)\big) = \big(0, \eta(c)-\mu\varphi^{-1}\lambda(c)\big) \\
c \arrow[r, mapsto] \arrow[u, mapsto] & \eta(c)-\mu\varphi^{-1}\lambda(c) \arrow[u, mapsto]
\end{tikzcd}$$

where the equality in the top right follows from simple algebra. In the final square,

$$\begin{tikzcd}
(0,e) \arrow[r, mapsto] & 0+\epsilon(e) = \epsilon(e) \\
e \arrow[r, mapsto] \arrow[u, mapsto] & \epsilon(e) \arrow[u, mapsto]
\end{tikzcd}$$

So we have verified that both Ψ and Φ are chain maps. It remains to show that they define a homotopy equivalence, that is: $\Psi\Phi \simeq Id_{B_\bullet}$ and $\Phi\Psi \simeq Id_{A_\bullet}$. Recall this means we need to produce homotopy equivalences (degree -1 "chain maps" which don't necessarily commute with differentials) $h^A : A_\bullet \to A_\bullet$, $h^B : B_\bullet \to B_\bullet$ such that for every n, we have

$$d^A_{n-1} \circ h^A_n + h^A_{n+1} \circ d^A_n = \Phi_n \circ \Psi_n - Id_n$$

and

$$d^B_{n-1} \circ h^B_n + h^B_{n+1} \circ d^B_n = \Psi_n \circ \Phi_n - Id_n$$

But note that $\Psi \circ \Phi = Id_{B_\bullet}$, so there is nothing to check here (we can set $h \equiv 0$. Exercise: Check if you want to show a complex is homotopy equivalent to itself, you should choose $h \equiv 0$.). So it remains to construct h^A showing $\Phi \circ \Psi \simeq Id_{A_\bullet}$. Clearly we will choose $h^A_n = 0$ away from the subcomplex in question. Specifically, since the maps are diagonal we (may) need make a non-trivial definition in 5 places, those maps which involve an element of the subcomplex in either the domain or the codomiain:

$$\begin{tikzcd}
A_n \arrow[r] \arrow[d] & A \arrow[r] \arrow[dl, "h^A_A"'] \arrow[d, bend right, "\Phi\Psi"'] \arrow[d, bend left, "Id"] & B \oplus C \arrow[r] \arrow[dl, "h^A_{B\oplus C}"'] \arrow[d, bend right, "\Phi\Psi"'] \arrow[d, bend left, "Id"] & D \oplus E \arrow[r] \arrow[dl, "h^A_{D\oplus E}"'] \arrow[d, bend right, "\Phi\Psi"'] \arrow[d, bend left, "Id"] & F \arrow[r] \arrow[dl, "h^A_F"'] \arrow[d, bend right, "\Phi\Psi"'] \arrow[d, bend left, "Id"] & A_{n+5} \arrow[d] \arrow[dl, "h^A_{A_{n+5}}"'] \\
A_n \arrow[r] & A \arrow[r] & B \oplus C \arrow[r] & D \oplus E \arrow[r] & F \arrow[r] & A_{n+5}
\end{tikzcd}$$

We inspect immediately that we can choose $h^A_{A_{n+5}} = h^A_A = 0$. This is forced because we chose each diagonal h to the side of these to be 0. Further, $\Psi_A = \Phi_A = Id_A$, so h^A_A must satisfy

$$d^A_n \circ h^A_A + \begin{pmatrix} *_A \\ \alpha \end{pmatrix} \circ h^A_{B\oplus C}$$

but because we chose $h^A_A = 0$, $h^A_{B\oplus C} = 0$. Similarly we must also choose $h^A_F = 0$. So it remains to define $h^A_{D\oplus E} : D \oplus E \to B \oplus C$. This will be a 2 x 2 matrix satisfying

$$h^A_{D\oplus E} \begin{pmatrix} \varphi & \lambda \\ \mu & \eta \end{pmatrix} (b, c) = (\Phi\Psi - Id)(b, c)$$

Evaluating,

$$h^A_{D\oplus E}\big(\varphi(b) + \lambda(c), \mu(b) + \eta(c)\big) = (-\varphi^{-1}\lambda(c) - b, 0)$$

requiring that we choose $h^A_{D\oplus E} = \begin{pmatrix} \varphi^{-1} & 0 \\ 0 & 0 \end{pmatrix}$.

We have shown the collection of h^A and h^B's, Φ, and Ψ defined in this way constitute a homotopy equivalence between the two chain complexes $A_\bullet \simeq B_\bullet$.

4/8/2023 The Schubert cells of $Gr(2,4)$

Let $\mathcal{V} = V_0 \subset V_1 \ldots V_4 = \mathbb{C}^4$ denote the standard complete flag in $\mathbb{C}^4$. For a generic element $V_{gen} \subset Gr(2,4)$, we can expect the intersection numbers (dimension of intersection) to be

	V_0	V_1	V_2	V_3	$V_4 = \mathbb{C}^4$
dim $V_{gen} \cap V_i$	0	0	0	1	2

The general pattern to recognize is that $\dim(\mathbb{C}^{n-k} \cap V^k) = 0$, since this is sort of like V^k and its orthogonal complement. A typical k-plane will have vanishing intersection number to the left of $n-k$ and increase by 1 starting at $n-k+1$ until it reaches k. Let's look at some specific k-planes: denote $V_{ij} = \text{span}(e_i, e_j)$:

	V_0	V_1	V_2	V_3	$V_4 = \mathbb{C}^4$
$\dim V_{gen} \cap V_i$	0	0	0	1	2
$\dim V_{12} \cap V_i$	0	1	2	2	2
$\dim V_{24} \cap V_i$	0	0	1	1	2
$\dim V_{34} \cap V_i$	0	0	0	1	2

To each $V \in Gr(2,4)$, we associate a partition of length 2 (the same 2 from $Gr(2,4)$) by choosing the first entry to be the distance of the first appearance of 1 in the table from the standard, and the second entry to be distance of the first appearance of 2 from the standard.

$$V_{12} \rightsquigarrow (2,2)$$
$$V_{24} \rightsquigarrow (1,0)$$
$$V_{34} \rightsquigarrow (0,0)$$

For each such partition, λ, we define the (open) Schubert cell Σ_λ° as the subset of $Gr(2,4)$ whose associated partition is λ. Claims: In general, the Schubert cells Σ_λ° form an open affine cover of the projective variety $Gr_n(\mathbb{C}^m) \subset \mathbb{P}^{\binom{m}{n}-1}$ (the existence of one such is guaranteed by the theory of quasiprojective varieties). To see they are affine, one should really work with the matrix definition, from which it is obvious that they are affine. This collection covers because every V must be associated to *some* partition. To count the number of charts, one should look to the associated Young tableaux: To each cell Σ_λ, we have an associated YT arising from the partition λ. I don't know what the general formula for this is (the number of YT fitting inside a fixed "universe" box), but in this case it is easy to see: The universe is the 2 x 2 square, and one can check that there are 6 ways, corresponding to the partitions $(0,0), (1,0), (1,1), (2,0), (2,1), (2,2)$.

4/10/2023 *The motive of $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$

First recall the Grothendieck ring, $K_0(Var_k)$, is generated additively by isomorphism classes of varieties over k, subject to the "cut-and-paste" relation, that is: $[X] = [Y] + [U]$ if $Y \subset X$ is a closed subvariety, $U = Y^C$, and U is open in X, and multiplication given by Cartesian product of representatives. The <u>motive</u> of a variety is just its class in the Grothendieck ring.

Definition: The <u>Lefschetz motive</u> is the class $[\mathbb{A}_k^1] \equiv \mathbb{L} \in K_0(Var_k)$.

Example: We have $\mathbb{A}_k^n \cong \mathbb{A}_k^1 \times \cdots \times \mathbb{A}_k^1$, so that $[\mathbb{A}_k^n] = \mathbb{L}^n$.

Example: In general, we have: $\mathbb{P}^n_k = \mathbb{P}^{n-1}_k \cup \mathbb{A}^n_k$ (This is just choosing one element of the standard affine cover). So by the cut-and-paste relation, we have

$$\begin{aligned}
[\mathbb{P}^n_k] &= [\mathbb{P}^{n-1}_k] + \mathbb{L}^n \\
&= [\mathbb{P}^{n-2}_k] + \mathbb{L}^{n-1} + \mathbb{L}^n \\
&\vdots \\
&= 1 + \mathbb{L} + \cdots + \mathbb{L}^n \text{ "="} \frac{\mathbb{L}^{n+1} - 1}{\mathbb{L} - 1}
\end{aligned}$$

Of course, the final equality doesn't really make sense, as we don't know what it means to divide by $\mathbb{L} - 1$ in the Grothendieck ring. However there is a ring where it does make sense, $R^{mot}(Var_k)$, which is in essence just the Grothendieck ring with some denominators and square roots formally adjoined.

Example: We know that $Gr_2(\mathbb{C}^4)$ admits an open disjoint affine cover via Schubert varieties (generally any $Gr(n, m)$ is covered by these). So in particular,

$$Gr_2(\mathbb{C}^4) = \mathbb{A}^3_k \cup$$

$\boxed{\text{FINISH}}$

4/15/2023 Examples of Motivic invariants

A property of varieties is called a motivic invariant if it is an invariant and satisfies the cut-and-paste relation. So suppose $V(X)$ is some quantity you can compute out of a variety X, we say it is a motivic invariant if

$$V([X]) = V([Y]) + V([U])$$

is well-defined (i.e. $V(X)$ is an isomorphism invariant) and the equality holds, where Y is a closed subvariety and U is its open complement.

Euler Characteristic: I think this is just a statement about disjoint unions really. The homology of a disjoint union is the direct sum of homologies. In cases where it makes sense, one can just use the singular homology to define the Euler characteristic. Then

$$\chi(X \sqcup Y) \equiv \sum_{i=0}^{\infty} (-1)^i h^i(X \sqcup Y) = \sum_{i=0}^{\infty} (-1)^i h^i(X) + h^i(Y) = \chi(X) + \chi(Y)$$

Further, χ is an isomorphism invariant so χ is a motivic invariant.

Cardinality of points over a finite field: This is just because $X(\mathbb{F}_{q^m}) = Y(\mathbb{F}_{q^m}) \sqcup U(\mathbb{F}_{q^m})$. This is an isomorphism invariant because if $X \cong X'$, then

$$Hom(Spec\ k, X) \cong Hom(Spec\ k, X')$$

by composition with the isomorphism in question.

4/20/2023 Real points vs complex points

It is known that the complex points of a complex scheme suffice to reconstruct the scheme. This relies on the fact that $\mathbb{C}$ is algebraically closed. For example, consider the projective variety

$$Z(x_1^2 + x_2^2 + \cdots + x_{n+1}^2) \subset \mathbb{P}^n_{\mathbb{R}}$$

This variety has no real points, so in particular its real points cannot be distinguished from the real points of the empty variety (which is indeed a variety, but not irreducible, amusingly).

4/22/2023 *Virtual Hodge polynomial

Define

$\mathbb{L} \mapsto xy$.

4/28/2023 Simple representations of a quiver with no oriented cycles

Recall

Lemma (Nakayama's Lemma): *Let R be a unital ring and J its Jacobson radical. If M is a f.g. R-module, then JM is a proper submodule of M.*

Now if M is a simple representation of Q, then it is a simple kQ-module. In particular, it is finite dimensional. The Jacobson radical as identified above is the two-sided ideal generated by the arrows of Q (and thus includes all paths). (this involves some commutative algebra to prove which I haven't looked through yet). $kQ \cong J(kQ) \oplus kQ_0$ where kQ_0 is the subspace with basis of paths of length 0, all the e_α's.

Thus by Nakayama's lemma, for any simple module M, JM is a proper submodule of M and thus is 0. In terms of quiver representations, this means that in the representation, all linear maps are 0 (to see this one must understand the equivalence of categories $kQ - Mod \cong Rep(Q)$, but this is *mostly* follow your nose). Therefore any simple representation of Q (simple object of the category $Rep(Q)$) is an assignment of vector space to each vertex and the assignment of the 0 map to every arrow.

Suppose that at a vertex i of Q, the vector space at i in the representation M has dimension greater than 1. Then there is a proper subrepresentation consisting of the same vector space at every vertex $j \neq i$ and a proper subspace of the vector space at i. The condition of compatibility with arrows is fulfilled automatically since all arrows are 0. This contradicts the simplicity of M. So every vector space must have dimension 1 or 0.

Suppose there is more than one non-zero space in M. Then of course there is a subrepresentation consisting of setting one of those non-zero spaces to 0 and leaving the rest the same, which is a contradiction.

We conclude that any simple representation of Q (acyclic) must be of the form $S(i)$, the "delta" representation, consisting of a single copy of k at the vertex i and 0 at every other vertex, automatically requiring every map to be 0.

Recall that in 3/11/2023 we provided a projective resolution (and a short one) of each such $S(i)$ in the type A case:

$$0 \to P(i-1) \to P(i) \to S(i) \to 0$$

4/25/2023 Global (homological) dimension of type A path algebra

Recall that the path algebra of a quiver is the algebra with basis given by paths in Q, multiplication is given by concatenation where applicable and 0 where not and addition is formal. The path algebra is denoted kQ. We focus on the case of type A, where we may refer to Q as Q_ℓ, the type A quiver with length ℓ.

Definition: The global (homological) dimension of a ring A, *gl dim A*, is defined as the supremum over the set of projective dimensions of all left A-Modules: Given $M \in A-Mod$, the projective dimension, $pd(M)$, is defined as the minimal length finite projective resolution of M, assuming one finite length one exists, and if not it is assigned ∞. Sanity check: A module is projective dimension 0 iff $pd(M) = 0$.

Example: $\mathbb{Z}$ has global dimension 1, since every $\mathbb{Z}$-module (abelian group), admits a 3 term free resolution consisting of generators and relations (note we do not require M to be finitely generated).

Example: The path algebra is an algebra over k, and in particular is a ring, forgetting scalar multiplication, so we may consider its global dimension. It is not a commutative ring. Let's compute its dimension:

Theorem (black box): *If A is a f.d. associative, unital k-algebra, then gl dim A is equal to the maximum*[16] *of the projective dimensions of its simple modules.*

From this, combined with 4/28/2023 which characterizes the simple representations of acyclic quivers and with 3/11/2023 which provides a length 1 projective resolution of each such simple representation in the case of type A, we conclude that the global dimension of kQ_ℓ is 1. There is also a notion of dimension in an abelian category, or maybe even more generally. I suspect there is some theorem that says the dimension of R-Mod is the dimension of R, but I haven't checked. I'm not even sure what the right notion of

[16] I believe we may say maximum here because of some result which should say that there are finitely many simple modules. It follows from 4/28/2023 that this is the case when the algebra itself is the path algebra of a type A quiver, as we are dealing with in this case. It also holds for Gabriel's theorem for Dynkin quivers, which says there are finitely many indecomposables. But it holds in this more general setting as well. We do not prove it.

"dimension" is for a category, I just know that this is something that exists.

Finally, I believe this holds for any quiver, not just type A, but we don't prove it here.

5/5/2023 Indecomposable representations of type A quivers

Let Q_ℓ be a type A quiver. A representation of any quiver is called indecomposable if it cannot be written as a direct sum of two representations. Generally, simple implies indecomposable: If a representation could be written as a non-trivial direct sum, then each summand would constitute a non-trivial subrepresentation. However, indecomposable representations need not be simple. This is obvious as we characterized all simple representations of type A quivers above, but there are indecomposables which do not fit this characterization. Even within type A, we can take $Q_2 = \bullet \to \bullet$.

Now we state the classification of indecomposable representations of type A quivers without proof. For any two integers $1 \leq a \leq b \leq \ell$, denote the representation $V_{a,b}$ as

$$0 \to 0 \to \cdots \to 0 \to k \to k \to \cdots \to \cdots \to k \to 0 \to \cdots \to 0$$

where the initial k is placed at vertex a and the final is placed at vertex b. All maps which can be non-zero are the identity. We claim that each $V_{a,b}$ is indecomposable. Indeed this is obvious because the maps are all identities, and it is also obvious that no such $V_{a,b}$ is isomorphic to $V_{c,d}$ unless $a = c, b = d$. The content of Gabriel's theorem for type A quivers is to prove that every such indecomposable representation of Q_ℓ is isomorphic to $V_{a,b}$ for some a, b.

5/10/2023 Schubert calculus in $Gr(2,4)$

For precise definitions of the terms appearing here, see
https://www.mathematicalgemstones.com/gemstones/sapphire/schubert-calculus/

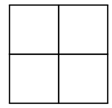
We already discussed the Schubert cells Σ_λ° in 4/8/2023. Taking the closures, $\overline{\Sigma_\lambda^\circ} \equiv \Sigma_\lambda$ gives the Schubert varieties, and considering their cohomology classes (the Schubert classes), $\sigma_\lambda \in H^2(Gr(2,4))$ (PD dual of the homology class $[\Sigma_\lambda]$) gives meaning to $\sigma_\lambda \cdot \sigma_\mu$ via the cup product:

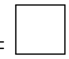
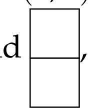
Theorem (Littlewood-Richardson rule): *For any two partitions λ and μ contained in the Important Box (Rimanyi refers to this as the "universe" I believe)*

$$\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^\nu \sigma_\nu$$

where the sum ranges over all ν in the universe and $c_{\lambda\mu}^\nu$ is the number of SSYT of shape ν/λ having content μ and whose reading word is lattice.

