Homological Algebra

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CHAPTER 1

Homological Algebra/Derived Functors/Ext and Tor

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Title: An introduction to Homological Algebra and Derived Functors

Abstract: Homological algebra can be loosely said to be the algebra needed to rigorously construct and compute (co)homology. We encountered small versions of it in our course with categorical language and diagram chases. In commutative algebra, there are important derived functors called Ext and Tor. In this presentation, we will define the notion of a derived functor and define Ext and Tor as special cases of this. Then we will present some computations and results about these particular functors/groups. The main source is Weibel Introduction to Homological Algebra.

Throughout this talk, the definitions we use will apply to an arbitrary abelian category, where objects are typically defined via a universal property such as (co)kernel and projective/injective objects. However, one can feel free to think of $\mathcal{A} = \mathcal{B} = R - Mod$ as a typical abelian category[1] with concrete definitions.

Definition: Given an object $A \in \mathcal{A}$, a projective resolution of A is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

where all the objects P_i are projective objects. If our category $\mathcal{A}$ has enough projectives, then every object admits a projective resolution. Similarly one can define an injective resolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \dots$$

Lemma: *The data of a projective resolution of A as described above is equivalent to the data of a chain complex $P_\bullet$ of projective objects satisfying $H_i(P_\bullet) = 0$ for $i > 0$ and $H_0(P_\bullet) = A$. An analogous statement holds for injective resolutions. This transition is sometimes referred to as "chopping off" or "deleting" the A term.*

Proof of Lemma: $\Rightarrow$: If $P_\bullet$ is as above, then we may consider the chain complex obtained by chopping off the A term, $P'_\bullet$:

$$\cdots \to P_2 \to P_1 \to P_0 \to 0$$

clearly by the exactness of $P_\bullet$, we have $H_i(P'_\bullet) = 0$ for $i > 0$. However the cohomology at 0 is now given by

$$H_0(P'_\bullet) = \frac{ker\ d_0}{im\ d_1} = \frac{P_0}{im\ d_1}$$

and what is $im\ d_1$? By exactness at index 0 of $P_\bullet$, it is equal to, denoting the map $P_0 \to A$ as ϵ, $im\ d_1 = ker\ \epsilon$, so

$$\frac{P_0}{im\ d_1} = \frac{P_0}{ker\ \epsilon}$$

But because ϵ is surjective, again by exactness of $P_\bullet$,

$$\frac{P_0}{ker\ \epsilon} \cong A$$

$\Leftarrow$: If we have a chain complex $P'_\bullet$

$$\cdots \to P_2 \to P_1 \to P_0 \to 0$$

satisfying $H_i(P'_\bullet) = 0$ for $i > 0$ and $H_0(P'_\bullet) = ker\ d_0/im\ d_1 = P_0/im\ d_1 = A$, then we may consider the new complex

$$\cdots \to P_2 \to P_1 \to P_0 \to P_0/im\ d_1 \to 0$$

[1]In a sense, all (small) abelian categories are of this form, search Freyd-Mitchell Embedding.

where $P_0 \to P_0/im\ d_1$ is the "quotient map"[2]. One can easily check that this sequence is now exact.

□

Definition: Given a left exact functor $F : \mathcal{A} \to \mathcal{B}$, if $\mathcal{A}$ has enough injectives, we may define the right derived functor $R^iF : \mathcal{A} \to \mathcal{B}$ as:

$$R^iF(A) := H^i(F(I^\bullet))$$

where $I^\bullet$ is an injective resolution of the object A with deleted A term. Dually, given a right exact functor $F : \mathcal{A} \to \mathcal{B}$, we may define the left derived functors $L_iF(A) : \mathcal{A} \to \mathcal{B}$ as

$$L_iF(A) := H_i(F(P_\bullet))$$

where $P_\bullet$ is a projective resolution of A with deleted A term.

Lemma: The functors L_iF and R^iF are well-defined (in the sense of natural isomorphism) under choice of projective or injective resolution.

□

I think it's easier to process this definition as a sequence of instructions: Say you have a right exact functor $F : \mathcal{A} \to \mathcal{B}$ and you want to compute left derived functors $L_iF(A)$. Then you take your object A and first find a projective resolution of it:

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

Then chop off A to get a chain complex:

$$\cdots \to P_2 \to P_1 \to P_0 \to 0$$

Then apply F to this complex:

$$\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0$$

and you take homology of this chain complex to compute the ith left derived functor applied to A.

Proposition: $L_0F(A) \cong F(A)$ *(similarly* $R^0F(A) \cong F(A)$*).*

Proof: Assuming $\mathcal{A}$ has enough injectives, find a projective resolution of A,

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} A \to 0$$

chop off A:

$$\cdots \to P_2 \to P_1 \to P_0 \to 0$$

[2] Be careful about what this means if not working in $R - Mod$.

Applying F yields the complex

$$\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0$$

and we want to compute 0th homology:

$$L_0F(A) = H_0(F(P_\bullet)) = ker\ F(d_0)/im\ F(d_1) = F(P_0)/im\ F(d_1)$$

Now we can split $P_\bullet$ into a bunch of short exact sequences, as we discussed in the class (I think this was assigned as a homework problem as well). In particular, $0 \to P_1 \to P_0 \xrightarrow{\epsilon} A \to 0$ is exact, so

$$F(P_1) \to F(P_0) \xrightarrow{F(\epsilon)} F(A) \to 0$$

is exact, thus $ker\ F(\epsilon) = im\ F(d_1)$. So the homology is

$$\frac{F(P_0)}{im\ F(d_1)} = \frac{F(P_0)}{ker\ F(\epsilon)}$$

but because $F(\epsilon)$ is surjective,

$$\cong F(A)$$

as desired.

