$D^b Coh(X)$

Reese Lance

Fall 2022

Chapter 1

Derived Categories of Coherent Sheaves

Created Fall 2022

Abstract: This talk will have 3 sections: the category of coherent sheaves, the derived category, and understanding what happens when you combine the two, and why it may appear for someone who is interested in AGT. There will not be a complete construction of the derived category, as this alone would take more than a lecture to do properly. I will remind people about the basics of sheaves, but this will go fast, so it will be helpful to have seen them before, and I'll assume some familiarity with basics of category theory. I'm presenting a streamlined path to $D^bCoh(X)$, so we will try not to introduce anything unnecessary. As a result, this talk will not be as robust as possible, nor will it contain maximum generality, but everything in life is a sacrifice. I will skip over many proofs in this document, and many more proofs in the talk itself. However, for almost every claim, but certainly not every, I myself have gone through and proved it to ensure I understand how it works. Therefore if you're interested to know how to justify any of these claims, I recommend first trying it yourself (because if I did it then you can too) and if you can't get it, then you can talk to me and we can work it out together. Further, there is one glaring omission from this talk, namely the role of derived functors. Some of the omitted proofs in the third section rely heavily on derived functors, but I've skipped over this story for the sake of brevity. If this lecture ever gets a part II, it will be mainly about how to think about and use derived functors on this derived category. The main reference is Huybrechts: Fourier-Mukai transforms in Algebraic Geometry, though I used the internet/google as much as I used this book.

Sheaves: A <u>presheaf</u> $\mathcal{F}$ on a topological space X, say of abelian groups, is an assignment, for every open set U, an abelian group, denoted $\mathcal{F}(U)$, and for every $U \subset V$, a "restriction morphism" $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$. We often abbreviate the notation of the restriction morphism as if it were a literal restriction, while remembering all along that it is not. Such that

$\rho_{UU} = Id_{\mathcal{F}(U)}$, and $\rho_{WV}\rho_{VU} = \rho_{WU}$.

If further, we have the two conditions
i) Locality: If U is open, and covered by $\{U_i\}$ open, $s, t \in \mathcal{F}(U)$ such that $s|_{U_i} = t|_{U_i}$ for all i implies $s = t$.
ii) Gluing: If U is open, and covered by $\{U_i\}$ open, if we have a collection $s_i \in \mathcal{F}(U_i)$ satisfying

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall\ i, j$$

then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.
Then $\mathcal{F}$ is called a sheaf.

The presheaf data is conveniently packaged by saying that a presheaf is a contravariant functor $\mathcal{F} : Op(X) \to AbGrp$, and a sheaf is such a functor satisfying conditions i) and ii). Wherever possible, we will adopt categorical language and techniques to make our life easier.

Example (Structure sheaf): If our topological space X is a projective variety, then we can define the sheaf of holomoprhic functions: $f : X \to \mathbb{C}$ is holomorphic if for every $p \in X$, there exists a neighborhood $U \ni x$ such that $f|_U = \frac{g}{h}$, where g, h are homogeneous polynomials of the same homogeneous degree, and $h(p) \neq 0$. It is readily verified that this is a sheaf, which we denote as $\mathcal{O}_X$, and often refer to as the structure sheaf.

Given a topological space, X, there is a category of sheaves over X, denoted $Shv(X)$, whose objects consist of sheaves on X and whose morphisms are natural transformations between those sheaves.

The title contains an adjective in front of sheaves, so clearly we need to study some specific kinds of sheaves.

Notice that the structure sheaf, $\mathcal{O}_X$, evaluated on an open set U, is more than just an abelian group with respect to addition of functions, but is also a ring with respect to multiplication. Thus it makes sense to consider modules over $\mathcal{O}_X(U)$.

Definition: More generally, a topological space X is called a ringed space if it is equipped with a sheaf of rings, denoted $\mathcal{O}_X$. The motivating example is what we have described above, and so one should always think in terms of the structure sheaf. From here, we will use the language of schemes, but one should feel free to just replace the word scheme with projective variety throughout, or even $\mathbb{P}^n$, as this is one such example. Or maybe just think of a scheme as a certain ringed space which enjoys some nice properties.