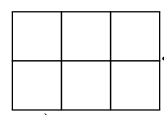
Example: Consider the Schubert classes $\sigma_{(1,1)}$ and $\sigma_{(2,0)}$ in $H^*(Gr(2,4))$, and suppose we want to compute its product. First we identify the universe as

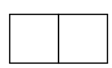


the first 2 comes from the 2 in $Gr(2,4)$ and the second 2 is 4-2. Then LR rule tells us to look at all ν contained in the universe and such that $\nu/(1,1)$ has content $(2,0)$. For $\nu/(1,1)$ to be non-empty ν must be either $(2,1)$ or $(2,2)$, so that $\nu/\lambda =$  and , respectively. So the sum on the RHS of the LR rule may have at most 2 non-zero terms, $\sigma_{(2,1)}$ and $\sigma_{(2,2)}$. However we see that both coefficients c are 0 since there is no way to fill in these boxes with content $(2,0)$, independent of reading word being lattice. Thus

$$\sigma_{(1,1)} \cdot \sigma_{(2,0)} = 0$$

Geometrically, $\sigma_{(1,1)}$ is the set of 2-planes in $\mathbb{C}^4$ whose corresponding partition is $(1,1)$, so its first 2 appears under V_3, referencing the table we have in 4/8/2023. Thus it is contained in V_3, the chosen 3-plane, whereas $\sigma_{(2,0)}$ has its first 1 appearing under V_1, so it consists of 2-planes containing that chosen line V_1. But choosing a generic 3-plane in $\mathbb{C}^4$ and a line through the origin, there is no plane which is contained in the chosen 3-plane and contains the chosen line, since the line need not be contained in the 3-plane. As one can see, this depends heavily on the fact that we are in $\mathbb{C}^4$, since it means there is another dimension in which we may place the line. So it stands to reason that the same product should be 0 when letting $4 \to \infty$ while 2 stays fixed.

Example: However if we consider $Gr(3,5)$, the new universe is . Then for $\nu/(1,1)$ to be non-empty, ν must be $(2,1), (2,2), (3,1), (3,2)$, or $(3,3)$. In order to have content $(2,0)$, $\nu/(1,1)$ must have exactly 2 boxes, so ν cannot be $(2,1), (3,2)$ or $(3,3)$. We see that $(2,2)/(1,1)$ cannot have content $(2,0)$, since verticals must be strictly increasing. Thus only $(3,1)$ can possibly be non-zero, and we see it must appear with coefficient 1 since there is a SSYT of the form

$$(3,1)/(1,1) =$$ 

with SSYT

1	1

whose reading word is lattice. Therefore

$$\sigma_{(1,1)} \cdot \sigma_{(2,0)} = \sigma_{(3,1)} \in H^*(Gr(3,5))$$

5/20/2023 Cohomology ring (Chow ring) of $Gr(2,4)$.

Someone on mathoverflow claimed these two rings are isomorphic for toric varieties and flag varieties, which $Gr(2,4)$ certainly is, but i don't have a link to it anymore.

We know the schubert classes σ_λ form an additive basis, and thus form a generating (with respect to multiplication) set. In this case, these are all length 2 partitions that fit inside the 2 x 2 universe: $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$, $(2,2)$. However when allowing for multiplication, these are not all independent anymore: For example,

$$\sigma_{(2,1)} = \sigma_{(1,1)} \cdot \sigma_{(1,0)}, \quad \sigma_{(2,1)} \cdot \sigma_{(2,0)} = 0$$

follows easily from the LR rule cited above. One can use the rule above to write down all the multiplication rules in the cohomology ring, leading to the table:

$\cdot$	1	σ_1	σ_2	$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$
1	1	σ_1	σ_2	$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$
σ_1	σ_1	$\sigma_2 + \sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$	0
σ_2	σ_2	$\sigma_{2,1}$	$\sigma_{2,2}$	0	0	0
$\sigma_{1,1}$	$\sigma_{1,1}$	$\sigma_{2,1}$	0	$\sigma_{2,2}$	0	0
$\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$	0	0	0	0
$\sigma_{2,2}$	$\sigma_{2,2}$	0	0	0	0	0

which I stole from the textbook "Enumerative Geometry and String Theory" by Sheldon Katz. According to the formula for Littlewood-Richardson coefficients, in $Gr(n,m)$, one should expect the product to tend to 0 as one goes to the right or down in the table.

By inspection, the classes $\sigma_{(1,0)}, \sigma_{(2,0)}$ and $\sigma_{(1,1)}$ do not appear in the multiplication table, therefore they should be treated as ring generators, while the terms $\sigma_{(2,1)}$ and $\sigma_{(2,2)}$ should be regarded as relations:

$$H^\bullet(Gr(2,4)) \cong \frac{\mathbb{Z}[\sigma_{(1,0)}, \sigma_{(1,1)}, \sigma_{(2,0)}]}{\left(1 - \sigma_{(1,0)} + \sigma_{(1,1)}\right)\left(1 + \sigma_{(1,0)} + \sigma_{(2,0)}\right) - 1}$$

This kind of ring will appear for any cohomology ring of flag variety: it will have additive basis indexed by the Schubert varieties and ring structure given by polynomials in some of the Schubert varieties (though at the time of writing I don't know the general rule for choosing which ones) and then quotiented by some polynomial relations in those Schubert varieties. I hope to understand the general case soon, to recover what is called the coinvariant algebra in the case of full flag varieties. Unfortunately the Grassmannian cannot be seen as an example of this unless looking at $Gr(1,2)$, which is too trivial to see the general structure.

5/22/2023 Cohomology ring of Grassmannians

Over the Grassmannian there exists a natural exact sequence of vector bundles

$$0 \to T \to V \to Q \to 0$$

where T is the tautological bundle over $Gr(k,n)$, V is the trivial bundle $G(k,n) \times \mathbb{C}^n$ and Q is the quotient vector bundle.

Proposition:

$$c_i(Q) = \sigma_{(i,0,\ldots,0)}, \quad c_i(T) = (-1)^i \sigma_{(1,\ldots,1)}$$

I don't know how to prove the above. I found a proof in a textbook but it is quite involved. I wonder if there is an elementary way to see this.

The Grothendieck axiom of additivity implies that

$$1 = c(V) = c(T)c(Q)$$

from which we recover the result in 5/20/2023 (the above example), and indeed gives the general rule for which additive generators appear in the ring generators:

$$H^\bullet(Gr(k,n)) \cong \frac{\mathbb{Z}[c_1(T),\ldots,c_k(T),c_1(Q),\ldots,c_k(Q)]}{c(T)c(Q)-1}$$

5/23/2023 *27 lines on a cubic surface via Schubert calculus

Taken from Wikipedia page for Schubert Calculus.

In a talk I gave in Spring 2023, we proved that there are 27 straight lines on a cubic surface via equivariant localization, integrating

$$\int_{Gr(2,4)} c_4(Sym^3(\mathcal{O}(1))) = \int_{Gr(2,4)} e(Sym^3(\mathcal{O}(1))) = 27$$

by rewriting the integral, using Atiyah-Bott localization, as a sum over fixed points of the torus action on $Gr(2,4)$ with certain weights. Another way to get this enumerative answer is to evaluate the integral directly by computing what c_4 is using Schubert calculus. The total chern class is given by

$$c(\mathcal{O}(1)) \equiv 1 + c_1(\mathcal{O}(1)) + c_2(\mathcal{O}(1))$$

with subsequent terms 0 since the tautological bundle, thus the dual tautological bundle, is a rank 2 bundle over $Gr(2,4)$. We have

$$c(\mathcal{O}(1)) = 1 + \sigma_{(1,0)} + \sigma_{(1,1)}$$

from the above example, 5/22/2023.

$\boxed{\text{FINISH}}$

5/27/2023 kG is a Hopf algebra

I encountered the statement of this example in the Youtube video "Introduction to quantized enveloping algebras - Leonardo Maltoni" from which I will probably have more journal entries in the near future.

Recall a Hopf algebra is an associative k-algebra with k-algebra morphisms 1) comultiplication: $\Delta : A \to A \otimes A$, 2) antipode: $S : A \to A^{opp}$ (A^{opp} is the algebra where multiplication order is reversed) and 3) counit: $\epsilon : A \to k$ satisfying the axioms

$$\text{coassociativity: } (1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$$
$$\text{counit axiom: } \varphi = (\epsilon \otimes 1) \circ \Delta, \quad \varphi^{opp} = (1 \otimes \epsilon) \circ \Delta$$
$$\text{antipode axiom: } m \circ (S \otimes 1) \circ \Delta = \iota \circ \epsilon, \quad m \circ (1 \otimes S) \circ \Delta = \iota \circ \epsilon$$

where φ is the canonical isomorphism $A \cong k \otimes A$ and φ^{opp} is the canonical isomorphism $A \cong A \otimes k$, and $\iota : k \to A$ is the map $x \mapsto x \cdot 1_A$.

If G is a group, define $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$ and $\epsilon(g) = 1$ and extend by linearity to maps on kG. Let's verify the axioms. First observe that it suffices to check on the basis elements $1 \cdot g \in kG$. This is due to the fact that each of the maps defined above are algebra morphisms.

$$\text{coassociativity: } g \mapsto (g \otimes g) \mapsto (g \otimes (g \otimes g))$$
$$g \mapsto (g \otimes g) \mapsto ((g \otimes g) \otimes g)$$

but because the tensor product of algebras is associative, these two are equal.

$$g \mapsto (g \otimes g) \mapsto (1 \otimes g) = \varphi(g)$$
$$g \mapsto (g \otimes g) \mapsto (g \otimes 1) = \varphi^{opp}(g)$$

because the canonical isomorphisms are $\varphi(g) = 1 \otimes g$ and $\varphi^{opp}(g) = g \otimes 1$.

$$g \mapsto (g \otimes g) \mapsto (g^{-1} \otimes g) \mapsto 1 \cdot 1_G$$
$$g \mapsto 1 \mapsto 1 \cdot 1_G$$

and

$$g \mapsto (g \otimes g) \mapsto (g \otimes g^{-1}) \mapsto 1 \cdot 1_G$$
$$g \mapsto 1 \mapsto 1 \cdot 1_G$$

so the antipode axiom is satisfied and kG is a Hopf algebra with the above maps.

Remark: Of course, kG is commutative iff G is commutative. A Hopf algebra is called <u>cocommutative</u> if $\sigma \circ \Delta = \Delta$, where $\sigma : A \otimes A \to A \otimes A$ is the map $\sigma(c \otimes d) = d \otimes c$. Note for $A = kG$,

$$g \mapsto g \otimes g$$
$$g \mapsto g \otimes g \mapsto g \otimes g$$

so kG is always cocommutative as a Hopf algebra.

5/28/2023 *$U(L)$ is a Hopf algebra

Here $U(L)$ denotes the universal enveloping algebra of L, an arbitrary Lie algebra. Note that if G is a Lie group such that $Lie(G) = L$, then kG is a Hopf algebra as above with maps $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$ and $\epsilon(g) = 1$. Therefore if $Lie(G) = L$ is the tangent space to the identity, a deformation of 1_G in G will look like $1_G + \varepsilon X$ for X the elements of L. Then

$$\Delta(1_G + \varepsilon X) = (1_G + \varepsilon X) \otimes (1_G + \varepsilon X)$$
$$\Delta(X) = 1_G \otimes X + X \otimes 1_G$$

the second line follows from looking at the first line and only considering terms linear in ε. Note that $U(L)$ also contains degree 0 part isomorphic to k, so we also have to define $1 \mapsto 1 \boxtimes 1$. Similarly,

$$S(1_G + \varepsilon X) = (1_G + \varepsilon X)^{-1} = 1_G - \varepsilon X$$
$$S(X) = -X$$
$$S(1) = 1$$

and

$$\epsilon(1_G + \varepsilon X) = 1_G$$
$$\epsilon(X) = 0$$
$$\epsilon(k) = k$$

The above is not really rigorous, but suggests that we define a Hopf algebra on $T(L)$, the tensor algebra of L (which doesn't know anything about Lie bracket) by the formulas

$$\Delta(x) = x \boxtimes 1 + 1 \boxtimes x, \quad \Delta(1) = 1 \boxtimes 1$$
$$S(x) = -x, \quad S(1) = 1$$
$$\epsilon(x) = 0, \quad \epsilon(1) = 1$$

for all $x \in L \cong T^1(L)$, and extend by homomorphism to all of $T(L)$. Here we must use $\boxtimes$ to denote the tensor product as in $T(L) \to T(L) \boxtimes T(L)$, the tensor product of algebras, to distinguish it from $\otimes$, the "internal" tensor product within $T(L)$.

Example: To see how the two tensor products interact, we have that

$$\Delta(x \otimes y) = \Delta(x) \cdot \Delta(y) = (x \boxtimes 1 + 1 \boxtimes x) \cdot (y \boxtimes 1 + 1 \boxtimes y)$$

Where $\cdot$ represents the canonical product structure on $T(L) \boxtimes T(L)$.

The distinction between tensor products is important here. For example $1 \otimes v = v \in T(L)$, but $1 \boxtimes v$ is not anything besides $1 \boxtimes v$.

$$= (x \boxtimes 1) \otimes (y \boxtimes 1) + (x \boxtimes 1) \cdot (1 \boxtimes y) + (1 \boxtimes x) \cdot (y \boxtimes 1) + (1 \boxtimes x) \cdot (1 \boxtimes y)$$

Let's look at the first term:

$$(x \boxtimes 1) \cdot (y \boxtimes 1) = (x \cdot y) \boxtimes 1$$

by definition of multiplication in $T(L) \boxtimes T(L)$. Thus we get

$$= (x \cdot y) \boxtimes 1 + x \boxtimes y + y \boxtimes x + 1 \boxtimes (x \cdot y)$$

This defines the desired maps, now we must check the Hopf algebra compatibility axioms:

$$\text{coassociativity} : (1 \boxtimes \Delta) \circ \Delta = (\Delta \boxtimes 1) \circ \Delta$$

It suffices to check on $V \subset TV$, since all maps involved are homomorphisms. Therefore we have

$$\begin{aligned} v \mapsto v \boxtimes 1 + 1 \boxtimes v &\mapsto v \boxtimes (1 \boxtimes 1) + 1 \boxtimes (v \boxtimes 1 + 1 \boxtimes v) \\ &= v \boxtimes 1 \boxtimes 1 + 1 \boxtimes v \boxtimes 1 + 1 \boxtimes 1 \boxtimes v \end{aligned}$$

while

$$\begin{aligned} v \mapsto (v \boxtimes 1 + 1 \boxtimes v) &\mapsto \Delta(v) \boxtimes 1 + 1 \boxtimes 1 \boxtimes v \\ &= v \boxtimes 1 \boxtimes 1 + 1 \boxtimes v \boxtimes 1 + 1 \boxtimes 1 \boxtimes v \end{aligned}$$

verifying coassociativity. For counit:

$$v \mapsto v \boxtimes 1 + 1 \boxtimes v \mapsto 0 + 1 \boxtimes v \cong v$$

and

$$v \mapsto v \boxtimes 1 + 1 \boxtimes v \mapsto v \boxtimes 1 + 0 \cong v$$

and finally antipode axiom:

$$\begin{gathered} v \mapsto (v \boxtimes 1 + 1 \boxtimes v) \mapsto (-v \boxtimes 1 + 1 \boxtimes v) \\ \mapsto -v \otimes 1 + 1 \otimes v \cong 0 \end{gathered}$$

while

$$v \mapsto 0 \mapsto 0 \boxtimes 1 \cong 0$$

We have skipped over some of the things required to check Hopf algebra, but there are too many to write down. These calculations should convince anyone that it does hold, or equip them with the necessary tools to check everything for themselves. Thus $T(L)$ is a Hopf algebra.

$U(L)$ is a quotient of $T(L)$, so we may induce a map on $U(L)$ if we verify that these definitions are compatible with the single (family of) relation introduced by the UEA. For Δ, we have to check $\Delta(x \otimes y - y \otimes x) = \Delta([x, y])$. The LHS is equal to

$$\begin{gathered} \Delta(x \otimes y - y \otimes x) = \Delta(x) \cdot \Delta(y) - \Delta(y) \cdot \Delta(x) \\ = (x \otimes y) \boxtimes 1 + x \boxtimes y + y \boxtimes x + 1 \boxtimes (x \otimes y) - (y \otimes x) \boxtimes 1 - y \boxtimes x - x \boxtimes y - 1 \boxtimes (y \otimes x) \\ = (x \otimes y - y \otimes x) \boxtimes 1 + 1 \boxtimes (x \otimes y - y \otimes x) \\ \equiv \Delta(x \otimes y - y \otimes x) \end{gathered}$$

$\boxed{\text{HOW TO SHOW } \Delta \text{ DESCENDS TO A MAP ON } U(L)?}$

6/1/2023 *Fixed locus of subtorus of residual G_W action on quiver variety is disjoint union of quiver varieties

This lemma is from Nakajima's paper "Quiver varieties and Tensor products". He provides a proof in the paper, which I now understand.