□

One reason that construction and consideration of these derived functors is desirable is that they measure how much a left-exact or right-exact functor fails to be exact. This is more or less evident from the definition. But further, they turn short exact sequences into long exact sequences via the snake lemma: For example given a right exact functor with left derived functors L_iF and SES $0 \to A \to B \to C \to 0$, there is a long exact sequence

$$\cdots \to L_2F(C) \to L_1F(A) \to L_1F(B) \to L_1F(C) \to L_0F(A) \to L_0F(B) \to L_0F(C) \to 0$$

which, in light of the above proposition, is isomorphic to

$$\cdots \to L_2F(C) \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0$$

So the derived functors can be seen as "completing" the one-sided short exact sequence provided by a one-sided exact functor into a full long exact sequence and of course there is an analogous statement for right derived functors.

Definition: As we saw in the course, given a ring R, the functor $Hom_R(A, -)$ from $R-Mod$ to $R-Mod$[3] is a covariant, left-exact functor. Thus we may consider the right derived functors $R^iHom_R(A, -)$. We define the Ext groups (in this case, modules):

$$Ext^i_R(A)(B) = R^iHom_R(A, -)(B)$$

[3]Indeed the Hom sets are R-modules when R is commutative, as it always is for us. If this is not the case, then it is just an abelian group.

which are computed by resolving B with an injective resolution.
However, $Hom_R(-,B)$ is a contravariant, left-exact functor. Thus taking a projective resolution of A can also be used to define Ext groups, since you will end up with a cochain complex in the end again. It is a theorem that these two constructions of Ext are well defined and agree with each other, so one may choose whichever is convenient for their setting:

$$Ext^i_R(A)(B) = R^i Hom_R(A,-)(B) = R^i Hom_R(-,B)(A)$$

Example: Let's do the simplest possible example with $R=\mathbb{Z}$. Then $R-Mod \cong AbGrp$. Say we want to compute $Ext^i_{\mathbb{Z}}(\mathbb{Z})(\mathbb{Z})$. First we must resolve $\mathbb{Z}$ with a projective resolution. But actually, $\mathbb{Z}$ itself is projective, using this fact

Lemma: *Any free R-Module is also projective.*

$\square$

Thus the resolution

$$0 \to \mathbb{Z} \xrightarrow{Id} \mathbb{Z} \to 0$$

is a valid projective resolution. Then we chop off $\mathbb{Z}$:

$$0 \to \mathbb{Z} \to 0$$

Apply $Hom_{\mathbb{Z}}(-,\mathbb{Z})$:

$$0 \to Hom_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$$

And we need to compute the cohomology of this sequence, which is clearly 0 except at 0, where it is $Hom_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$, which we already knew would happen. Let's dissect what facts we actually even used here. First, we only needed that $\mathbb{Z}$ was a free $\mathbb{Z}$ module to construct this projective resolution. So in fact, this would apply if we considered $Ext^i_{\mathbb{Z}}(\bigoplus_I \mathbb{Z})(\mathbb{Z})$. Of course, the second argument played no role either, so $Ext^i_{\mathbb{Z}}(\bigoplus_I \mathbb{Z})(G)$ is 0 unless $i=0$, when it is $Hom_{\mathbb{Z}}(\bigoplus_I \mathbb{Z}, G)$.

Now if we leave the world of free groups in the final argument, the projective resolution may no longer just contain two non-zero terms. However, for any abelian group G, there is a three term free (and thus projective) resolution constructed as follows: Consider

$$\bigoplus_{x\in G} \mathbb{Z}_x \xrightarrow{\epsilon} G \to 0$$

where $\epsilon_x : x\mathbb{Z} \to G$ sends $z \mapsto xz \in G$, and we extend by linearity wrt the direct sum. $\bigoplus_{x\in G} \mathbb{Z}_x$ is a free group, thus $ker\ \epsilon$ is a free subgroup: $ker\ \epsilon \cong \bigoplus_J \mathbb{Z}$. Because these ranks may be infinite, the proof is trickier than the standard one in finite rank from algebraic topology, but nevertheless the claim is true, but of course we do not prove it here. Thus there is an exact sequence

$$0 \to \bigoplus_{J} \mathbb{Z} \to \bigoplus_{x \in G} \mathbb{Z}_x \to G \to 0$$

this forms a projective resolution of G. So we chop off G then apply $Hom_{\mathbb{Z}}(-, G')$:

$$0 \to Hom_{\mathbb{Z}}\left(\bigoplus_{x \in G} \mathbb{Z}_x, G'\right) \to Hom_{\mathbb{Z}}\left(\bigoplus_{J} \mathbb{Z}, G'\right)$$

which trivially extends by 0 all the way to the right[4]. Thus the resulting cohomology (and thus Ext groups) could now possibly be non-zero in degree 1, but is definitely 0 for all $n \geq 2$. I'm not sure if we can say in general what this will be equal to. In Logan's presentation, she discussed the result in the case of finitely generated abelian groups, with a very nice proof strategy.

So

$$Ext^n_{\mathbb{Z}}(G)(G') = 0, \quad n \geq 2$$

But what properties did we even need to show this? All we used is that the subgroup of a free group is free. Groups arose in the first place because we chose $R = \mathbb{Z}$, so the module-theoretic statement is submodules of free $\mathbb{Z}$-modules are free. But this holds in higher generality as well: Any submodule of a free module over R is free as long as R is a PID. Thus we have shown

Theorem (Vanishing of Higher Ext on PIDs): *If R is a PID, then for any R-Modules M, N,*

$$Ext^n_R(M)(N) = 0 \quad n \geq 2$$

[4]Note we can extend to the right by 0 because this is only claimed to be a chain complex, not an exact sequence.