Definition: A sheaf of $\mathcal{O}_X$-modules on a scheme $(X, \mathcal{O}_X)$, is a sheaf $\mathcal{F} \in Shv(X)$ such that $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module. Note that $\mathcal{F}$ is valued in abelian groups per our convention, so it makes sense to ask if it is a module over a ring.

Example: Of course, the sheaf $\mathcal{O}_X$ itself is a sheaf of $\mathcal{O}_X$-modules.

We may also perform a direct sum operation on sheaves, which is defined exactly as one may imagine.

Example:
$$\mathcal{O}_X^{\oplus n} := \mathcal{O}_X \oplus \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X$$
is a sheaf of $\mathcal{O}_X$-modules. This is called the free $\mathcal{O}_X$-module of rank n.

Definition: A sheaf $\mathcal{F} \in Shv(X)$ is called locally free if, for every $x \in X$, there is an open $U \ni x$ such that $\mathcal{F}|_U \cong \bigoplus_I \mathcal{O}_X|_U$. We will denote $\mathcal{O}_X|_U$ as simply $\mathcal{O}_U$. We do not require I to be finite, but if I is finite and its cardinality does not depend on the point or neighborhood, then we say $\mathcal{F}$ is locally free of rank $|I|$.

This example is important because it allows us to have a pedestrian example:

Example: If X is a scheme, then locally free sheaves of rank n correspond to vector bundles of rank n over X.

From here, for safety, I will assume the scheme is Noetherian from here on out, which should alleviate most subtleties. For instance, projective varieties correspond to Noetherian schemes, so we can continue bringing that example along with us. One reason this condition is desirable is that it implies that the structure sheaf, $\mathcal{O}_X$, is coherent over itself.

Definition: A quasicoherent sheaf of $\mathcal{O}_X$-modules on a scheme $(X, \mathcal{O}_X)$ is a sheaf on X such that for every point $x \in X$, there is a neighborhood $U \ni x$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map
$$\bigoplus_J \mathcal{O}_U \to \bigoplus_I \mathcal{O}_U$$
i.e. there is an exact sequence
$$\bigoplus_J \mathcal{O}_U \to \bigoplus_I \mathcal{O}_U \to \mathcal{F}|_U \to 0$$

Definition: If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, we say $\mathcal{F}$ is coherent if
i) $\mathcal{F}$ is of finite type, i.e. for all $x \in X$, there exists $U \ni x$ open and an integer r such that there is a surjective sheaf morphism $\mathcal{O}_X^{\oplus r}|_U \to \mathcal{F}|_U$
ii) For every open $U \subset X$ and every finite collection $s_i \in \mathcal{F}(U)$, the sheaf
$$Ker\left(\bigoplus_I \mathcal{O}_U \to \mathcal{F}|_U\right)$$
is of finite type. Note the i in s_i belongs to the *same* indexing set as the I in $\bigoplus_I \mathcal{O}_U$, so there is a natural map $\bigoplus_I \mathcal{O}_U \to \mathcal{F}|_U$.

Proposition: *A coherent sheaf is quasi-coherent.*

Proof: Let $\mathcal{F}$ be a coherent sheaf on X. For any $x \in X$, there exists $U \ni x$ open such that there is a surjective sheaf morphism

$$\mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$$

and for any choice of r sections in $\mathcal{F}|_U$, the kernel sheaf is of finite type, so there is a neighborhood $x \in V \subset U$ such that

$$\mathcal{O}_V^{\oplus q} \to Ker\big(\mathcal{O}_V^{\oplus r} \to \mathcal{F}|_V\big)$$

is surjective. By the universal property of kernel, there is thus an exact sequence

$$\mathcal{O}_V^{\oplus q} \to \mathcal{O}_V^{\oplus r} \to \mathcal{F}|_V \to 0$$

so that $\mathcal{F}$ is quasi-coherent.