Setup: Let $\mathfrak{M}(\vec{v}, \vec{w})$ be a quiver variety. Fix a decomposition of the framing dimension vector $w = w^1 + w^2$, with according decomposition of framed spaces $W = W^1 \oplus W^2$. Define an action of one-parameter subgroup $\lambda(\mathbb{C}^*) \subset G_W \curvearrowright \mathfrak{M}(\vec{v}, \vec{w})$ by

$$\lambda(t) = id_{W^1} \oplus t \cdot id_{W^2} \subset G_{W^1} \times G_{W^2} \subset G_W$$

Lemma: *The fixed point set of $\lambda(\mathbb{C}^*)$ in $\mathfrak{M}(\vec{v}, \vec{w})$ is isomorphic to $\sqcup_{\vec{v}_1 + \vec{v}_2 = \vec{v}} \mathfrak{M}(\vec{v}_1, \vec{w}_1) \times \mathfrak{M}(\vec{v}_2, \vec{w}_2)$.*

$\square$

For example if we choose the quiver variety $T^*Gr(2,4) \cong \mathfrak{M}(\bullet, (2), (4))$ and define $\lambda(\mathbb{C}^*)$ action as above, let us first understand the residual action in terms of geometry. An element of the right hand side is acted on by G_W by sending

$$g \cdot (i, j) \mapsto (ig, g^{-1}j)$$

where $g \in GL(4, \mathbb{C}) \equiv G_W$. On the LHS, this corresponds to sending the element

$$j(\mathbb{C}^2) \in Gr(2, \mathbb{C}^4) \mapsto g^{-1}j(\mathbb{C}^2) \in Gr(2, \mathbb{C}^4)$$

we know $g^{-1}j(\mathbb{C}^2)$ belongs to $Gr(2, \mathbb{C}^4)$, i.e. is still 2-dimensional because g is full rank. Recall (I don't remember if I completed writing this out fully but if I did it would be in the EaD entry for the full quiver variety of a type A quiver) an element of $T^*Gr(2,4)$ has the form

$$\left\{ \begin{array}{l} j(\mathbb{C}^2) \in Gr(2,4) \\ i : \mathbb{C}^4 \to \mathbb{C}^2, \quad i|_{j\mathbb{C}^2} = 0 \end{array} \right\}$$

Thus the above G_W-action will send such an element to

$$\left\{ \begin{array}{l} g^{-1}j(\mathbb{C}^2) \in Gr(2,4) \\ ig : \mathbb{C}^4 \to \mathbb{C}^2, \quad (ig)|_{g^{-1}j\mathbb{C}^2} = 0 \end{array} \right\}$$

Geometrically, this signifies a new subspace $g^{-1}j(\mathbb{C}^2)$, and a new map on the ambient space ig. We choose a splitting of $W = \mathbb{C}^4$ into two pieces, say $\mathbb{C} \oplus \mathbb{C}^3 = \langle e_1 \rangle \oplus \langle e_2, e_3, e_4 \rangle$. Then an element of the cotangent bundle looks like

$$\left\{ \begin{array}{l} j(\mathbb{C}^2) \subset \mathbb{C} \oplus \mathbb{C}^3 \\ i : \mathbb{C} \oplus \mathbb{C}^3 \to \mathbb{C}^2, \quad i|_{j\mathbb{C}^2} = 0 \end{array} \right\}$$

with this setup, the element $\lambda(t) = id_{W^1} \oplus tid_{W^2}$ takes the form $g = diag(1, t, t, t)$, and sends the above element to

$$\left\{ \begin{array}{l} g^{-1}j(\mathbb{C}^2) \subset \mathbb{C} \oplus \mathbb{C}^3 \\ ig : \mathbb{C} \oplus \mathbb{C}^3 \to \mathbb{C}^2, \quad i|_{g^{-1}j\mathbb{C}^2} = 0 \end{array} \right\}$$

Let $j(\mathbb{C}^2)$ be spanned by the two vectors $\sum a_i e_i$, $\sum b_i e_i$ for $\{e_i\}$ the standard basis on $\mathbb{C}^4$. Then $g^{-1}j(\mathbb{C}^2)$ is spanned by $\langle \sum a_i g^{-1} e_i, \sum b_i g^{-1} e_i \rangle = \langle a_1 e_1 + \sum \frac{a_i}{t} e_i, b_1 e_1 + \sum \frac{b_i}{t} e_i \rangle$, and if $ig = i$, then

$$= \{\}$$

$\boxed{\text{HOW TO FIND FIXED POINTS??}}$

6/3/2023 Stirling numbers, complete homogeneous symmetric polynomials, and elementary symmetric polynomials

I read about these relations in Bruce Sagan's presentation "Stirling numbers for complex reflection groups".

Define the Stirling numbers of the second kind, $S(n,k)$, as the number of ways to partition $\{1,2,\ldots,n\}$ into k non-empty blocks, and recall the complete homogeneous symmetric polynomials have the form

$$h_k(x_1,\ldots,x_n) = \sum_{\deg = k} x_1^{k_1} \cdots x_n^{k_n}$$

Theorem:

$$S(n,k) = h_{n-k}(1,2,\ldots,k)$$

where the RHS indicates evaluation of the polynomial $h_{n-k}(x_1,\ldots,x_k)$.

Let's do the example of $n = 4, k = 3$.

Example: $S(4,3) = h_{4-3}(1,2,3)$. To compute LHS, we examine set $\{a,b,c,d\}$ and we seek the number of ways to partition it into 3 non-empty subsets: The first type of partition to consider is 1|1|2:

a	b	cd
a	c	bd
a	d	bc
b	c	ad
b	d	ac
c	d	ab

But in fact this is the only type of partition possible. Thus the total is 6.

To compute RHS we examine the polynomial

$$\sum_{\deg = 1} x_1^{k_1} x_2^{k_2} x_3^{k_3}$$

$$= x_1 + x_2 + x_3$$

$$\Rightarrow h_1(1,2,3) = 6$$

as desired.

We may also define the elementary symmetric polynomials by

$$e_k(x_1, \dots, x_n) = \sum_{\text{deg } = k, \text{ square free}} x_1^{k_1} \cdots x_n^{k_n}$$

So we see that the terms appearing in the elementary symmetric polynomials are subsets of those terms appearing in the complete homogeneous symmetric polynomial.

We define the Stirling numbers of the first kind by $s(n, k) = (-1)^{n-k} \#\{\sigma \in S_n \mid \sigma \text{ has } k \text{ disjoint cycles}\}$.

Theorem:

$$s(n, k) = (-1)^{n-k} e_{n-k}(1, 2, \dots, n-1)$$

Let's do the example $n = 5, k = 2$.

When writing this up, I accidentally computed the Stirling number of the second kind by accident, and typed it all before realizing it was the wrong thing. I'll just include it here so it doesn't go to waste:

Example: $S(5, 2) = 15$: we consider the set $\{a, b, c, d, e\}$ and seek the number of ways to partition it into two non-empty subsets. First consider the ways to break it up into 1 and 4:

a	bcde
b	acde
c	abde
d	abce
e	abcd

Then consider the ways to break it up into 2 and 3

ab	cde
ac	bde
ad	bce
ae	bcd
bc	ade
bd	ace
be	acd
cd	abe
ce	abd
de	abc

Then consider the ways to break it up into 3 and 2

abc | de

but notice this already appeared as the final entry in our previous case, and the same will happen when we go to 4 and 1. So the total is just the cases 1 | 4 and 2 | 3, of which we see there are 15.

Example: $s(5,1) = (-1)^{5-2}e_{5-2}(1,2,3,4) = (-1)e_3(1,2,3,4)$. For the RHS

$$e_4(x_1,\ldots,x_4) = x_1x_2x_3x_4$$

$$\Rightarrow e_3(1,2,3,4) = 1\cdot 2\cdot 3\cdot 4 = 24$$

For the LHS, we examine the action of S_5 on $\{1,2,3,4,5\}$ and ask for how many permutations there are in S_5 with 1 disjoint cycle, ie how many 5-cycles. To determine a 5-cycle, we must specify a choice of 1-5 for the first entry, a choice of the remaining 4 for the second entry and so on, leading to 5! choices. But also remember that cycles are invariant up to cyclic permutations, so we must also divide by the 5 equivalent rotations, leading to $5!/5 = 4! = 24$, as desired.

6/6/2023 Passing from partial to complete flags

I read about this in "Lectures on the geometry of flag varieties" by Michel Brion.

Let G/B denote the variety of complete flags in $\mathbb{C}^n$ and G/P denote the variety of partial flags in $\mathbb{C}^n$ of type $(d_1,\ldots,d_m)$. There is a $GL_n(\mathbb{C})$-equivariant map

$$G/B \to G/P$$

by sending the equivalence class mod B to the equivalence class mod P, since P contains B (parabolic and Borel subgroups of GL_n). Geometrically this corresponds to sending the complete flag

$$0 \subset V^1 \subset V^2 \subset \cdots \subset V^n = \mathbb{C}^n$$

to the partial flag of type $(d_1,\ldots,d_m)$ given by

$$0 \subset V^{d_1} \subset V^{d_2} \subset \cdots \subset V^{d_m} = \mathbb{C}^n$$

ie collect the first d_1 steps and put them together into one d_1-dimensional subspace, then collect the first d_2 steps and put them together into one d_1-dimensional subspace, and so on. In particular, given a fixed partial flag of type $(d_1,\ldots,d_m)$, $V^{d_1} \subset \cdots \subset V^{d_m} = \mathbb{C}^n$, the fiber over it under the above map is given by, for example, any complete flags whose d_1 entry defines the subspace V^{d_1}, and all the steps before V^{d_1} may be arbitrary. This is obviously isomorphic to the full flag variety in the space $\mathbb{C}^{d_1}$, $\mathcal{F}(\mathbb{C}^{d_1})$. Thus the fiber over the partial flag variety is given by

$$\prod \mathcal{F}(\mathbb{C}^{d_i})$$

This establishes the variety of partial flags as a fiber bundle with total space given by the variety of complete flags and fiber as above.

Apparently this is important because it indicates that to study partial flags it "suffices" in some ways to study complete flags, though I don't understand the exact nature of how.

6/7/2023 COHA of a point

Richard Rimanyi told me to try to calculate some things in this COHA and in Kontsevich and Soibelman's paper, they claim that this COHA is isomorphic to the exterior algebra on certain generators, to be identified below.

Consider the quiver $Q = \bullet$, and we want to understand its cohomological Hall algebra. Its additive structure is given by $\bigoplus_{\vec{v}} H^{\bullet}_{GL(\vec{v})}(Rep(Q, \vec{v}))$. As we showed in 3/25/2023, this is given by

$$\cong \bigoplus_{\vec{v}} \mathbb{C}[x_{11}, x_{12}, \ldots, x_{1\vec{v}_1}, \ldots, x_{\ell\vec{v}_\ell}]^{\prod S_n]}$$

In our case, there is only one vertex so the second index is superfluous, and we have

$$\cong \bigoplus_{v=1}^{\infty} \mathbb{C}[x_1, \ldots, x_v]^{S_v}$$

The graded multiplicative structure is given by, for $f \in H_{v_1}, g \in H_{v_2}$: One considers the symmetric polynomial $g(x_1, x_2, \ldots, x_{v_1}, x_{v_1+1}, \ldots, x_{v_1+v_2})$ given by symmetrizing the expression, for all partitions of the variables $x_1, \ldots, x_{v_1+v_2}$ into two subsets of sizes v_1 and v_2, denoting the first as x' and the second as x'',

$$f(x')g(x'')\frac{1}{x'' - x}$$

where $f(x')$ means plug in all the variables (formally) for the partition of size v_1 and similarly for $g(x'')$. This is the explicit equation for multiplication in COHA in its simplest form. There are more terms which arise when there are multiple vertices and multiple arrows which we will do in the following entry. For now we can compute[17], for example taking $(1 \in H_0) * (1 \in H_0)$. The result will be another element of H_0, and the denominator is empty since there are no elements to put in, so $1^2 = 1$. If we consider $(1 \in H_1) * (1 \in H_1)$, the result eats variables x_1, x_2 and is given by the formula

$$1(x_1)1(x_2)\frac{1}{x_2 - x_1} + 1(x_2)1(x_1)\frac{1}{x_1 - x_2} = 0$$

So $1^2 = 0$. In fact, for any $f, g \in H_1$:

$$f * g = \frac{f(x_1)g(x_2) - f(x_2)g(x_1)}{x_2 - x_1}$$

So in particular, for every $f \in H_1, f^2 = 0$. Recall that $H_1 \cong \mathbb{C}[x_1]$. Denote $x_1^i := \Psi_{2i+1} \in H_1$. We have observed $\Psi_{2i+1}^2 = 0$. In fact, with more argument one can show that this

[17]When multiplying elements of COHA explicitly, one must keep track of the degree of the elements which you are multiplying, since this controls how many variables the result will eat.

provides an isomorphism[18] of algebras $H(\bullet) \cong \bigwedge(\Psi_{2i+1})$.

I don't know how to prove this in full generality, but I have checked many examples:

$$\Psi_{2i_1+1} * \Psi_{2i_2+1} * \cdots * \Psi_{2i_n+1} = s_{(i_n+1-n, i_{n-1}+2-n, \ldots, i_1}(\overline{x})$$

$$0 \leq i_2 < i_2 < \cdots < i_n$$

where s_λ is the Schur polynomial corresponding to the partition λ. Calculating these things by hand is a real nightmare, especially given variances in notation across the internet. For that reason, we will use the ones calculated for us already on the wikipedia page for Schur polynomials. I understand there is software for computing these things but I distrust computers.

Example: Let's compute

$$\Psi_3 * \Psi_5 * \Psi_9 \equiv (x_1 \in H_1) * (x_1^2 \in H_1) * (x_1^4 \in H_1)$$

the result lies in H_3, and thus eats 3 variables, x_1, x_2, x_3. First we compute using our general formula for multiplication in H_1,

$$(x_1 \in H_1) * (x_1^2 \in H_1) = \frac{x_1 x_2^2 - x_1^2 x_2}{x_2 - x_1} = x_1 x_2$$

Then

$$(x_1 x_2 \in H_2) * (x_1^4 \in H_1)$$

$$= \frac{x_1 x_2 x_3^4}{(x_3 - x_1)(x_3 - x_2)} + \frac{x_1 x_2^4 x_3}{(x_2 - x_1)(x_2 - x_3)} + \frac{x_1^4 x_2 x_3}{(x_1 - x_2)(x_1 - x_3)}$$

$$= x_1 x_2 x_3 (x_1 + x_2 + x_3) \equiv s_{(2,1,1)} = \Psi_3 * \Psi_5 * \Psi_9$$

as required.

Example:

$$\Psi_1 * \Psi_7 * \Psi_9 \equiv (1 \in H_1) * (x_1^3 \in H_1) * (x_1^4 \in H_1)$$

First (note that $1 \in H_1$ does not act as a unit in COHA)

$$\Psi_1 * \Psi_7 = \frac{x_2^3 - x_1^3}{x_2 - x_1} = x_1^2 + x_1 x_2 + x_2^2$$

Then

$$(x_1^2 + x_1 x_2 + x_2^2 \in H_2) * (x_1^4 \in H_1)$$

[18] I'm a little confused by this. If we have an isomorphism of algebras, shouldn't multiplication coincide in the two? But in COHA we show multiplication is given by the Schur polynomials, whereas multiplication in the exterior algebra is "free". Nevertheless K&S claim this, unless I am reading it incorrectly (very possible). Perhaps they mean H is a certain quotient of the exterior algebra $\bigwedge(\Psi_{2i+1})$ identifying multiplication with the Schur polynomials.

$$= \frac{(x_1^2+x_1x_2+x_2^2)(x_3^4)}{(x_3-x_1)(x_3-x_2)} + \frac{(x_1^2+x_1x_3+x_3^2)(x_2^4)}{(x_2-x_1)(x_2-x_3)} + \frac{(x_2^2+x_2x_3+x_3^2)(x_1^4)}{(x_1-x_2)(x_1-x_3)}$$
$$= x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 \equiv s_{(2,2,0)} = \Psi_1 * \Psi_7 * \Psi_9$$

I think an in depth understanding of how to do these algebraic manipulations at the end may give rise to a general proof of the above isomorphism, but I don't know of one. I've just been plugging the final step into Wolfram.

6/8/2023 *COHA of Jordan quiver

Same as above: Introduced (to me) by Richard Rimanyi, claim of isomorphism found in Kontsevich and Soibelman, still don't know how to prove it because I have no experience working with these general combinatorial formulas. Is everything just proved by induction??

We investigate the COHA of the Jordan quiver, the quiver with one vertex and one arrow. The additive structure is the same as above, 6/7/2023. Indeed one can easily see from the definition of COHA that the additive structure depends only on the set of vertices, not on arrows. So the additive grading is the same. For any $f, g \in H_1$, we see

$$(f \in H_1) * (g \in H_1)$$
$$= f(x_1)g(x_2)\frac{x_2-x_1}{x_2-x_1} + f(x_2)g(x_1)\frac{x_1-x_2}{x_1-x_2} = f(x_1)g(x_2) + f(x_2)g(x_1)$$

In fact, for the Jordan quiver it is also obvious from the definition that multiplication is just given by shuffles in the coordinates x_i. The emergence of the shuffle product is not a coincidence, in fact multiplication in every COHA is intimately linked with shuffle products. This is spelled out for unframed COHA in the paper of Kontsevich and Soibelman and for framed COHA in the paper of Tommaso Botta. So the structure of this algebra is simpler: We have generators $\phi_{2i} \in H_1$ corresponding to the x_1^i, generators for $H^\bullet(\mathbb{C}P^\infty) \cong \mathbb{Z}[x_1]$, thus we have an isomorphism with the algebra of symmetric functions.

$\boxed{\text{IS IT REALLY SHUFFLES IN THE COORDINATES?. IT DOESN'T SEEM LIKE IT.}}$

6/9/2023 *framed COHA of a point

6/10/2023 Cohomology ring of a projective bundle

From Fulton's "Young Tableaux", appendix B.4.

Let E be a vector bundle of rank e over X and $\mathbb{P}(E) \to X$ the projective bundle associated to E. There is a tautological exact sequence of bundles over $\mathbb{P}(E)$

$$0 \to L \to p^*(E) \to Q \to 0$$

where L is the tautological line bundle, $\mathcal{O}(1) = L^\vee$, and Q is the quotient bundle. Unwinding the definitions, L consists of pairs $\big((x, \ell \in E_x), v \in \ell\big)$, while $p^*(E)$ consists of pairs $\big((x, \ell \in E_x), e \in E_x\big)$, so the morphism $L \to p^*(E)$ is defined fibre-wise by sending $v \in \ell \subset E_x \mapsto v \in E_x$, and $p^*(E) \to Q$ is the quotient map.

Let $\zeta = c_1(\mathcal{O}(1)) = -c_1(L) \in H^\bullet(\mathbb{P}(E))$ (second equality follows from the formula for chern class of a tensor product). On any $U \subset X$ a trivializing open set, $\mathbb{P}(E|_U) \cong U \times \mathbb{P}^{e-1}$ by the local trivializations and ζ restricts to an element of $H^\bullet(U \times \mathbb{P}^{e-1}) = H^\bullet(\mathbb{P}^{e-1})$. The axioms of Chern classes ensure that it restricts the hyperplane generator in degree 2: If $i : \mathbb{P}^e \hookrightarrow \mathbb{P}(E)$ denotes the inclusion of a fiber, then $i^*(\zeta) \in H^2(\mathbb{P}^e)$. Because Chern classes commute with pullbacks, this implies that the classes $i^*(\zeta)^k$ coincide with the standard basis of $H^2(\mathbb{P}^e)$ for $k \in \{0, \ldots, e-1\}$. By Leray-Hirsch theorem (I have no idea how to prove this, I'm pretty sure it needs spectral sequences which I haven't touched yet), then the classes $\{1, \zeta, \ldots, \zeta^{e-1}\}$ form a basis of $H^\bullet(\mathbb{P}(E))$ as a free module over $H^\bullet(X)$. In particular, there must be some polynomial equation

$$\zeta^e = a_1\zeta^{e-1} + a_2\zeta^{e-1} + \cdots + a_e$$
$$H^\bullet(\mathbb{P}(E)) \cong \frac{H^\bullet(X)[\zeta]}{(\zeta^e + a_1\zeta^{e-1} + \cdots + a_e)}$$

for some $a_i \in H^\bullet(X)$. In fact, a_i are the ith Chern classes $c_i(E) \in H^{2i}(X)$.

6/11/2023 Staircase construction of the full flag variety

This is sketched by Fulton in "Young Tableaux", pieces put together by me.

From 6/10/2023 (the above entry), we know how to compute the cohomology of the total space of a projective bundle in terms of its base. Inspired by this, we want to construct the full flag variety as some total space of a projective bundle over some concrete space. We do so iteratively (note that one often restricts to the standard vector space $V = \mathbb{C}^n$):

Define $X_1 = pt$. There is a trivial bundle over it, $V \times pt$ of rank n. Consider $\mathbb{P}(V)$, the projective bundle. This admits a tautological line bundle, U_1.

$$\begin{tikzcd}
 & & U_1 \subset V \arrow[dl] \\
V \arrow[dr] & \mathbb{P}(V) \arrow[d] & \\
 & pt &
\end{tikzcd}$$

Over $\mathbb{P}(V)$, we can consider the quotient bundle fitting into the exact sequence

$$0 \to U_1 \to V \to V/U_1 \to 0$$

So we have a vector bundle of rank $n-1$ over $\mathbb{P}(V)$,

$$\begin{tikzcd}
V/U_1 \arrow[dr] & & U_1 \subset V \arrow[dl] \\
V \arrow[dr] & \mathbb{P}(V) \arrow[d] & \\
& X_1 = pt &
\end{tikzcd}$$

and we can repeat the above construction: consider the projective bundle $\mathbb{P}(V/U_1)$ and its associated tautological bundle, which has the form U_2/U_1:

$$\begin{tikzcd}
 & & U_2/U_1 \arrow[dl] \\
V/U_1 \arrow[dr] & X_3 = \mathbb{P}(V/U_1) \arrow[d] & U_1 \subset V \arrow[dl] \\
V \arrow[dr] & X_2 = \mathbb{P}(V) \arrow[d] & \\
 & X_1 = pt &
\end{tikzcd}$$

Maybe some King James english to explain what is happening. In $n = 1$ step, U_1 is the bundle over $\mathbb{P}(V)$ such that the fiber over $(pt, \ell \subset V_x)$ is the set $v \in \ell$, so that the fiber is isomorphic to ℓ itself. Obviously this is a sub-bundle of the trivial bundle V, so we can consider the quotient bundle over $\mathbb{P}(V)$, $\mathbb{C}^n/U_1$. The fiber over the point $(pt, \ell \subset V) \in \mathbb{P}(V)$ is the space $V_{(pt,\ell)}/(U_1)_{(pt,\ell)} = V/\ell \cong \mathbb{C}^{n-1}$. Then there is a projective bundle, whose fiber over the point (pt, ℓ) is the space of lines in $V/\ell \cong \mathbb{P}(V/\ell)$, so $\mathbb{P}(V/U_1)$ consists of triples (pt, ℓ_1, ℓ_2) where $\ell_2 \subset V/\ell_1$ is a line, equivalently ℓ_2 is a 2-dimensional space in V containing ℓ_1. This projective bundle admits a tautological line bundle, L, whose fiber over the point (pt, ℓ_1, ℓ_2) is isomorphic to the line ℓ_2, and thus has the form U_2/U_1, where U_2 is a rank 2 bundle over $\mathbb{P}(V/U_1)$ whose fiber over the point (pt, ℓ_1, ℓ_2) is the 2-dimensional space $\ell_1 + \ell_2$.

Then we consider the rank $n-2$ bundle V/U_2 over $\mathbb{P}(V/U_1)$ and its tautological line bundle, U_3/U_2, and so on:

$$\begin{tikzcd}
 & & U_{n-1}/U_{n-2} \arrow[dl] \\
V/U_{n-2} \arrow[dr] & \mathbb{P}(V/U_{n-2}) \arrow[d] & \\
 & \vdots \arrow[d] & U_3/U_2 \arrow[dl] \\
V/U_2 \arrow[dr] & \mathbb{P}(V/U_2) \arrow[d] & U_2/U_1 \arrow[dl] \\
V/U_1 \arrow[dr] & X_3 = \mathbb{P}(V/U_1) \arrow[d] & U_1 \subset V \arrow[dl] \\
V \arrow[dr] & X_2 = \mathbb{P}(V) \arrow[d] & \\
 & X_1 = pt &
\end{tikzcd}$$

where an element of $\mathbb{P}(V/U_{n-2})$ is a collection $(pt, \ell_1, \ell_2, \ldots, \ell_{n-1})$ where ℓ_{n-1} is a line in the 2-dimensional space $V/(\ell_1 + \ell_2 + \cdots + \ell_{n-2})$, equivalently ℓ_{n-2} is an $n-1$-dimensional space in V.

The isomorphism

$$\mathbb{P}(V/U_{n-2}) \cong Fl(n)$$

is now clear: Send $(pt, \ell_1, \ldots, \ell_{n-1})$ to the flag $(0 \subset \ell_1 \subset \ell_2 \subset \cdots \subset \ell_{n-1} \subset V)$.

6/12/2023 The cohomology ring of the full flag variety

Now we can combine (6/11/2023) and (6/10/2023) to compute the cohomology ring of the full flag variety.

Denote $-c_1(U_1) := x_1$. Then

$$H^\bullet(\mathbb{P}(V)) \cong \frac{H^\bullet(pt)[x_1]}{(x_1^n + a_1 x_1^{n-1} + \cdots + a_n)} = \frac{\mathbb{Q}[x_1]}{(x_1^n + a_1 x_1^{n-1} + \cdots + a_n)}$$

ie x_1^k for $0 \leq k \leq n-1$ forms a basis of $H^\bullet(\mathbb{P}(V))$.

Denote $-c_1(U_2/U_1) := x_2$. Then

$$H^\bullet(\mathbb{P}(V/U_1)) \cong \frac{H^\bullet(\mathbb{P}(V))[x_2]}{(x_2^{n-1} + a_1' x_1^{n-2} + \cdots + a_{n-1}')}$$

In particular, x_2^k for $0 \leq k \leq n-2$ is a basis for $H^\bullet(\mathbb{P}(V/U_1))$ as a $H^\bullet(\mathbb{P}(V))$-algebra, so $x_1^{i_1} x_2^{i_2}$ is a basis for $H^\bullet(\mathbb{P}(V/U_1))$ as a $\mathbb{Q}$-algebra, where $i_j \leq n-j$. Following this pattern, $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ is a basis for $H^\bullet(\mathbb{P}(V/U_{n-1})) \cong H^\bullet(Fl(n))$ where $i_j \leq n-j$.

6/13/2023 The Fundamental groupoid of S^1 is monoidal

This exercise was given at LAWRGE 2023. There will be many entries coming from this workshop as I work back through the notes and exercises post-meeting.

The fundamental groupoid of a space X is a category $\pi_{\leq 1} X$ whose objects are points of X and whose morphisms are homotopy classes of paths between points. In the case of S^1, quotienting paths between two points by homotopy equivalence means that we may assume every morphism between two points is just the "constant speed" path between the two points, ie the path doesn't turn around on the way there. The group structure on S^1 (adding angles, if you like) induces a monoidal structure on $\pi_{\leq 1}(S^1)$: To tensor points, we simply apply the group operation, identifying S^1 with the unit sphere in $\mathbb{C}$:

$$e^{i_1\theta} \otimes e^{i_2\theta} := e^{(i_1+i_2)\theta}$$

A monoidal structure also requires the ability to tensor maps. Suppose we have two homotopy classes of paths $[f_1] : e^{i_1\theta} \to e^{i_2\theta}$, $[f_2] : e^{j_1\theta} \to e^{j_2\theta}$, meaning that f_1 is a path in S^1 from $e^{i_1\theta}$ to $e^{i_2\theta}$ and same for f_2. To describe the monoidal structure, we must define a morphism

$$\begin{aligned} f_1 \otimes f_2 &: e^{i_1\theta} \otimes e^{j_1\theta} \to e^{i_2\theta} \otimes e^{j_2\theta} \\ &: e^{(i_1+j_1)\theta} \to e^{(i_2+j_2)\theta} \end{aligned}$$

which will be defined by starting at the point $i_1 + j_1$ apply the map "$f_1 \otimes e^{j_1\theta}$", ie

$$t \mapsto f_1(t) \otimes e^{j_1\theta} i$$

In particular, $f_1(t) \otimes e^{j_1\theta}$ is a path

$$f_1(0)e^{j_1\theta} = e^{(i_1+j_1)\theta} \longrightarrow f_1(1)e^{j_1\theta} = e^{(i_2+j_1)\theta}$$

Compose this with the morphism $f_2 \otimes e^{i_2\theta}$ to obtain a morphism

$$i_1 + j_1 \to i_2 + j_1 \to i_2 + j_2$$

as desired. The point $e^0 = 1$ is the monoidal unit in this category. It is also symmetric monoidal.

6/14/2023 A $\pi_{\leq 1}(S^1)$-module category is equipped with a natural automorphism of the identity functor

LAWRGE 2023 Exercise.

First we must say what it means to be a module over a category. This is a general feature of monoidal categories, hence the previous entry:

Definition (nLab): Let $\mathcal{M}$ be a monoidal category. A module category over $\mathcal{M}$ is a category $\mathcal{C}$ equipped with a functor $\triangleright : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ (with the category $\mathcal{M} \times \mathcal{C}$ defined as expected) and natural isomorphisms $\alpha_{A,B,X} : A \triangleright (B \triangleright X) \to (A \otimes B) \triangleright X$ (these are the components of the natural isomorphism) satisfying a pentagon axiom involving the associator of $\mathcal{M}$, and natural isomorphisms $\lambda_X : I \triangleright X \to X$, where I is the monoidal unit of $\mathcal{M}$.

This data is equivalent to the choice of monoidal category $\mathcal{C}$ and monoidal functor $\mathcal{M} \to End(\mathcal{C})$ where the monoidal product on the functor category $End(\mathcal{C})$ is composition.

Thus a $\pi_{\leq 1}(S^1)$-module category is a category $\mathcal{C}$ and a functor $\triangleright : \pi_{\leq 1}(S^1) \times \mathcal{C} \to \mathcal{C}$ with natural isomorphisms satisfying the action axioms. Let $\gamma(t) = e^{it} \in Hom_{\pi_{\leq 1}(S^1)}(1, 1)$ denote the morphism winding around the circle once at constant speed. Then by functoriality,

$$\gamma : 1 \to 1$$

$$\gamma \triangleright A : 1 \triangleright A \to 1 \triangleright A$$

So we may construct an isomorphism

$$A \cong 1 \triangleright A \xrightarrow{\gamma \triangleright A} 1 \triangleright A \cong A$$

as the components $h_A : Id_{\mathcal{C}} \to Id_{\mathcal{C}}$. Each h_A is an isomorphism due to the action axioms, and naturality follows from naturality of $1 \triangleright (-)$.

6/15/2023 Fibers over configuration spaces

LAWRGE 2023 Exercise.

Let $Conf_k(\mathbb{R}^n)$ denote the configuration space[19] of k distinct points in $\mathbb{R}^n$. There is a natural map $Conf_k(\mathbb{R}^n) \to Conf_{k-1}(\mathbb{R}^n)$ given by forgetting the kth point. The fiber over a configuration of $k-1$ points is all configurations of k points whose first $k-1$ points agree with the chosen configuration, and whose kth point is anywhere except at those $k-1$ chosen points (because in the configuration space, points must all be distinct). Thus

[19] For those in the know, $Conf_k(\mathbb{R}^n)$ is homotopy equivalent to $\mathbb{E}_n(k)$, a certain algebra over the "little disk operad"? I am not really in the know, it was taught at LAWRGE 2023 but was over my head at the time. This homotopy equivalence is one reason why physics-adjacent people care about configuration spaces.

the fiber is equivalent to $\mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\}$, which is homotopy equivalent to the wedge of $k-1$ spheres S^{n-1} by deformation retract. Indeed this establishes the fibration

$$F_k \hookrightarrow Conf_k(\mathbb{R}^{n-1}) \to Conf_{k-1}(\mathbb{R}^n)$$

One should note that this does apply to more general spaces than $\mathbb{R}^n$, such as topological manifolds.

6/16/2023 Homology of configuration spaces

LAWRGE 2023 Exercise.

By some abstract nonsense (Leray Spectral sequence, I have no idea how it works), one has an isomorphism

$$H_\bullet(Conf_k(\mathbb{R}^n), \mathbb{Q}) \cong H_\bullet(Conf_{k-1}(\mathbb{R}^n), \mathbb{Q}) \otimes H_\bullet(F_k; \mathbb{Q})$$

where F_k is the fiber described above. We want to compute $H_\bullet(\mathbb{E}_n(k); \mathbb{Q})$ for $i = 1, 2, 3$.

As mentioned in 6/15/2023 (the above entry), $\mathbb{E}_n(k)$ is homotopy equivalent to $Conf_k(\mathbb{R}^n)$, so we may use the formula above (this is just to get practice using it I suppose (and to remember that tensor product of chain complexes is not just tensoring the pieces)).

Example: For $k = 1$, we have

$$H_\bullet(\mathbb{E}_n(1); \mathbb{Q}) \cong H_\bullet(Conf_0(\mathbb{R}^n); \mathbb{Q}) \otimes H_\bullet(\bigvee^0 S^{n-1}; \mathbb{Q})$$
$$\cong H_\bullet(\{pt\}, \mathbb{Q}) \otimes H_\bullet(\{pt\}, \mathbb{Q}) \cong \mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$$

using the homotopy equivalence established in 6/15/2023, the above entry.

Example: For $k = 2$, we have

$$H_\bullet(\mathbb{E}_n(2); \mathbb{Q}) \cong H_\bullet(Conf_1(\mathbb{R}^n); \mathbb{Q}) \otimes H_\bullet(S^{n-1}; \mathbb{Q})$$
$$\cong H_\bullet(\mathbb{R}^n; \mathbb{Q}) \otimes H_\bullet(S^{n-1}; \mathbb{Q})$$

the first term has non-zero homology only in degree 0 and the second has non-zero homology only in degree 0 and $n-1$, with all non-zero values being $\mathbb{Q}$. Then

$$H_\bullet(\mathbb{E}_n(2); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & deg\ 0 \\ \mathbb{Q} & deg\ n-1 \\ 0 & else \end{cases}$$

Example: For $k = 3$,

$$H_\bullet(\mathbb{E}_n(3); \mathbb{Q}) \cong H_\bullet(Conf_2(\mathbb{R}^n); \mathbb{Q}) \otimes H_\bullet(S^{n-1} \bigvee S^{n-1}; \mathbb{Q})$$

We just calculated $H_\bullet(Conf_2(\mathbb{R}^n);\mathbb{Q})$ and we know what homology does to wedge sums:

$$\cong \begin{cases} \mathbb{Q} & deg\ 0 \\ \mathbb{Q} & deg\ n-1 \\ 0 & else \end{cases} \otimes \begin{cases} \mathbb{Q} & deg\ 0 \\ \mathbb{Q}^2 & deg\ n-1 \\ 0 & else \end{cases}$$

$$\cong \begin{cases} \mathbb{Q} & deg\ 0 \\ \mathbb{Q}^3 & deg\ n-1 \\ 0 & else \end{cases}$$

So we see in principle that one can compute the homology of $\mathbb{E}_n(k)$ for any n and k just knowing homology of spheres, which is easy.

6/17/2023 *The homology of the little disk operad is a free $\mathbb{P}_n$-algebra

LAWRGE 2023 Exercise.

Recall a $\mathbb{P}_n$-algebra is a dg commutative algebra A equipped with a bracket $\{-,-\}$ of cohomological degree $1-n$, inducing a Lie structure on $A[n-1]$, satisfying the relation $\{a, bc\} = \{a,b\}c + (-1)^{|b||c|}\{a,c\}b$. Let $\mathbb{P}_n(k)$ be the subspace of the free $\mathbb{P}_n$-algebra on degree 0 variables $x_1, \ldots, x_k$ consisting of expressions where each x_i appears only once.