$\square$

Proposition: *For a scheme $(X, \mathcal{O}_X)$, the structure sheaf is coherent.*

Proof: Clearly, the structure sheaf satisfies condition i), since we may select $r = 1$ and the identity morphism for every x and U. To check ii), let U and s_i be as described. A subtle point, which I'm not even sure is correct, but I believe it is: We have to think of $\mathcal{O}_U$ as a sheaf on the topological space U, and then show that this sheaf is of finite type, so there will be several layers of open sets taken: Let $x \in U$ and V be some open subset of U with $x \in V$. Then

$$Ker\left(\bigoplus_I \mathcal{O}_U \to \mathcal{O}|_U\right)\Bigg|_V = Ker\left(\bigoplus_I \mathcal{O}_V \to \mathcal{O}_V\right)$$

(is the above true?)
Then to describe a sheaf morphism, we have for every V' open in V,

$$Ker\left(\bigoplus_I \mathcal{O}_V \to \mathcal{O}_V\right)(V') = Ker\left(\bigoplus_I \mathcal{O}_V(V') \to \mathcal{O}_V(V')\right)$$

The right hand side of this equation is a submodule of the Noetherian module $\bigoplus_I \mathcal{O}_V(V')$, since I is finite, and is thus finitely generated, as a $\mathcal{O}_V(V')$ module. Thus there exists a surjective map of modules

$$\mathcal{O}_V(V')^{\oplus q} \to Ker\left(\bigoplus_I \mathcal{O}_V(V') \to \mathcal{O}_V(V')\right)$$

Doing this for every open $V' \subset V$ defines a surjective sheaf morphism (one must also check naturality)

$$\mathcal{O}_V^{\oplus q} \to Ker\left(\bigoplus_I \mathcal{O}_V \to \mathcal{O}_V\right)$$

Doing this for every V shows that the sheaf $Ker\left(\bigoplus_I \mathcal{O}_U \to \mathcal{O}_U\right)$ is of finite type.

□

Lemma: *A locally free sheaf is quasicoherent. Further, if the locally free sheaf is of finite rank, then it is coherent.*

Proof: Routine.

□

So we could think about the category of locally free sheaves, a subcategory of the category of sheaves. When we get to the derived category, we will see that the idea of cohomology of chain complexes is a crucial one. One may hope that locally free sheaves form an Abelian category, i.e. one which is suitable to consider cohomology, but this is not the case. What the category of locally free sheaves lacks is cokernels.

Example: Consider $\mathbb{P}^1$ equipped with the structure sheaf, $\mathcal{O}_{\mathbb{P}^1}$. Let $p \in \mathbb{P}^1$. Then there is an exact sequence

$$\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_p \to 0$$

It suffices to work locally with an affine coordinate x: The left map is multiplication by x, and $\mathcal{O}_p$ is the skyscraper sheaf concentrated at p, so the right map is evaluation at p. The idea is: the cokernel is the target mod the image. The image of the left map consists of holomorphic functions which vanish at p. So if U contains p, then the quotient results in $\mathbb{C}$, while if U does not contain p, then the condition "vanishes at p" is vacuous, and thus we are quotienting by all functions, yielding 0. Thus the cokernel is given by the skyscraper sheaf. However, this is not locally free.

It turns out that cokernels are the *only* thing that the category of locally coherent sheaves is missing to be abelian. So if we consider the category of locally free sheaves and throw in the extra objects of cokernels of all morphisms, we will obtain an abelian category. i.e. the category of quasicoherent sheaves is an enlargement of the category of locally free sheaves, is actually the smallest abelian category which contains all locally free sheaves[1]. Further, if we restrict attention to locally free sheaves of finite rank, then the category of coherent sheaves is the smallest abelian category containing these.

So now we understand the category of coherent sheaves over a scheme. Now that we've constructed an abelian category, we can begin to talk about the derived category. Of course, we will ignore all size-related considerations.

There is an explicit construction of the derived category, with lots of details to fill in and check. No individual step is very difficult, there are just so many steps one can lose the forest for the trees. In the end, this is all used to prove existence and uniqueness for the following theorem:

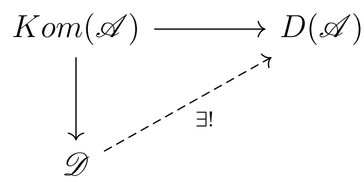
Theorem (UP of DC): *Let $\mathscr{A}$ be an abelian category and $Kom(\mathscr{A})$ be the chain complex category of $\mathscr{A}$. Note that when $\mathscr{A}$ is abelian, so too is $Kom(\mathscr{A})$. Then there exists a*

[1]There is a notion of abelianization of a category, but I don't know if that's what is being done here.

category $D(\mathscr{A})$, the derived category of $\mathscr{A}$, and a functor

$$Q : Kom(\mathscr{A}) \to D(\mathscr{A})$$

such that
i) $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism implies $Q(f)$ is an isomorphism, and
ii) Any functor $F : Kom(\mathscr{A}) \to \mathscr{D}$ satisfying the above condition factors uniquely over Q:

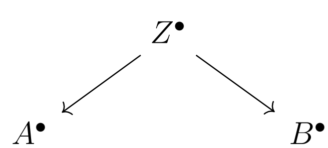


ii) as above is referred to as the universal property of the derived category, and it is a particular instance of the universal property of localisation of a category with respect to quasi-isomorphisms.