Describe the graded vector space $\mathbb{P}_n(k)$ for $k = 1, 2, 3$ and find an isomorphism

$$H_\bullet(\mathbb{E}_n(k);\mathbb{Q}) \cong \mathbb{P}_n(k)$$

for $n \geq 2$.

$\boxed{\text{FINISH}}$ (maybe, I'm pretty confused about the definition of $\mathbb{P}_n(k)$).

6/18/2023 Hilbert series of coinvariant algebra

Definition: If $R = \bigoplus R_k$ is a graded algebra, then its <u>Hilbert series</u> is

$$Hilb\ R := \sum_{k \geq 0} \dim R_k q^k$$

Theorem (Chevalley, 1995)

$$Hilb(R_n) = [n]_q! \equiv [1]_q[2]_q \cdots [n]_q$$

where

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

Example: for $n = 3$, we have

$$Hilb(R_3) \equiv Hilb\left(\frac{\mathbb{Q}[x_1, x_2, x_3]}{p_1(3), p_2(3), p_3(3)}\right) = [2]_q[3]_q$$

The RHS is

$$\begin{aligned}&(1+q)(1+q+q^2)\\&=1+2q+2q^2+q^3\end{aligned}$$

so that

$$\begin{cases}\dim\,(R_3)_0,\ (R_3)_3=1\\ \dim\,(R_3)_1,\ (R_3)_2=2\end{cases}$$

For example, the degree 2 piece of R_3 has dimension 2. There are 6 additive generators $x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3$, before considering the relations. The relation $p_1(3)$, $x_1+x_2+x_3=0$ means that any generator containing, for example, x_3, can always be regarded as a sum of monomials in the other variables:

$$x_1x_3 = x_1(-x_1-x_2) = -x_1^2 - x_1x_2$$

and so on. So the generating set reduces to x_1^2, x_2^2, x_1x_2. The relation $p_2(3)$ says we can remove either x_1^2 or x_2^2, yielding 2 generators as desired.

One way to prove the theorem of Chevalley is to prove that the Artin basis of sub-staircase monomials[20]

$$\{x_1^{a_1}\cdots x_n^{a_n} \mid 0 \le a_i \le n-i\}$$

indeed forms a $\mathbb{C}$-basis for R_n. If one does prove this, then we have a homogeneous basis for the graded VS. In such a case,

$$Hilb(A;q) = \sum_{b\in\mathcal{B}} q^{\deg b}$$

$$Hilb(R_n) = \sum_{(a_1,\ldots,a_n)} q^{a_1+\cdots+a_n} = [n]_q!$$

(I don't understand the final equality here).

6/19/2023 Hilbert series of super coinvariant algebra

From Bruce Sagan's presentation.

Let $t_n := \{\theta_1,\ldots,\theta_n\}$ be a set of anticommuting variable (fermions) which commute with the x_j's. For $k\ge 0$, define the super power sum symmetric polynomials

$$sp_k(n) = x_1^k\theta_1 + \cdots + x_n^k\theta_n$$

Definition: The super coinvariant algebra is

$$SR_n := \frac{\mathbb{Q}[x_n, t_n]}{\langle p_i(n), sp_0(n),\ldots, sp_{n-1}(n)\rangle}$$

[20]For those truly avid readers, this coincides with the basis of chern classes in $H^\bullet(Fl(m))$ which we found in 6/12/2023. There the condition $a_i \le n-i$ comes from the fact that the bundle generating the ith chern class has rank $n-i$.

Definition: The q-Stirling numbers of the second kind are defined by the recurrence relation

$$S_q(n,k) = [k]_q! S_q(n-1,k) + S_q(n-1,k)$$

and $S_q(0,k) = \delta_{0,k}$, $S_q(n,0) = \delta_{n,0}$.

Theorem[21] **(Rhoades-Wilson, 2023)**:

$$Hilb(SR_n) = \sum_{k=1}^{n} t^{n-k} \cdot [k]_q! S_q(n,k)$$

Example:

$$Hilb(SR_2) = tS_q(2,1) + [2]_q! S_q(2,2) = t + 1 + q$$

Example:

$$\begin{aligned} Hilb(SR_3) &= t^2 S_q(3,1) + t[2]_q! S_q(3,2) + [3]_q! S_q(3,3) \\ & t^2 + t(1+q)(2+q) + (1+q)(1+q+q^2) \\ &= q^3 + q^2 t + 2q^2 + 3qt + 2q + t^2 + 2t + 1 \end{aligned}$$

6/20/2023 Topological twists in $3d$ $\mathcal{N} = 4$ SUSY

LAWRGE 2023 Exercise.

Worth pointing out: The exercise was given as an example of $3d$ $\mathcal{N} = 4$ SUSY, but for 4 parts, of which this is only the first. This part doesn't appear to actually depend on the choice $\mathcal{N} = 4$, but maybe I just don't see it.

In the case of $3d$ supersymmetry, the supersymmetry (super-Poincare) algebra is the Lie super-algebra

$$\mathfrak{siso}(3,\mathbb{C}) = \mathfrak{iso}(3,\mathbb{C}) \oplus \Pi\Sigma$$

where Σ is a spinorial representation[22] of $Spin(3,\mathbb{C}) \cong SL(2,\mathbb{C})$. Such a representation has the form $S \otimes W$, where S is the two-dimensional defining representation of $SL(2,\mathbb{C})$. The non-trivial Lie bracket is given by a map[23] $Sym^2(\Sigma) \to \mathbb{C}^3 \cong Sym^2(S) \cong V$, where V is the 3-dimensional adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$. Let's denote the isomorphism[24]

[21] The proof is due to Rhoades-Wilson, conjectured by Sagan-Swanson. I'm going to write the theorem in the notation of Rhoades-Wilson, since I originally wrote this entry using the notation of Sagan-Swanson and arrived at an expression for $Hilb(SR_3)$ that was insanely incorrect, so I was interpreting something about their notation wildly incorrectly. For example, somehow I ended up with a term $q^9 t^3$ which obviously does not make sense.

[22] Odd dimensions are good because spinorial representations are are just fundamental spin representations, otherwise in even dimension you get semi-spin reps which are more complicated.

[23] Symmetric instead of antisymmetric because the overall Lie bracket is graded antisymmetric, and we take two elements from the odd portion in the Lie superalgebra, thus the bracket here is symmetric.

[24] One way to see that there is such an isomorphism is to note that the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$ is irreducible, thus it must have the form $Sym^k(S)$ for some k, which indexes the family of f.d. irreps of $\mathfrak{sl}(2,\mathbb{C})$. The dimension of the latter is $\binom{2+k-1}{k}$ (in general, 2 is the dimension of the underlying space). The only way to get 3 out of this is with $k = 2$, so there is such an isomorphism.

$Sym^2(S) \cong V$ as ρ. A map $Sym^2(S \otimes W) \to V$ is equivalent to a map

$$Sym^2(S) \otimes Sym^2(W) \oplus \bigwedge^2(S) \otimes \bigwedge^2(W) \to V$$

So given an element of $Sym^2(\Sigma)$, we may project onto the $Sym^2(S) \otimes Sym^2(W)$ component. Then $Sym^2(S)$ is already isomorphic to V, so a choice of map

$$g : Sym^2(W) \to \mathbb{C}$$

will uniquely determine such a bracket. Non-degeneracy of this g makes it equivalent to an inner product on W. Concretely, the bracket $[s \otimes w, s' \otimes w']$ is computed by sending $[s \otimes s']$ through the isomorphism to $Sym^2(S) \cong V \cong \mathbb{C}^3_{E,F,H}$, then multiplying by the inner product $\langle w, w' \rangle$. Thus the dimension of W (determining the specific spinorial representation) and the data of an inner product on W determines a 3d SUSY theory.

A topological twist of a SUSY is determined by a BRST operator $Q \in \Pi\Sigma$ such that $[Q, Q] = 0$. The variety $\{Q \in \Pi\Sigma \mid [Q, Q] = 0\}$ is called the nilpotence variety.

If we choose a basis $\{u, v\}$ for S, and $Q := Q_1 \otimes u + Q_2 \otimes v \in \Sigma$, by standard (super) Lie bracket properties

$$\begin{aligned}[Q, Q] &= [Q_1 \otimes u, Q_1 \otimes u] + 2[Q_1 \otimes u, Q_2 \otimes v] + [Q_2 \otimes v, Q_2 \otimes v] \\ &= \rho[u \otimes u]\|Q_1\|_W + 2\rho[u \otimes v]\langle Q_1, Q_2 \rangle + \rho[v \otimes v]\|Q_2\|_W\end{aligned}$$

Where $\rho : Sym^2(S) \to V$. If e_1, e_2 is the standard basis of S, then ρ sends

$$\begin{aligned} e_1 \otimes e_1 &\mapsto E \\ e_1 \otimes e_2 &\mapsto \frac{H}{2} \\ e_2 \otimes e_2 &\mapsto F \end{aligned}$$

Letting $u = u^i e_i$ and $v = v^i e_i$,

$$\begin{aligned} u \otimes u &\mapsto (u^1)^2 E + u^1 u^2 H + (u^2)^2 F \\ u \otimes v &\mapsto u^1 v^1 E + \frac{u^1 v^2 + u^2 v^1}{2} H + u^2 v^2 F \\ v \otimes v &\mapsto (v^1)^2 E + v^1 v^2 H + (v^2)^2 F \end{aligned}$$

So $[Q, Q] = 0$ implies

$$\begin{aligned}
0 = &\Big[(u^1)^2E + u^1u^2H + (u^2)^2F\Big]\|Q_1\| \\
&+2\Big[u^1v^1E + \frac{u^1v^2 + u^2v^1}{2}H + u^2v^2F\Big]\langle Q_1, Q_2\rangle \\
&+\Big[(v^1)^2E + v^1v^2H + (v^2)^2F\Big]\|Q_2\|
\end{aligned}$$

$$\begin{aligned}
0 = &E\Big[(u^1)^2\|Q_1\| + 2u^1v^1\langle Q_1, Q_2\rangle + (v^1)^2\|Q_2\|\Big] \\
&+F\Big[(u^2)^2\|Q_1\| + 2u^2v^2\langle Q_1, Q_2\rangle + (v^2)^2\|Q_2\|\Big] \\
&+H\Big[u^1u^2\|Q_1\| + (u^1v^2 + u^2v^1)\langle Q_1, Q_2\rangle + v^1v^2\|Q_2\|\Big]
\end{aligned}$$

Because E, F, H form a basis of $\mathbb{C}^3$,

$$\begin{aligned}
(u^1)^2\|Q_1\| + 2u^1v^1\langle Q_1, Q_2\rangle + (v^1)^2\|Q_2\| &= 0 \\
(u^2)^2\|Q_1\| + 2u^2v^2\langle Q_1, Q_2\rangle + (v^2)^2\|Q_2\| &= 0 \\
u^1u^2\|Q_1\| + (u^1v^2 + u^2v^1)\langle Q_1, Q_2\rangle + v^1v^2\|Q_2\| &= 0
\end{aligned}$$

is the nilpotence variety for 3d super symmetry, which parameterizes topologically twisted 3d super symmetric theories.

6/21/2023 The super-Artin basis of SR_n

I read these ideas in "q-Stirling numbers in type B" (though ironically I am only interested in type A for now) by Sagan-Swanson.

Recall SR_n defined in 6/19/2023. The conjectural (as of 2022, and with some experimental evidence) basis of this algebra is defined as:

$$SA_n = \{x^\alpha \theta_T \mid T \subset [2, n],\ \theta_T = \theta_{t_1} \cdot \theta_{t_2} \cdots \theta_{t_k},\ \alpha \le \alpha(T)\}$$

where $T = \{t_1 < \cdots < t_k\}$ and $\alpha(T)$ is the composition defined recursively by, for any $T \subset [2, n]$, $\alpha(T)_1 := 0$ and

$$\alpha(T)_i := \alpha(T)_{i-1} + \begin{cases} 0 & i \in T \\ 1 & i \notin T \end{cases}$$

and inequality of compositions is defined as expected.

Sanity check: The Artin basis A_n is the case where $T = \emptyset$, since in this case $\alpha(T)$ is just the staircase diagram and θ's do not appear, so SA_n reduces to the sub-staircase monomials, which is a standard presentation of Artin basis.

Example: $n = 2$: In this case, $T \subset [2, 2]$ is just a bit which can be denoted as $0, 1$. In general, the basis is doubly-indexed, first by choice of T then by choice of $\alpha \leq \alpha(T)$. In this case, there are two possible T's, corresponding to the diagrams

$$T_1 = \{2\},\ T_2 = \emptyset$$

As we already mentioned $T = \emptyset$ corresponds to the diagram $(0, 1)$. The possible $\alpha \leq \alpha(T)$ are $(0, 0)$ and $(0, 1)$. These correspond to the terms

$$x_1^0 x_2^0, \quad x_1^0 x_2^1$$

Remark: Despite the situation, we may always choose $T = \emptyset$, so the standard sub-staircase monomial basis should always appear.

We also have $T = \{2\}$, corresponding to the term

$$x_1^0 x_2^0 \theta_2$$

to make the basis $1, x_2, \theta_2$.

Example: $n = 3$: Now $T \subset \{2, 3\}$. Note that T is always meant to be always increasing, so we don't need to consider different orders.
$T = \emptyset$: The Artin basis[25] is $1, x_2, x_3, x_2x_3, x_3^2, x_2x_3^2$. (One could check that this coincides with the dimensions we found in 6/18/2023 where we write out examples of the Hilbert series for the coinvariant algebra)

We may also choose $T = \{2\}$. This corresponds to $\theta_2, x_3\theta_2$. There is also $T = \{3\}$. This corresponds to $\theta_3, x_3\theta_3, x_2\theta_3, x_2x_3\theta_3$. Finally we have $T = \{2, 3\}$, carrying the term $\theta_2\theta_3$. Altogether,

$$SR_3 = \left\langle 1, x_2, x_3, \theta_2, \theta_3, x_2x_3, x_3^2, x_3\theta_2, x_2\theta_3, x_3\theta_3, \theta_2\theta_3, x_2x_3^2, x_2x_3\theta_3 \right\rangle$$

For both examples, we may verify that they agree with the dimensions given in 6/19/2023, where we write down examples of the doubly graded Hilbert series for the supercoinvariant algebra.

6/22/2023 Translating between bases of $H^\bullet(Fl(\mathbb{C}^3))$

We have discussed the "algebraic basis", given by sub-staircase monomials for a full flag variety. This is referred to as algebraic basis but to me it is still geometric. After all, its

[25]Sagan-Swanson use the convention to always eliminate x_1, θ_1 when possible. This is actually very annoying to me because the natural construction of $Fl(m)$ as a sequence of projective bundles eliminates the highest index. In the case $n = 3$, one can just replace x_1 with x_3 for the Artin basis, but I doubt it is always this easy and I don't know if one can directly substitute x_1 with x_3 in the super Artin basis.

elements are given by chern classes of canonical bundles over the space. What could be more geometric than that?

The "geometric basis" is given by the Schubert classes, which are indexed by the symmetric group S_3. It is introduced by first considering dimensions of intersections, a natural generalization of what we defined in 4/8/2023. In particular, if $F_\bullet$ denotes the standard coordinate full flag in $\mathbb{C}^3$, we can define the Schubert cell X_ω as the equivalence class of the coordinate flag corresponding to $\omega(123)$ under the relation

$$E_\bullet \sim E'_\bullet \iff dim(E_p \cap F_q) = dim(E'_p \cap F_q), \quad \forall\, p, q$$

To unpack this, first note that there is an induced S_n-action on $Fl(\mathbb{C}^n)$ which preserves the subset of coordinate flags. In fact, this leads to a bijection

$$S_n \cong \{\text{complete coordinate flags in } \mathbb{C}^n\}$$

For example, if [123] represents the standard coordinate flag, then $(12) \cdot [123] = [213]$, which is the coordinate flag $0 \subset \langle e_2 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle = \mathbb{C}^3$.

Consider the Schubert cell $X_{(12)}$, the equivalence class of the coordinate flag [213]. By the equivalence relation, any other flag, $E_\bullet$, in this class satisfies the relation

$$\begin{aligned} dim(E_1 \cap F_1) &= 0 \\ dim(E_1 \cap F_2) &= 1 \\ dim(E_2 \cap F_1) &= 1 \\ dim(E_2 \cap F_2) &= 2 \end{aligned}$$

Given such a flag, $dim(E_1 \cap F_1) = 0$ implies that E_1 is not $\langle e_1 \rangle$. But $dim(E_1 \cap F_2) = 1$ means that E_1 is contained in $\langle e_1, e_2 \rangle$, so it can be any line in this plane besides the line $\langle e_1 \rangle$, call this $E_1 = \langle ae_1 + be_2 \rangle$, with $b \neq 0$.

$dim(E_2 \cap F_2) = 2$ means $E_2 = \langle e_1, e_2 \rangle$, so E_2 is pre-determined. Thus an element of $X_{(12)}$ is determined by a choice of one-dimensional space in $\langle e_1, e_2 \rangle$, which is given by $\mathbb{P}^1$. However it cannot be the line $\langle e_1 \rangle$, so $X_{(12)} = \mathbb{P}^1 - \{pt\} = \mathbb{A}^1$.

We could carry out such a computation for every X_ω and obtain an isomorphism to affine spaces of various dimensionsx. Thus the Schubert cells provide an affine open cover of the full flag manifold, thus its cohomology ring gets a basis consisting of the Schubert classes σ_ω, the cohomology classes of the closures of the Schubert cells. This basis is the geometric basis. The question addressed in this example is what is the change of basis formulas?

From general theory, we always have

$$\sigma_e = 1, \quad \sigma_{(12)} = x_1, \quad \sigma_{(23)} = x_1 + x_2$$

$$\sigma_{(13)} = \sigma_{\omega_0} = x_1^2 x_2$$

And we must use the divided difference operators ∂_i to calculate the remaining two:

$$\sigma_{(123)} = \partial_1 \sigma_{(13)} = \partial_1(x_1^2 x_2) = x_1 x_2$$

$$\sigma_{(132)} = \partial_2 \sigma_{(13)} = x_1^2$$

NOTE: This calculation with divided difference operators

$$\sigma_{(123)} = \frac{x_1^2 x_2 - x_2^2 x_1}{x_1 - x_2} = x_1 x_2$$

is identical to the COHA multiplication

$$(x_1 \in H_1(\bullet)) * (x_1^2 \in H_1(\bullet)) = \frac{x_1 x_2^2 - x_2 x_1^2}{x_2 - x_1} = x_1 x_2$$

This is probably just a coincidence though, you get more denominators in COHA multiplication, but not in Schubert polynomials.

Note that this is the Schubert polnomials expressed in terms of the additive basis. To express in terms of the algebra-basis, we should apply the relations to obtain only those terms in the Artin basis:

$$\sigma_{(e)} = 1$$

$$\sigma_{(12)} = x_2 + x_3, \quad \sigma_{(23)} = 2x_2 + x_3$$

$$\sigma_{(123)} = x_3^2, \quad \sigma_{(132)} = x_2 x_3$$

$$\sigma_{(13)} = -x_2 x_3^2$$

Note: I used SageMath (after watching about 3 hours worth of tutorials in 1.5 hours) to calculate $\sigma_{(13)}$ in terms of the Artin basis. This notes the first appearance of any software to perform a computation in this document, and it surely will not be the last. I don't like computers. I wish I could do all the math I wanted without ever having to touch software. Alas, I am not so lucky.

We have expressed the Schubert polynomials (geometric basis) in terms of the chern classes (algebraic basis). Finding a combinatorial formula to compute the Schubert polynomials for arbitrary full flag varieties is a massive open problem in algebraic combinatorics.

Recall that the basis of $H^\bullet(Gr(k,n))$ can be indexed by the Young tableau associated to the partition in σ_λ. The basis of $H^\bullet(Fl(m))$ is instead indexed by S_m which has a nice bijection with pairs of same-shape standard Young Tableaux (SYT). This correspondence is known as the Robinson-Schensted algorithm. A SYT is a YT with boxes filled in with distinct natural numbers so that the columns and rows form increasing sequences. I don't want to write down the full algorithm, just google it if you need to know how to do it

$$e \longleftrightarrow \begin{ytableau} 1 & 2 & 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 3 \end{ytableau}$$

$$(12) \longleftrightarrow \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$$

$$(23) \longleftrightarrow \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$

$$(123) \longleftrightarrow \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$

$$(132) \longleftrightarrow \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$$

$$(13) \longleftrightarrow \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$$

So in this example, we have (I will write the pairs of SYT in parenthesis instead of subscripts, since it will be too small to read).

$$\sigma_e = \sigma\left(\begin{ytableau} 1 & 2 & 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 3 \end{ytableau}\right) = 1$$

$$\sigma_{(12)} = \sigma\left(\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\right) = x_2 + x_3$$

$$\sigma_{(23)} = \sigma\left(\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\right) = 2x_2 + x_3$$

$$\sigma_{(123)} = \sigma\left(\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\right) = x_3^2$$

$$\sigma_{(132)} = \sigma\left(\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\right) = x_2 x_3$$

$$\sigma_{(13)} = \sigma\left(\begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}\right) = -x_2 x_3^2$$

Actually now that I think about it, I don't know what the geometric meaning of this bijection is. I should probably investigate that soon. For Grassmannians, the YT indexing of the basis is geometrically significant because the partition of the Schubert cell corresponds to the dimensions of intersections of the subspaces. In the case of flags, the dimensions of an Schubert class are indexed by pairs, $d_{i,j} \equiv dim(E_i \cap F_j)$ with $F_\bullet$ the standard flag.

6/24/2023 Hyperkahler coordinates on $\mathbb{C}^2$

LAWRGE 2023 Exercise.

Let $\mathbb{C}^2$ be the hyperkahler space with complex coordinates (x, y) in the complex structure I, real Kahler form

$$\omega_I = -\frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})$$

and holomorphic symplectic form

$$\Omega_I = dx \wedge dy$$

We may find the formulas for the other two Kahler and holomorphic symplectic forms by the formula

$$\Omega_I = \omega_J + i\omega_K$$

So if we write $dx \wedge dy$ into real and imaginary parts, we may identify the other two Kahler forms: Let $x = x_1 + ix_2$, $y = y_1 + iy_2$. Then

$$\begin{gathered} dx \wedge dy \\ = (dx_1 + idx_2) \wedge (dy_1 + idy_2) \\ = dx_1 \wedge dy_1 + idx_1 \wedge dy_2 + idx_2 \wedge dy_1 - dx_2 \wedge dy_2 \\ = (dx_1 \wedge dy_1 - dx_2 \wedge dy_2) + i(dx_1 \wedge dy_2 + dx_2 \wedge dy_1) \\ \Rightarrow \omega_J = dx_1 \wedge dy_1 - dx_2 \wedge dy_2, \quad \omega_K = dx_1 \wedge dy_2 + dx_2 \wedge dy_1 \end{gathered}$$

We were given coordinates in terms of x, y so if we wanted to give the answers back in those terms, we would use

$$x_1 = \frac{x + \bar{x}}{2}, \quad x_2 = \frac{x - \bar{x}}{2i}$$
$$y_1 = \frac{y + \bar{y}}{2}, \quad y_2 = \frac{y - \bar{y}}{2i}$$

Then, for example,

$$\omega_J = d\left(\frac{x+\bar{x}}{2}\right) \wedge d\left(\frac{y+\bar{y}}{2}\right) - d\left(\frac{x-\bar{x}}{2i}\right) \wedge d\left(\frac{y-\bar{y}}{2i}\right)$$
$$= \frac{1}{4}(dx + d\bar{x}) \wedge (dy + d\bar{y}) + \frac{1}{4}(dx - d\bar{x}) \wedge (dy - d\bar{y})$$
$$= \frac{1}{4}\Big[dx \wedge dy + dx \wedge d\bar{y} + d\bar{x} \wedge dy + d\bar{x} \wedge d\bar{y} + dx \wedge dy - dx \wedge d\bar{y} - d\bar{x} \wedge dy + d\bar{x} \wedge d\bar{y}\Big]$$
$$= \frac{1}{4}\Big[2dx \wedge dy + 2d\bar{x} \wedge d\bar{y}\Big]$$
$$= \frac{1}{2}(dx \wedge dy + d\bar{x} + d\bar{y})$$

Applying the exact same process we obtain

$$\omega_K = \frac{1}{2i}(dx \wedge dy - d\bar{x} \wedge d\bar{y})$$

So we have formulas for all the Kahler forms. We also obtain formulas for the holomorphic symplectic forms

$$\Omega_J = \omega_K + i\omega_I, \quad \Omega_K = \omega_I + i\omega_J$$

These are straightforward computations yielding, for example,

$$\Omega_K = \frac{i}{2}\big(dx \wedge dy + dx + d\bar{x} + dy \wedge d\bar{y} + d\bar{x} \wedge d\bar{y}\big)$$

Using this formula, we can guess what the holomorphic coordinates in the complex structure K should be: It should be some coordinates a, b on $\mathbb{C}^2$ such that $\Omega_k = da \wedge db$. One can see by computation that

$$\Omega_K = d(x - \bar{y}) \wedge d(\bar{x} + y)$$

which gives our holomorphic coordinates $(x - \bar{y}, \bar{x} + y)$ in K.

One may suspect that this is not unique to dimension 2. This is correct, and one would obtain the holomorphic coordinates and above formulas for the hyperkahler space $\mathbb{C}^{2n}$ in the same way, replacing x with $\vec{x}$ and so on. These formulas are uglier because you have to write $\vec{\bar{x}}$ for example.

6/26/2023 Doubled coordinates on a hyperkahler space

LAWRGE 2023 Exercise.

On the same hyperkahler space $\mathbb{C}^2$ considered as above, there is an explicit matrix[26] in $SU(2)$ rotating the complex structure I into K:

$$\frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

[26] This is Hermitian but doesn't have determinant 1. I'm not sure what's happening here, but that's how the problem was stated.

Introduce "doubled coordinates" on $\mathbb{C}^2$:

$$Z^{i+} = \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow Z^{ia} = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}$$

where $a \in \{+,-\}$. indicating the first and second columns. The $SU(2)$ hyperkahler rotations act on the doubled coordinates on the right, so they should also rotate the holomorphic coordinates of I into the holomorphic coordinates of K. Indeed, (you have to delete the overall factor to make it work out correctly. Another mysterious aspect of this problem.)

$$\begin{pmatrix} x & \bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} x - \bar{y} & x + \bar{y} \\ y + \bar{x} & y - \bar{x} \end{pmatrix}$$

So that the holomorphic coordinates (x, y) in I have been replaced by the holomorphic coordinates $(x - \bar{y}, \bar{x} + y)$ that we found in the above entry. I still don't really understand the role of doubled coordinates in the big picture sense.

6/27/2023 Geometry behind indexing set of the basis of $H^\bullet(Fl(3))$ I

Recall we discussed two bases of the cohomology ring $H^\bullet(Fl(3))$, $\{\sigma_\omega\}_{\omega \in S_3}$ and $x_1^{i_1} x_2^{i_2} x_3^{i_3}$ with $i_j \leq 3 - j$ (using the relations in the coinvariant algebra, one can replace the "staircase" in the sub-staircase monomial basis with the "opposite staircase" yielding the other basis which is also called sub-staircase monomial basis, $\{x_1^{i_1} x_2^{i_2} x_3^{i_3} \mid i_j \leq j - 1\}$). I will put the formulas from that entry here, along with the additional geometric information, the coordinate flag representative $[ijk]$ and the (2,2) dimension matrix $d_{i,j} = (d)_{ij} = \begin{pmatrix} dim(E_1 \cap F_1) & dim(E_1 \cap F_2) \\ dim(E_2 \cap F_1) & dim(E_2 \cap F_2) \end{pmatrix}$, as well as λ, the shape of the SYT's.

$$\lambda = (3,0,0) \quad \sigma_{[123]} = \sigma_e = \sigma\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}, \; \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \right) = 1 \quad d = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\lambda = (2,1,0) \quad \sigma_{[213]} = \sigma_{(12)} = \sigma\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) = x_2 + x_3 \quad d = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\lambda = (2,1,0) \quad \sigma_{[132]} = \sigma_{(23)} = \sigma\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) = 2x_2 + x_3 \quad d = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\lambda = (2,1,0) \quad \sigma_{[231]} = \sigma_{(123)} = \sigma\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) = x_3^2 \quad d = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\lambda = (2,1,0) \quad \sigma_{[312]} = \sigma_{(132)} = \sigma\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) = x_2 x_3 \quad d = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\lambda = (1,1,1) \quad \sigma_{[321]} = \sigma_{(13)} = \sigma\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\,,\quad \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = -x_2x_3^2 \quad d = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

and now I will stare at this page and try to see a pattern. In particular I'm looking for some way to translate between the algebraic data (partition) and the geometric data (intersection dimension matrix) in the way that you can do for Grassmannians. Some things to notice: All the entries are bits: They can only take on two values. The bottom right must be 1 or 2, while the others must be 0 or 1.

The only classes with 1 in the top left of d are also the only classes whose both reading words (starting top to bottom) are in order, but $\sigma_{(13)}$ is an exception.

The pairs are equal for $\sigma_{(13)}, \sigma_{(23)}, \sigma_{(12)}$ and σ_e, while they are not for the others. In other words, all the transpositions and identity satisfy this.

6/28/2023 Geometry behind indexing set of the basis of $H^\bullet(Fl(3))$ II

Taken from mathoverflow user Igor Makhlin, who also appears to be a real person.

Continuing from the above entry, we wanted to know if there is some way to, given the pairs of SYT for a certain element of $H^\bullet(Fl(3))$, determine what Schubert cell that element will belong to. There is an indirect method of doing so (this isn't exactly what I'm looking for, but it is better to have than not having it): Given such an element σ and its pairs of SYT, work backwards through the RS correspondence to obtain its permutation, $\omega \in S_3$ (not fun, and already we see that it is not a direct method like I hoped). Then given ω, $dim(E_i \cap F_j)$ is equal to the number of values $\leq j$ among the set $\omega(1), \ldots, \omega(i)$.

Example: So really this is an algorithm to get from $\omega \in S_n$ to the partition, which is still useful. So choosing σ whose pairs of SYT correspond to the permutation (13), we write in two-line notation as $\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$. Then $dim(E_1 \cap F_1)$ is either 1 or 0, depending on $\omega(1) = 3$. In this case, it is not lesser or equal to 1, so $d = \begin{pmatrix}0 & *\\ * & *\end{pmatrix}$. The others are filled in similarly, $d = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$, which agrees with the manual computation we did above. So in theory, we now know how to translate between the algebraic indexing of the geometric basis (SYT) and the geometric indexing ($dim(E_i \cap F_j)$.

6/30/2023 Pauli matrices as intertwiners

(part of a) LAWRGE 2023 Exercise.

Define the Pauli matrices

$$(\sigma^{\mu}) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

These may be interpreted as maps

$$V_2 \otimes \bar{V}_2 \to V_3$$

where V_2 is the fundamental representation of $SU(2)$, $\bar{V}_2$ is the antifundamental representation where X acts as $\bar{X} = X^T$, and V_3 is the adjoint representation of $SU(2)$.

IM NOT SURE HOW TO VIEW THE PAULI MATRICES AS MAPS BETWEEN THESE TWO SPACES

I SEE HOW THEY ACT ON THE LEFT BUT WHY IS THE IMAGE IN V_3?

THIS IS SOME PHYSICS MALARKEY

7/3/2023 Two 3d mirror dual quiver gauge theories

LAWRGE 2023 Exercise.

For an explanation of how to do these brane diagram manipulations, one can consult Yiyan Shou's PhD thesis titled "Bow varieties - Geometry, combinatorics, characteristic classes". I believe there is a paper with the author and Richard Rimanyi by a similar name, but probably the thesis is more beginner friendly.

Consider the canonical quiver corresponding to $T^*Gr(2,4)$, that is A_2 framed with weight 4 (I'm too lazy to learn how to tikz quivers nicely). This has an associated brane diagram (I don't know why my NS5 branes are so much smaller than the D5 branes and in this giant document I can't be bothered to figure it out)

$$T^*Gr(2,4) \rightsquigarrow /2\backslash 2\backslash 2\backslash 2\backslash 2/$$

There is also a quiver gauge theory associated to this quiver, which we will not discuss how it works here. We may apply 3DMS followed by 2 HW moves and then another 2 to obtain:

$$/2\backslash 2\backslash 2\backslash 2\backslash 2/$$

3DMS

$$\backslash 2/2/2/2/2\backslash$$

2 HW moves at the ends

$$/1\backslash 2/2/2\backslash 1/$$

2 HW moves in the middle

$$/1/2\backslash 2\backslash 2/1/$$

This is exactly the brane diagram associated to the quiver A_3 with dimensions 1,2,1 and framing 0,2,0. This shows that the associated quiver gauge theories are 3D mirror dual to each other. In particular, their Higgs and Coulomb branches are isomorphic. Note that the D5 charge vector of the first and last brane diagrams are (1,1,1,1) and (2,2), which is consistent with them being cobalanced. The third brane diagram is not cobalanced, but it is HW equivalent to one which is, meaning its charge vector should be nonstrictly decreasing, which we see it is, (2,2).

7/4/2023 Cobalanced brane diagrams and charge vectors

Same as above entry for a source of expoisition.

Theorem: *A brane diagram is HW equivalent to a balanced brane diagram iff its NS5 charge vector is weakly increasing.*

Example: The brane diagram

$$/1\backslash 3/4\backslash$$

has NS5 charge vector $(1,2)$, so there should exist some sequence of HW moves to make it balanced. In this case that means we need to change the 3 to a 4 or vice versa. Apply HW on the rightmost pair:

$$/1\backslash 3\backslash (0+3-4+1)/ = /1\backslash 3\backslash$$

which is vacuously balanced, so that wasn't that interesting.

Example: $\boxed{\text{FINISH}}$

7/7/2023 Affine paving of projective space

I read this from Pawlowski-Rhoades.

Definition: A <u>paving by affines</u> (or affine paving) of a complex algebraic variety is a stratification/filtration

$$X_\bullet = (X_0 = X \supset X_1 \supset \cdots \supset X_m = \varnothing)$$

where X_i is a closed subvariety and $X_i - X_{i+1}$ is a disjoint union of affine spaces:

$$X_i - X_{i+1} = \sqcup_j A_{ij}$$

A_{ij} are the *cells* of this paving (also referred to as cellular decomposition). It is a result that each successive difference can be refined to contain only a single cell.