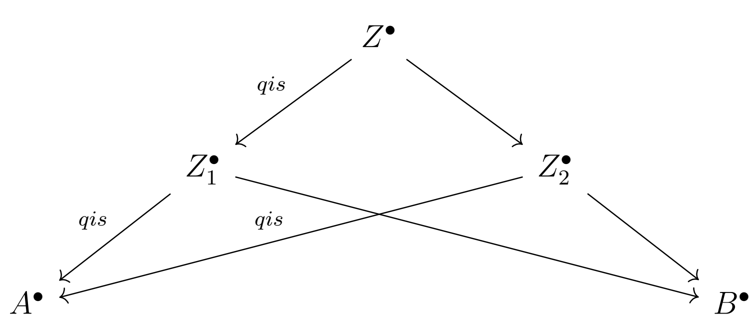
Recall that any morphism of chain complexes $f : A^\bullet \to B^\bullet$ induces a map on all cohomology groups $H^n(f) : H^n(A^\bullet) \to H^n(B^\bullet)$. Such a morphism f is called a quasi-isomorphism if all the f_*'s are isomorphisms.

At this point, we should also denote the homotopy category of $\mathscr{A}$, $\mathcal{K}(\mathscr{A})$, as the category whose objects consist of chain complexes of objects in $\mathscr{A}$ and whose morphisms are chain morphisms considered up to homotopy equivalence.

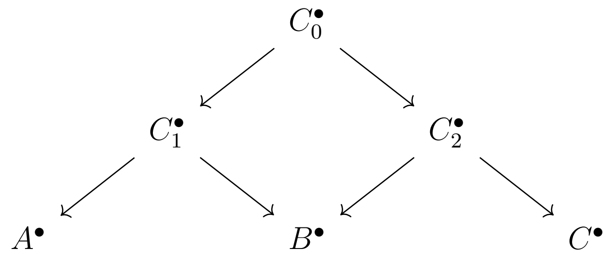
If you go through the construction of the derived category, which I feel everyone should do at least once, you will find that the objects of $D(\mathscr{A})$ consist of chain complexes of objects in $\mathscr{A}$, and a morphism $A^\bullet \to B^\bullet$ consist of a "roof", of the form



where $Z^\bullet \to A^\bullet$ is a quasi-isomorphism, modulo the equivalence relation that two morphisms $A^\bullet \to B^\bullet$ are equivalent if they are dominated by one roof in the homotopy category, i.e. chain morphisms considered up to homotopy:



We define composition of morphisms as one may hope it is done:



though there is a bit of content in how to construct $C_0^\bullet$. Again if interested, one should dig into the details on their own, and I can provide some resources for doing so.

Further, the derived category is always triangulated, i.e. equipped with a shift functor and class of distinguished triangles obeying TR1 (existence), TR2 (shifting), TR3 (completion), and TR4 (octahedral axiom), with the shift functor shifting chain complexes forward by 1, often denoted by [1], and distinguished triangles given by the mapping cone:

$$A^\bullet \to B^\bullet \to C(f) \to A[1]$$

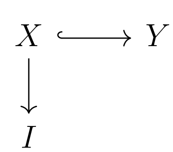
The triangulated structure is quite important, as the bounded derived category of coherent sheaves is an invariant when considered as a triangulated category, i.e. we consider "exact" equivalences of categories, equivalences which respect the triangulated structure.

The bounded derived category is constructed similarly, but whose objects contain only those chain complexes of finite length, and is denoted $D^b(Coh(X))$.

So what kinds of things are in this category? First note that because the objects are chain complexes, we may view objects and morphisms of $Coh(X)$ as objects and morphisms of $D(Coh(X))$, i.e. there is a fully faithful embedding given by sending an object to a complex supported in degree 0. So we may think of the simplest objects in $D(Coh(X))$, or $D^b(Coh(X))$, as simply vector bundles over X.