(I'm not sure how much of this requires choosing $k = \mathbb{C}$)
Consider projective space $\mathbb{P}^{k-1}$ with its homogeneous coordinates $[z_0 : \cdots : z_k]$. One affine paving is given by

$$X_\bullet = \Big(\{[\star : \cdots : \star]\} \supset \{[0 : \star : \cdots : \star]\} \supset \cdots \supset \{[0 : 0 : \cdots : 0 : \star]\} \supset \varnothing\Big)$$

In this case, $X_0 - X_1$ is the set $\{[z_0 : \cdots : z_k] \mid z_0 \neq 0\}$, which is the familiar first chart when describing projective space as a smooth manifold. It is isomorphic to affine space $\mathbb{A}^{k-1}$ by the map

$$[z_0 : z_1 : \cdots : z_k] \mapsto \left(\frac{z_1}{z_0}, \ldots, \frac{z_k}{z_0}\right)$$

$X_1 - X_2$ is isomorphic to $\mathbb{A}^{k-2}$ by the map

$$[0 : z_1 : z_2 : \cdots : z_k] \mapsto \left(\frac{z_2}{z_1}, \ldots, \frac{z_k}{z_1}\right)$$

and so on. Such a paving by affines leads to an additive basis of singular cohomology $x_i = [\bar{A}_i]$. To obtain its ring structure, we have to multiply using intersection for cup product:

$$x_i \smile x_j = x_{i+j}$$

because x_i is PD to a $n-i$ plane, x_j is PD dual to an $n-j$ plane, which intersect transversely in an $n-i-j$ plane in $\mathbb{C}P^n$. Note $x_{i+j} = 0$ if $i + j \geq k$. Thus as a ring

$$H^\bullet(\mathbb{P}^{k-1}) \cong \mathbb{Z}[x]/x^k$$

where $x = x_1 \in H^2(\mathbb{P}^{k-1})$.

7/8/2023 Products of affine pavings

Last time we saw an affine paving of $\mathbb{P}^{k-1}$, leading to an easy computation $H^\bullet(\mathbb{P}^{k-1})$. There is an induced affine paving on $(\mathbb{P}^{k-1})^n$ where cells are given by all possible n-wise products of cells A_i, so the cells in the n-fold product are labelled by words $w = w_1 \cdots w_n$ for $w \in [k-1]$ leading to an additive basis of $H^\bullet((\mathbb{P}^{k-1})^n)$ $\{A_w\}$. In the $n = 1$ case, the cells A_i were represented by column vectors whose first i entries are 0 and whose $i+1$ entry is non-zero. The product cell A_w is thus represented by a matrix m_{ij} with $m_{ij} = 0$ when $i < w_j$ and $m_{w_j j} \neq 0$, so the set $\{A_w\}$ indexes the additive basis of $H^\bullet((\mathbb{P}^{k-1})^n)$. Again this leads to the presentation (think about how cup product/intersections work in this case)

$$H^\bullet((\mathbb{P}^{k-1})^n) = \mathbb{Z}[x_1, \ldots, x_n]/\langle x_1^k, \ldots, x_n^k\rangle$$

ie if we assign $A_w \equiv x_1^{w_1} \cdots x_n^{w_n}$, then the cup product is just multiplication of polynomials

$$[\bar{A}_w] \smile [\bar{A}_{w'}] = [\bar{A}_{w+w'}] \equiv x_1^{w_1+w_1'} \cdots x_n^{w_n+w_n'}$$

so that it suffices to generate using x_i's, which correspond to hyperplanes in the individual copies of $\mathbb{P}^{k-1}$.

7/10/2023 An affine paving on a product which is not a product of affine pavings

This is going to be a long one, but I am essentially copying from Pawlowski-Rhoades for my own understanding. I want to understand their constructions/calculations in this paper to adapt for my own nefarious purposes.

Here we seek to provide a new affine paving of $X := (\mathbb{P}^{k-1})^n$ different from the natural product affine paving described above. If $(\ell_1, \ldots, \ell_n) \in X$, let F_j denote the span of all the lines ℓ_1 to ℓ_j, and $\{j_1 < \cdots < j_m\}$ be the set of indices j such that $F_{j-1} \neq F_j$ (so we take only those lines which genuinely increase the dimension by 1. This means, for example if ℓ is a fixed line, the element $(\ell, \ldots, \ell) \in X$ corresponds only to the partial flag of type $(1, k)$). Thus $F_{j_1} \subset F_{j_2} \subset \cdots \subset F_{j_m}$ is a partial flag variety in $\mathbb{C}^k$ (If my understanding is correct it is a partial flag of type $(1, \ldots, 1)$ and only the length can vary. In particular it is a full flag in $\mathbb{C}^m$. I don't know if interpreting it in this way will have it corresponding to the same Schubert polynomial though). Therefore it belongs to a cell in the corresponding flag variety, whose cells are indexed by S_m. This element of S_m, written in one-line notation ie a word in $[k]^m$, will label the cell containing $\ell_\bullet$.

Definition: If m is a $k \times n$ matrix, define the <u>rank function</u>

$$r(m) : [k] \times [n] \to \mathbb{Z}_{\geq 0}$$

so that (i, j) is sent to the rank of the submatrix of m whose rows are indexed by $[i]$ and whose columns are indexed by $[j]$. The map

$$[k]^n \hookrightarrow Mat_{k \times n}$$

allows defining of the rank of a word $r(w)$. Any element $\ell_\bullet \in (\mathbb{P}^{k-1})^n$ can be viewed as an element of $\mathcal{U}_{n,k}/T$ where T is the diagonal torus and

$$\mathcal{U}_{n,k} = \{m \in Mat_{k \times n} \mid m \text{ has no 0 columns}\}$$

so that r descends to a map $r : \mathcal{U}_{n,k}/T \to Hom([k] \times [n], \mathbb{Z}_{\geq 0})$. Thus every element of $(\mathbb{P}^{k-1})^n$ has an associated "rank" and we may stratify $(\mathbb{P}^{k-1})^n$ into level sets of the rank function:

Definition: If $w \in [k]^n$, define $\Omega_w \subset (\mathbb{P}^{k-1})^n$ by

$$\Omega_w := \{\ell_\bullet \mid r(\ell_\bullet) = r(w)\}$$

It is clear that X is the disjoint union over all possible words. There is a lemma stating that it suffices to only vary w over convex words, that is words which do not repeat letters of the alphabet with some other letter in between. Thus the Ω's partition X but it remains to identify their cells. Remember a paving by affines is a collection of increasing subvarieties whose differences are disjoint unions of affine spaces. These affine spaces are the cells.

9/1/2023 The brane diagram of a quiver

I took a long break to work on a project, but I'm back now. Let Q be the quiver such that

$$\mathcal{N}(Q) = T^* Fl(\vec{v}, \mathbb{C}^n)$$

ie Q is the (if $\vec{v}$ has length ℓ)type A_ℓ quiver with framing $w = (0, \dots, 0, n)$. Then the corresponding brane diagram is

$$D = /v_1/v_2 \cdots /v_{\ell-1}/v_\ell \underbrace{\backslash v_\ell \backslash \cdots \backslash}_{n} v_\ell /$$

Ie the bow variety $\mathcal{C}(D)$ is the same variety.

9/2/2023 Equivariant cohomology of projective space

This was mostly told to me by Andrey Smirnov.

Let $X = \mathbb{P}^n$ and let $T = (\mathbb{C}^\times)^{n+1}$ act on X in the natural way. For now let's choose $n = 2$. There are homogeneous coordinates $[x_0 : x_1 : x_2]$, and thus 3 fixed points corresponding to placing a 1 in each spot and a 0 otherwise. Call these p_1, p_2, p_3 corresponding to the placement of 1. We construct a "moment graph"

$$\begin{tikzcd} & p_1 \arrow[dl, no head] \arrow[dr, no head] & \\ p_2 \arrow[rr, no head] & & p_3 \end{tikzcd}$$

depicting the fixed points and fixed curves containing pairs of fixed points. For example, there is a fixed curve containing p_1 and p_2 given by $[x_2 = 0]$, and so on. If we choose a "chamber" $z_1 \leq z_2 \leq z_3$ (I don't really know what this means), then in a neighborhood of p_2 intersecting the fixed curve also containing p_1, the action of T is by $(z_1, z_2, z_3) \cdot [x_0 : x_1 : 0] = [z_1 x_0 : z_2 x_1 : 0] = [\frac{z_1}{z_2} x_0 : x_1 : 0]$. Because $z_1 \leq z_2$, this means that for the points in this neighborhood intersected with the fixed curve, the magnitude of their first coordinate is decreasing as T is acting on it, thus p_2 is an attracting point in this "chamber". Thus we may label the arrow (and the rest in the same manner) as

$$\begin{tikzcd} & p_1 \arrow[dl] \arrow[dr] & \\ p_2 \arrow[rr] & & p_3 \end{tikzcd}$$

and we may denote the weight of the T action at p_2 in the direction of p_1 additively as $u_1 - u_2$, while at p_1 in the direction of p_2 it would be denoted additively as $u_2 - u_1$, and similarly for the other points.

Proposition[27]**:** *If X is equivariantly formal wrt a T action then there is a natural $H_T^*(pt)$-module isomorphism $H_T^*(X) = H^*(X) \otimes H_T^*(pt)$.*

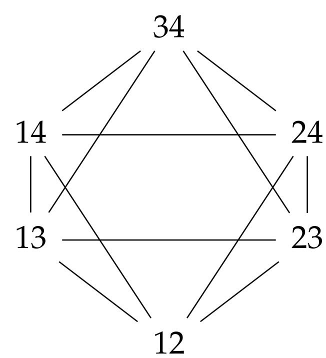
The ordinary cohomology of projective space is $\mathbb{Z}[h]/h^{n+1}$, and the equivariant cohomology of a point is determined by the classifying space (see 2/10/2023) $H_T^*(pt) = H^*(BT) = H^*((\mathbb{P}^\infty)^n) = \mathbb{Z}[u_1, \ldots, u_n]$.

What's the tensor product of these things?

9/3/2023 Equivariant cohomology of $Gr(2,4)$

This was told to me by both Andrey Smirnov and Luke Conners because I couldn't understand it the first time.

We want to construct a similar moment graph as in the example above. Here the fixed points of the (induced) T-action are given by the coordinate subspaces: $V_{ij} = \text{span}(e_i, e_j)$, of which there are $\binom{4}{2}$. Then the moment graph has vertices and fixed points given by (we abbreviate the names of the fixed points)



There is a fixed curve containing any two points providing they share an index. For example V_{123} contains the fixed points V_{13} and V_{23}, and is G-invariant, and indeed a 3 dimensional space can be represented by a path of planes, so V_{123} can be seen as a path in $Gr(2,4)$. This shows why it is important to share an index: If they didn't, the space which contained the two points would be 4-dimensional, which is just the whole space, and is no longer a path in $Gr(2,4)$ but a surface[28].

Edge labels?

The general moment graph for $Gr(k,n)$ is: Vertices are given by k-subsets, I, of $[n]$ and two vertices I_1, I_2 are joined by an edge iff $|I_1 \cap I_2| = (k-1)$. The edge is labelled by $e_i - e_j$, where i, j are the

[27] https://arxiv.org/pdf/math/0503369.pdf

[28] Does that mean there's a fixed surface in $Gr(2,4)$ containing 12 and 34? The answer is yes.

9/4/2023 Grassmann coordinates as a commutative superalgebra

Taken from "Mathematical Foundations of Supersymmetry" on arxiv.

Let $A = k[t_1, \ldots, t_p, \theta_1, \ldots, \theta_q]$ have 'odd" or "Grassmann" coordinates. We show it is a commutative superalgebra. The $\mathbb{Z}_2$ grading is given by considering those polynomials which contain no odd coordinates or those which contain an even number of odd coordinates (then minus signs will cancel). In symbols

$$A_0 = \left\{ f_0 + \sum_{|I| \text{ even}} f_I \theta_I \;\middle|\; I = \{i_1 < \cdots < i_r\} \right\}$$

Some even polynomials are

$$1, \quad t_1 t_2^3 t_3, \quad t_1 \theta_1 \theta_2, \quad \theta_1 \theta_2 \theta_3 \cdots \theta_{2n} + t_5^5, \quad t_7^2 + \theta_4 \theta_9 + t_3^4$$

and

$$A_1 = \left\{ \sum_{|J| \text{ odd}} f_J \theta_J \;\middle|\; J = \{j_1 < \cdots < j_s\} \right\}$$

some odd polynomials are

$$\theta_n, \quad t_1 \theta_1, \quad t_5 \theta_3 \theta_4 \theta_5, \quad t_2^3 t_6 t_{15}^3 \theta_9$$

It is easy to show that multiplication of polynomials is indeed a morphism of superalgebras $A \otimes A \to A$. With this grading, supercommutativity is granted because

$$\begin{aligned}
&\left(f_0 + \sum_{|I| \text{ even}} f_I \theta_I \right) \left(f_0' + \sum_{|I| \text{ even}} f_I' \theta_I' \right) \\
&= f_0 f_0' + f_0 \left(\sum_{|J|} f_J' \theta_J' \right) + \left(\sum_{|I|} f_I \theta_I \right) f_0' + \sum_{|I|,|J|} f_I \theta_I f_J' \theta_J' \\
&= f_0' f_0 + \left(\sum_{|J|} f_J' \theta_J' \right) f_0 + f_0' \left(\sum_{|I|} f_I \theta_I \right) + \sum_{|I|,|J|} f_J' \theta_J' f_I \theta_I
\end{aligned}$$

The simplication of the final term comes from the lemma

Lemma:

$$\theta_I \theta_J = (-1)^{|I||J|} \theta_J \theta_I$$

Proof: Begin by moving θ_{j_1} through θ_I. This yields

$$(-1)^{|I|} \theta_{j_1} \theta_I \theta_{J - j_1}$$

Continuing

$$(-1)^{|I| + |I|} \theta_{j_1} \theta_{j_2} \theta_I \theta_{J - j_1 - j_2}$$

Crucial observation is that each term of θ_J moved through θ_I picks up exactly $|I|$ minus signs, and you must do this $|J|$ times. Formalize with induction if desired to obtain the result.

$\square$

In fact the above lemma just finishes the full result for supercommutativity, because it shows that the term in question will only pick up a minus sign when $|I|, |J|$ are both odd, which means choosing both odd elements.

There are derivations $\partial_{t_i}, \partial_{\theta_j}$ acting as formal derivatives.

Proposition:

$$Der(A) = Span_A(\partial_{t_i}, \partial_{\theta_j})$$

Proof: If $f \in Der(A)$, then f vanishes on constants:

$$f(a) = f(a \cdot 1) = f(a) \cdot 1 + (-1)^{|f||1|} a \cdot f(1) = f(a) + af(1)$$

which is a contradiction unless $f(1) = 0$, and k-linearity implies f vanishes on constants. That seems like something a derivative would do!...

$\boxed{\text{IDK HOW TO FINISH THIS}}$

$\square$

Lemma: *In the setting above, $Der(A)$ is a Lie superalgebra with bracket given by graded commutator:*

$$[D_1, D_2] = D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1$$

Proof: A derivation is first and foremost a morphism of super vector spaces. As such, $Der(A)$ inherits a $\mathbb{Z}_2$ grading from $Hom_{SuperVect_k}(\varphi(A), \varphi(A))$ where φ is the forgetful functor $\varphi : SuperAlg_k \to SuperVect_k$. One can check that the bracket defined above satisfies the graded antisymmetry and graded Jacobi identities.

$\square$

As one can plainly see, we did not reference the structure of A at all in the above proof:

Proposition: *For any superalgebra A, $Der(A)$ is a Lie superalgebra with bracket defined as above.*

$\square$

9/6/2023 Nilpotent orbits

In part from the wiki page on transversality and the proposition proof from Tiger Cheng.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $X \in \mathfrak{g}$ nilpotent.

Proposition: *If $X \in \mathfrak{g}$ is nilpotent, then $g \cdot X$ is nilpotent for every $g \in G$ (adjoint action).*

Proof: Suppose x is nilpotent. For any $y \in \mathfrak{g}$, we have (use Ado theorem to consider $\mathfrak{g}$ and G as matrix Lie algebras/groups)

$$\begin{aligned} ad_{g\cdot x}(y) &= gxg^{-1}y - ygxg^{-1} \\ &= g(xg^{-1}yg - g^{-1}ygx)g^{-1} \\ &\equiv g(ad_x(g^{-1}yg))g^{-1} \\ \Rightarrow ad^2_{g\cdot x}(y) &= g(ad_x(g^{-1}g(ad_x(g^{-1}yg)g^{-1}g)g \\ &= g(ad_x(ad_x(g^{-1}yg)g \\ &= g(ad^2_x(g^{-1}yg))g \\ &\vdots \\ \Rightarrow ad^n_{g\cdot x}(y) &= g(ad^n_x(g^{-1}yg))g \\ &\Rightarrow ad_{g\cdot x} \text{ is nilpotent} \end{aligned}$$

□

Example: $\mathfrak{sl}_2(\mathbb{C})$ is generated by e, f, h.

$$ad_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}$$

which is nilpotent. For example its characteristic polynomial is

$$c_f(\lambda) = det(f - \lambda I) = -\lambda^3$$

For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$, we have

$$g \cdot f = gfg^{-1} = \begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix}$$

so that we can directly compute

$$ad_{g\cdot f} = \begin{pmatrix} 2bd & 0 & 2b^2 \\ 0 & -2bd & 2d^2 \\ -d^2 & -b^2 & 0 \end{pmatrix}$$

which is also nilpotent. In fact it has the same characteristic polynomial.

Bracketing an elt of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ can yield h or e, so $\mathfrak{g} \cong [\mathfrak{g}, e] \oplus \mathfrak{g}_f$.

9/8/2023 Combinatorial Reciprocity

I learned this example from Richard Rimanyi.

Consider size 5 (could be replaced with any positive integer) subsets of $[n]$. The number of such is represented by n choose 5. Thought of as a polynomial in n,

$$P(n) = \frac{n^5}{120} - \frac{n^4}{12} + \frac{7n^3}{24} - \frac{5n^2}{12} + \frac{n}{5}$$

This is a degree 5 polynomial with 5 zeros located at 0,1,2,3,4 because of course there are no size 5 subsets. This means that as $x \to -\infty$, this polynomial has no more zeros. Combinatorial reciprocity conjectures (roughly) that the $x < 0$ regime corresponds (after absolute value) to solving some combinatorial problem which is a "friend" to the one we started with, corresponding to the $x > 0$ regime: For $x > 0$, $P(n)$ is the number of 5-subsets of $[n]$. For $n < 0$, $|P(-n)|$ is the number of 5-subsets of $[n]$ allowing repetition. Thus these two combinatorial problems are related by combinatorial reciprocity.

9/13/2023 Stable envelopes for $T^*\mathbb{P}^1$

I learned this in Andrey Smirnov's stable envelopes seminar.

A stable envelope is a map of $H^*_T(pt)$-modules:

$$Stab : H^*_T(X^T) \to H^*_T(X)$$

satisfying some conditions, where X is acted on by some algebraic torus T. In the case that X has finitely many T-fixed points,

$$H^*_T(X^T) \cong \bigoplus H^*_T(pt) \cong \bigoplus \mathbb{Q}[u_1, \ldots, u_{dim\ X}]$$

So if the fixed point set is $\{p_i\}$, then we can identify $[p_i] = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in H^*_T(X^T)$, where 1 is in the ith position. Then we want an expression for $Stab(p_i) \in H^*_T(X)$. There are also restriction maps for each fixed point $H^*_T(X) \to H^*_T(pt)$, and we may send the stable envelope through these maps

$$Stab(p_i) \mapsto Stab(p_i)|_{p_j}$$

Yielding a matrix that describes all stable envelopes for X, since it suffices to know an equivariant cohomology class at every fixed point to determine the class.

Let $X = T^*\mathbb{P}^1$, which is acted on by a torus $T = (\mathbb{C}^\times)^2 \times \mathbb{C}^\times_\hbar$. There are 2 fixed points, $p_1 = [1 : 0], p_2 = [0 : 1]$, and there is a fixed curve whose closure contains these two points, $\gamma(t) = [1 : t]$, contained entirely in $\mathbb{P}^1$ (zero section). The closure of this curve is compact, but at each fixed point there is also an unbounded direction (in the cotangent direction). If we choose a cocharacter $\sigma : \mathbb{C}^\times \to A = (\mathbb{C}^\times)^2$, $\sigma(z) = (z, z^2)$, then we can specify directions of flow lines. For example, for any point in the image of γ, we have

$$\begin{aligned} \lim_{z\to 0} \sigma(z) \cdot [1 : t] = \lim_{z\to 0} (z, z^2) \cdot [1 : t] &= \lim_{z\to 0} [z : z^2] \\ &= \lim_{z\to 0} [1 : z] \\ &= [1 : 0] = p_1 \end{aligned}$$

Which shows all points in the (open) image of γ are attracted to p_1 with this choice of cocharacter. One should note that we also made a choice of curve γ whose closure contains p_1, p_2. We could have also chosen $\gamma = [t : 1]$, but one can check (by just calculating the limit again) that the direction of the arrow didn't depend on this choice. The tangent space at each fixed point breaks up into weight spaces of the T action. This leads to the moment graph

$$\begin{array}{c} \uparrow \\ p_1 \\ \uparrow \\ p_2 \\ \uparrow \end{array}$$

The arrows on the unbounded directions are determined because the unbounded directions are isomorphic to the curve γ as T-reps. Reading from top to bottom, the weights are $u_1 - u_2 + \hbar, u_2 - u_1, u_1 - u_2, u_2 - u_1 + \hbar$. The weight of the unbounded direction is the reciprocal of the weight in the bounded direction because the induced map on contangent bundle must be a symplectomorphism. The conditions for the stable envelope maps are abbreviated as i) support, ii) normalization, iii) "smallness", iv) eqCoh. We seek to fill out the 2 x 2 matrix

$$Stab(p_i)|_{p_j} = \begin{pmatrix} \_ & \_ \\ \_ & \_ \end{pmatrix}$$

(here the ith row is $Stab(p_i)$). We know by i) that the matrix is upper triangular, so we can put a 0 in the lower left. The diagonal elements are determined by condition ii), normalization. It says that

$$Stab(p_i)|_{p_i} = e(N_-)|_{p_i}$$

where $e(N_-)|_{p_i}$ is the Euler class of the repelling bundle restricted to the fixed point p_i. In the case of p_1, the repelling bundle is the cotangent fiber, so its Euler class is $u_1 - u_2 + \hbar$:

$$Stab(p_1)|_{p_1} = u_1 - u_2 + \hbar$$

This is the upper left entry. The lower right entry is the restriction

$$Stab(p_2)|_{p_2} = e(N_-)|_{p_2} = u_1 - u_2$$

By iii), the upper right entry must satisfy the condition that its degree in the u_i parameters must be $< \frac{1}{2}dim(X) = 1$, so it must be constant in u_i, and it must be linear in $\hbar$. So it must have the form $a\hbar + b$. To apply condition iv), recall the equivariant cohomology ring for $T^*\mathbb{P}^n$:

$$H_T^\bullet(T^*\mathbb{P}^n) \cong \{(f_1, f_2) \in \mathbb{C}[u_1, u_2]^2 \mid u_1 - u_2 \mid f_1 - f_2\}$$

In other words,

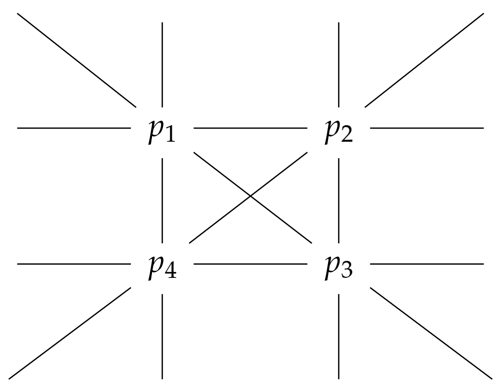
$$\big(Stab(p_1)|_{p_2}\big)|_{u_1=u_2} = \big(Stab(p_1)|_{p_1}\big)|_{u_1=u_2}$$
$$\Rightarrow a = 1, b = 0$$

So the stable envelope for $T^*\mathbb{P}^1$ is

$$Stab(p_i)|_{p_j} = \begin{pmatrix} u_1 - u_2 + \hbar & \hbar \\ 0 & u_1 - u_2 \end{pmatrix}$$

9/15/2023 Stable envelopes for $T^*\mathbb{P}^3$

Let's redo the above example for $n = 3$. The moment graph is



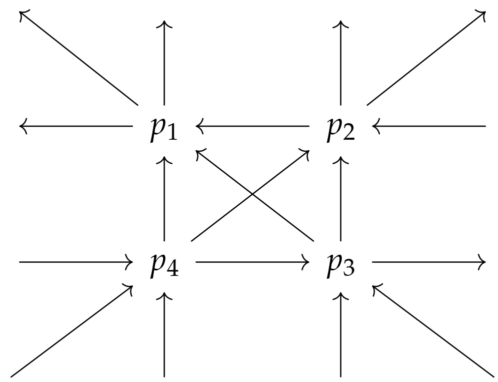
Signifying that at every fixed point p_i, the tangent space is 6-dimensional, and we already know three of the directions are towards other fixed points. As before, this has the weight $u_1 - u_2$ near p_2 and weight $u_2 - u_1$ near p_1. This pattern can be used to write down the weights in all the bounded directions (within $\mathbb{P}^2$), and there are three unbounded (cotangent) directions at every fixed point, with weights that are the reciprocals of the bounded direction weights. To calculate the directions, we again choose a generic cocharacter $\sigma : \mathbb{C}^\times \to A = (\mathbb{C}^\times)^4$, $\sigma(z) = (z, z^2, z^3, z^4)$. The curve between, say, p_1 and p_2 is the same curve (as a T-rep) as in the example above, so it points to p_1. For p_2, p_3, the curve is $\gamma = [0 : 1 : t]$.

$$\lim_{z\to 0} \sigma(z) \cdot [0 : 1 : t] = \lim_{z\to 0} [0 : z^2 : z^3 t] = \lim_{z\to 0} [0 : 1 : zt] = [0 : 1 : 0] = p_2$$

and the curve between p_1, p_3 is $\gamma = [1 : 0 : t]$, so

$$\lim_{z\to 0} \sigma(z) \cdot [1 : 0 : t] = \lim_{z\to 0} [z : 0 : z^3 t] = \lim_{z\to 0} [1 : 0 : z^2 t] = p_1$$

and so on. The general rule is clear: Under this cocharacter, the direction of the arrows is determined by the opposite total order on integers. So the moment graph with directions looks like (again we can reason about inverses of weights and comparing isomorphic T-reps to label the unbounded directions)



The weight space near p_i going towards p_j is given by $u_j - u_i$. Now we can begin to write down the matrix elements of the stable envelope. The diagonal elements are again determined by ii (normalization)

$$\begin{gathered}
Stab(p_1)|p_1 = (u_1 - u_2 + \hbar)(u_1 - u_3 + \hbar)(u_1 - u_4 + \hbar) \\
Stab(p_2)|_{p_2} = (u_1 - u_2)(u_2 - u_4 + \hbar)(u_2 - u_3 + \hbar) \\
Stab(p_3)|_{p_3} = (u_2 - u_3)(u_3 - u_1)(u_3 - u_4 + \hbar) \\
Stab(p_4)|_{p_4} = (u_1 - u_4)(u_2 - u_4)(u_3 - u_4)
\end{gathered}$$

Because the moment graph is connected, the support condition implies that the stable envelope matrix is upper triangular. The "smallness' condition implies that all strictly upper triangular terms are at most quadratic in the u_i's. Unfortunately the matrix is too large to put on the screen even in this small example. It's almost too large to fit on a piece of paper. The non-determined element in row 3, $Stab(p_3)|_{p_4}$, satisfies

$$(Stab(p_3)|_{p_4})|_{u_1=u_4} = (Stab(p_3)|_{p_1})|_{u_1=u_4}$$

$$\begin{gathered}
(Stab(p_3)|_{p_4})|_{u_1=u_4} = 0 \\
\Rightarrow (Stab(p_3)|_{p_4}) = n(u_1 - u_4)
\end{gathered}$$

for n some degree 1 polynomial. By the same reason applied to $u_2 - u_4$,

$$Stab(p_3)|_{p_4} = n(u_1 - u_4)(u_2 - u_4)$$

where n is now a number. To determine that number, we compare with 3:

$$\begin{gathered}
(Stab(p_3)|_{p_4})|_{u_3=u_4} = (Stab(p_3)|_{p_3})|_{u_3=u_4} \\
n(u_1 - u_4)(u_2 - u_4)|_{u_3=u_4} = \hbar(u_2 - u_3)(u_3 - u_1)|_{u_3=u_4} \\
\Rightarrow n = -\hbar \\
\Rightarrow Stab(p_3)|_{p_4} = -\hbar(u_1 - u_4)(u_2 - u_4)
\end{gathered}$$

To determine $Stab(p_2)|_{p_3}$, we do the same. It must have a factor of $f(u_1 - u_3)$ by comparing it to 1. Comparing to 2 we have

$$\begin{gathered} f(u_1 - u_3)|_{u_2=u_3} = (u_1 - u_2)(u_2 - u_4 + \hbar)\hbar|_{u_2=u_3} \\ f|_{u_2=u_3} = \hbar(u_2 - u_4 + \hbar)|_{u_2=u_3} \\ \Rightarrow f = \hbar(u_{2,3} - u_4 + \hbar) \\ \Rightarrow Stab(p_2)|_{p_3} = \hbar(u_1 - u_3)(u_{2,3} - u_4 + \hbar) \end{gathered}$$

where the comma indicates there are two possibilities. Similarly

$$Stab(p_2)|_{p_4} = \hbar(u_1 - u_4)(u_{2,4} - u_3 + \hbar)$$

Comparing 3 and 4:

$$\hbar(u_1 - u_3)(u_{2,3} - u_4 + \hbar)|_{u_3=u_4} = \hbar(u_1 - u_4)(u_{2,4} - u_3 + \hbar)|_{u_3=u_4}$$

$$\Rightarrow u_{2,3} = u_{2,4} = u_2$$

So

$$\begin{gathered} Stab(p_2)|_{p_3} = \hbar(u_1 - u_3)(u_2 - u_4 + \hbar) \\ Stab(p_2)|_{p_3} = \hbar(u_1 - u_4)(u_2 - u_3 + \hbar) \end{gathered}$$

For the first row, we have, just by comparing with 1,

$$\begin{aligned} Stab(p_1)|_{p_2} &= \hbar(u_{1,2} - u_3 + \hbar)(u_{1,2} - u_4 + \hbar) \\ Stab(p_1)|_{p_3} &= \hbar(u_{1,3} - u_2 + \hbar)(u_{1,3} - u_4 + \hbar) \\ Stab(p_1)|_{p_4} &= \hbar(u_{1,4} - u_2 + \hbar)(u_{1,4} - u_3 + \hbar) \end{aligned}$$

Within a single stable envelope, there are two independent choices to be made. Comparing 2 and 3:

$$\hbar(u_{1,2} - u_3 + \hbar)(u_{1,2} - u_4 + \hbar)|_{u_2=u_3} = \hbar(u_{1,3} - u_2 + \hbar)(u_{1,3} - u_4 + \hbar)|_{u_2=u_3}$$

For example, if the first choice on the LHS is u_1, then the first choice on RHS must be u_1, so that the terms containing u_2, u_3 agree. In this case, the second choice on LHS must agree with the second choice on RHS, so there are possibilities if the first choice on LHS is u_1: Reading left to right, the choices can be u_1, u_1, u_1, u_1 or u_1, u_2, u_1, u_3. Similarly if the first choice on LHS is u_2, then the first choice on RHS must be u_3, and there are two possibilities following this: u_2, u_1, u_3, u_1 or u_2, u_2, u_3, u_3. So we reduced the 8 possibilities (if they were all independent) to 4.

Comparing 3 and 4:

$$\hbar(u_{1,3} - u_2 + \hbar)(u_{1,3} - u_4 + \hbar)|_{u_3=u_4} = \hbar(u_{1,4} - u_2 + \hbar)(u_{1,4} - u_3 + \hbar)|_{u_3=u_4}$$

This leads to the four possibilities: $u_1, u_1, u_1, u_1, — u_1, u_3, u_1, u_4, — u_3, u_1, u_4, u_1, — u_3, u_3, u_4, u_4$. Comparing 4 and 2

$$\hbar(u_{1,4} - u_2 + \hbar)(u_{1,4} - u_3 + \hbar)|_{u_2=u_4} = \hbar(u_{1,2} - u_3 + \hbar)(u_{1,2} - u_4 + \hbar)|_{u_2=u_4}$$

Leading to the possibilities: u_1, u_1, u_1, u_1 — u_1, u_4, u_2, u_1 — u_4, u_1, u_1, u_2 — u_4, u_4, u_2, u_2.

One can check that the choices for $Stab(p_1)|_{p_1}$ determine the remaining choices. For example if we choose u_1, u_2 for $Stab(p_1)|_{p_1}$, then we must choose u_1, u_3 for $Stab(p_1)|_{p_2}$ and we must choose u_1, u_4 for $Stab(p_1)|_{p_4}$. So there are 4 possibilities left. Note that the possibilities beginning with u_1, u_2 and u_2, u_1 are not consistent, leaving only u_1, u_1 and u_2, u_2. So far I have not been able to eliminate either of these. Hmmm...

Editor's note: This example was finished later but never written up within the EaD cinematic universe.

9/18/2023 A separated brane diagram for $T^*\mathbb{P}^n$

The quiver presentation for $T^*\mathbb{P}^n$ is the framed quiver of type A_1 with framing $v = 1$ and $w = n$. The associated brane diagram is

$$/1\underbrace{\backslash 1 \backslash 1 \cdots 1 \backslash}_{w \text{ times}} 1/$$

To separate this brane diagram, we have to move the NS5 from the right to left w times. After the first HW move,

$$/1\backslash 1\backslash 1 \cdots 1\backslash 1\backslash 1/1\backslash$$

After the second

$$/1\backslash 1\backslash 1 \cdots 1\backslash 1\backslash 1/2\backslash 1\backslash$$

$$/1\backslash 1\backslash 1 \cdots 1\backslash 1/3\backslash 2\backslash 1\backslash$$

After w moves,

$$/1/w\backslash w-1\backslash w-2 \cdots 2\backslash 1\backslash$$

The above is a separated brane diagram whose bow variety is isomorphic to $T^*\mathbb{P}^n$.

9/24/2023 A separated brane diagram for $T^*\mathcal{F}_{\vec{v}}$

To fix notation, $\mathcal{F}_{\vec{v}}$ for $\vec{v}$ length ℓ is the flag variety consisting of flags

$$F_\bullet = \left(F_0 = 0 \subset F_1 \subset \cdots \subset F_\ell = \mathbb{C}^{|\vec{v}|}\right)$$

(so really there is only a choice of $\ell - 1$ subspaces, since the final choice is just the whole space), where dim $F_{i+1}/F_i = \vec{v}_i$.

Example: The quiver of the flag $T^*\mathcal{F}_{111}$ is the quiver A_2 with weights 1,2 and framed at the end with weight 3. The associated brane diagram is

$$/1/2\backslash 2\backslash 2\backslash 2/$$

$$/1/2\backslash 2\backslash 2/1\backslash$$

$$/1/2\backslash 2/2\backslash 1\backslash$$

$$/1/2/3\backslash 2\backslash 1\backslash$$

The general form for a flag $T^*\mathcal{F}_{\vec{v}}$ is

$$/\vec{v}_1/\vec{v}_1+\vec{v}_2/\cdots(\vec{v}_1+\cdots+\vec{v}_{\ell-1})/|\vec{v}|\backslash|\vec{v}|-1\backslash|\vec{v}|-2\backslash\cdots 2\backslash 1\backslash$$