The morphisms are a bit mysterious though, at least to me. Composition is a bit complicated, and the equivalence relation of being mutually covered by one roof is similarly opaque. Under some nice conditions, this reduces significantly, though.

Definition: $I \in \mathscr{A}$ is called injective if for every pair of morphisms



with $X \to Y$ a monomorphism (think injective map), there exists a (not necessarily unique) morphism $Y \to I$. Equivalently (in an abelian category), if $Hom_{\mathscr{A}}(-, I)$ is exact.

Definition: $\mathscr{A}$ is said to have enough injectives if for every $A \in \mathscr{A}$, there exists an injective morphism $A \to I$, for I some injective obejct.

Theorem: *If an abelian category $\mathscr{A}$ has enough injectives, then there is an exact equivalence of categories*

$$K^+\big(Inj(\mathscr{A})\big) \to D^+(\mathscr{A})$$

Note there is always such a functor, on the full homotopy category $K(\mathscr{A})$, given by sending a chain complex to the same chain complex, and a sending a morphism

$$\big([f^\bullet] : A^\bullet \to B^\bullet\big) \longmapsto \quad A^\bullet \xleftarrow{\;Id\;} A^\bullet \xrightarrow{\;f^\bullet\;} B^\bullet$$

Morphisms in $\mathcal{K}^+\big(Inj(\mathscr{A})\big)$ are considered up to homotopy, so we should check that this map on morphisms is well defined, but this is the case because homotopy equivalences are quasi-isomorphisms. There is also a similar dual statement with bounded below and enough projectives. The above theorem states that when the functor above is restricted to the subcategory $K^+(Inj(\mathscr{A}))$, it becomes an equivalence.

The functor $D^+(\mathcal{A}) \to K^+(Inj(\mathcal{A}))$ which realizes this equivalence is given by sending a complex to any[2] injective resolution of the complex: A complex $I^\bullet$, which is bounded below 0, whose objects are all injective objects.

Fact: For general ringed spaces, and even general schemes, the category $Coh(X)$ does not necessarily have enough injectives. In fact it almost always does not:

Example: Consider the affine scheme $Spec(\mathbb{Z})$[3]. Then $Coh(Spec(\mathbb{Z})) \cong FgAbGrp$. Consider an injective object in $FgAbGrp$, I: Choose any $n \in \mathbb{Z}$ and $i \in I$. Then we have the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xhookrightarrow{\;\cdot n\;} & n\mathbb{Z} \\ {\scriptstyle 1\mapsto i}\downarrow & \swarrow{\scriptstyle \exists\varphi} & \\ I & & \end{array}$$

which we claim implies $nI = I$ for every $n \in \mathbb{Z}$. To see this, choose any $i_0 \in I$. Then there is a diagram as above with φ such that $\varphi(n) = i_0 \Rightarrow n\varphi(1) = i_0 \Rightarrow n \mid i_0$. So I must be a divisible group. But there are no finitely generated abelian divisible groups due to the structure theorem.

□

[2]It is a result that any two injective resolutions are homotopy equivalent, and are thus isomorphic objects in the homotopy category, which makes this functor well defined up to isomorphism.

[3]My favorite 3-manifold.

In fact, this example shows that the problem of $Coh(X)$ not having enough injectives is quite fundamental, not a one-off, as $Spec(\mathbb{Z})$ is a very important scheme: $\mathbb{Z}$ is initial in the category of rings, so $Spec(\mathbb{Z})$ is final in the category of rings, so every scheme is a scheme over $Spec(\mathbb{Z})$.

However, if one enlarges to $QCoh(X)$, then there are enough injectives. Further, $QCoh(X)$ is also abelian, and thus we could consider $D(QCoh(X))$. Often we must work with both. Grothendieck's words echoed....

> It is better to have a good category with bad objects than a bad category with good objects.

From what I can gather, $D(QCoh(X))$ is a good category with bad objects, while $D^b(Coh(X))$ is a bad category with good objects. So from a categorical perpsective, $D(QCoh(X))$ is preferable to study. But from a geometric perspective, $D^b(Coh(X))$ is preferred for some reason. I don't fully understand this. I think there's arguments for $D(QCoh(X))$ being a good or bad category: On the one hand, it is much better behaved with respect to the 6 functors, and $QCoh(X)$ actually has enough injectives, so $D(QCoh(X))$ has a characterization as complexes of injectives. On the other hand, injective resolutions are much harder to construct.

One of the most useful applications of the fact that $QCoh(X)$ has enough injectives is the result:

Proposition: *For any scheme, there is a natural equivalence:*

$$D^*(QCoh(X)) \cong D^*_{qcoh}(Sh_{\mathcal{O}_X}(X))$$

where $D^*_{qcoh}(Sh_{\mathcal{O}_X}(X))$ is the full triangulated subcategory of the derived category of sheaves of $\mathcal{O}_X$-modules whose cohomology objects are all quasi-coherent.

In the coherence case we have a different result.

Proposition: *For any scheme there is an equivalence*

$$D^b(X) \cong D^b_{coh}(QCoh(X))$$

the full triangulated subcategory of the derived category of quasi-coherent sheaves with coherent cohomology.

Now we discuss some of the geometric implications of the above techniques, particular when $D^b(X)$ can be used as an isomorphism invariant. For the latter case, we will further restrict back down to smooth projective varieties:

Proposition: *For any scheme X, the category $D^b(X)$ is indecomposable iff X is connected.*

Here indecomposable refers to a particular type of semiorthogonal decomposition.

Proposition (B,O): *If X is a smooth projective variety with ample (anti)canonical bundle, then $D^b(X)$ is a complete invariant, i.e. any derived equivalence $D^b(X) \cong D^b(Y)$ induces an isomorphism $X \cong Y$.*

In particular, for a smooth projective curve with genus not equal to 1, i.e. not an elliptic curve, either the canonical bundle or the anticanonical bundle is ample, and the above applies. So for non-elliptic curves, the derived category is a complete invariant. In fact, it is true even for elliptic curves, but it is not due to this result.

In general it is not the case that the derived category is a complete invariant: There are smooth projective varieties X, Y which are derived equivalent but not isomorphic. These cases are interesting, because the derived category is picking up on some level of "sameness" which isomorphisms are too strict to detect. This is a popular area of study, particularly in the case of $X = Y$ and studying autoequivalences of $D^b(X)$ which do not descend to automorphisms of X.

One result in this area is

Proposition (B,O): *If X is a smooth, projective variety with ample (anti)canonical bundle,*

$$Aut(D^b(X)) \cong \mathbb{Z} \times (Aut(X) \ltimes Pic(X))$$

accounting for shifts by the translation functor, actual automorphisms of X, and twists by line bundles. So this subcase of varieties is closed. A current open subcase which is quite active, or was open in 2006 and I don't know if it's open anymore, is the case of $K3$ surfaces, which are those for which the canonical bundle is trivial.

Outside the area of isomorphisms of varieties, there are other interesting applications. If one is interested in quivers, there is a large class of cases when the category $D^b(X)$ is equivalent to the derived category of quiver representations. For example, this is the case for $\mathbb{P}^1$ whose corresponding quiver is the Kronecker quiver, two vertices and two edges, while the abelian categories $Coh(\mathbb{P}^1)$ and $Rep(K)$ are not equivalent.

Further directions to study:
i) Certain functors (of the 6 functor formalism) do not restrict to functors on $D^b(Coh(X))$. Which do and don't, does it depend on X, and can this then tell me about the geometry of X?

ii) Semi orthogonal decomposition of $D^b(Coh(X))$.

iii) Application to D-brane categories (i.e. the real reason that I care).

iv) Application to minimal model program and birationality (I will probably cover this somewhat since it is the most accessible application for geometers).

v) proofs of the statements of section 3, namely involving derived functors and spectral sequences (this would be a good talk topic for another member of the seminar, or possibly myself in the future).

vi) We can discuss many AG-related concepts which build off of the derived category: for example, the Fourier Mukai transform is a certain derived equivalence, a perverse sheaf is a certain object in a bounded derived category of sheaves (though replacing coherence with a different condition) and a tilting sheaf is a certain type of coherent sheaf (and thus a certain element of $D^b(X)$, which can be said to "classically generate" $D^b(X)$. These tilting sheaves, and the associated tilting modules, shed light on the relationship between $D^b(X)$ and $D^b(Rep(Q))$.