Kahler Geometry/Hodge Theory

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Notes for a talk I'm giving in my algebraic geometry class.

Overview/Intro:

To start with, why is Kahler geometry of interest to me? Naturally, physics is at least part of the answer. There are two important types of Kahler manifolds which play a role in string theory, Calabi-Yau's and hyperkahler: In particular, those who study and are proponents of M-theory claim that the important object to study is an 11-dimensional manifold, 6 of whose dimensions belong to a Calabi-Yau threefold and 4 of whose dimensions belong to a hyperkahler manifold[1](for concreteness, the standard example is T^*S^3 as the CY and the Taub-NUT space as the hyperkahler). But even for those who may have no interest in physics, Kahler geometry is of intrinsic interest to complex geometers. The definition of a Kahler manifold will make this idea clear. It is the natural way to combine the ideas of Riemannian, complex, and symplectic geometry. I'll say some sources for people to look into if they'd like to learn more, although I would also suggest just talking to me, if possible[2]. These are mainly what I was reading while preparing for this talk:

i) Lectures on Kahler Geometry, Moroianu
ii) Complex Geometry, Huybrechts
iii) Hodge theory and Complex algebraic geometry I, Voison

Though it is very interesting in its own right, I will completely skip over the story of almost complex manifolds (and the subsequent concept of almost Kahler) for the sake of time, and assume everything is integrable. It pains me greatly to do this, because I love the story so much, and I was recently reading a paper developing Hodge theory for almost complex manifolds[3].

The talk corresponding to these notes is only 45 minutes, which explains the many details being skipped over. Nevertheless, these notes will be more detailed than the talk itself, while still attempting to be succinct, and does not attempt to tell the complete story.

[1]For those mathematically astute in the audience, this only adds up to 10. The final dimension is time, which is just a copy of $\mathbb{R}$ or S^1, if you're feeling adventurous.

[2]Not because I am all-knowing, and able to answer your question, but because I am selfish and if someone is learning about Kahler geometry, I want to be part of it because I want to understand it well.

[3]https://arxiv.org/abs/2203.09274

Kahler Geometry:

We will take the complex/Hermitian viewpoint[4]. So we start with a complex manifold[5], and we need to upgrade it to a Hermitian manifold.

First we need a construction: If M is a complex manifold, then it comes naturally equipped with a global endomorphism of its tangent bundle, J, which squares to -1, which we will construct in the subsequent proposition.

Definition: Given a smooth (possibly complex) manifold M, an endomorphism[6] of TM which squares to -1 is called an almost complex structure.

Proposition: *Every complex manifold comes equipped with an almost complex structure*

Proof: Around any point, consider a local chart (φ, U), with $\varphi : U \to V \subset \mathbb{C}^n$. Then $\varphi_* : T_U \to T_*V \cong \mathbb{C}^{2n}$. From this isomorphism, we import an almost complex structure on T_U by sending a tangent vector through the isomorphism, multiplying by i, and sending it back. Clearly this composition squares to -1, as desired. Then use a partition of unity to patch this together into a global endomorphism[7].

□

On a complex manifold, we may also consider complex-valued differential forms, which can be thought of as formal sums $\omega + i\tau$. Formally these are defined as sections of the complexified exterior algebra $\bigwedge T^*M \otimes_{\mathbb{R}} \mathbb{C}$, and the space of such is denoted as $\Omega^\bullet_{\mathbb{C}} M$. These forms act on (copies of) the complexified tangent bundle, $TM_{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$. Because this is a complex bundle, $TM_{\mathbb{C}}$ now splits into i and $-i$ eigenspaces of J:

Lemma:

$$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$$

where the first summand is the i-eigenspace and the second is the $-i$ eigenspace. Accordingly

Lemma[8]:

$$\bigwedge_{\mathbb{C}}^1 M = \bigwedge^{1,0} M \oplus \bigwedge^{0,1} M$$

[4]Already there is something to be said here. The meaning of 'viewpoint' is that there are three definitions of a Kahler manifold, each one focusing on a certain structure: Riemannian, complex, or symplectic. Your own motivation and experience will inform which approach is most suitable for you. I assume that if one wants to actually work in Kahler geometry, they should know all these aspects completely and be able to switch between them according to what the problem requires. Although it all ends up in the same point, so I don't know if it is actually valuable to do this exercise.

[5]Real smooth manifold where charts are valued in open sets in $\mathbb{C}^n$ rather than $\mathbb{R}^n$ and transition functions are required to be holomorphic, instead of just smooth.

[6]Note this is a bundle morphism, considering TM as a vector bundle over M. In particular, it is required to commute with the projection, so we may think of such an endomorphism as an endomorphism of each tangent space, varying smoothly from point to point.

[7]Recall a complex manifold does not have holomorphic bump functions, because of the identity theorem. Nevertheless, it does still have smooth bump functions which forget the complex structure, which we may use to construct smooth partitions of unity, which suffices here.

[8]I don't know what's going wrong with the formatting here.

where the first summand annihilates the $-i$ eigenspace and the second summand annihilates the i eigenspace.

More generally,

$$\bigwedge_{\mathbb{C}}^{k} M = \bigoplus_{p+q=k} \bigwedge^{p,q} M$$

where

$$\bigwedge^{p,0} M := \left(\bigwedge^{1,0} M\right)^{\otimes p}, \quad \bigwedge^{0,q} M := \left(\bigwedge^{0,1} M\right)^{\otimes q}$$

and

$$\bigwedge^{p,q} M := \bigwedge^{p,0} M \otimes \bigwedge^{0,q} M$$

Of course we can consider the sections of these bundles, defined as $\Omega^{p,q}M$. Because we have a complex manifold[9], we can say concretely exactly what this means in local coordinates. The i-eigenspace is generated by the ∂_{z_α}'s, the $-i$ eigenspace is generated by the $\partial_{\bar{z}_\alpha}$'s, so that the $(1,0)$ forms are given by dz_α and the $(0,1)$ forms are given by $dz_{\bar{z}_\alpha}$'s, and higher order forms are just wedges of these.

Example:

$$\bigwedge_{\mathbb{C}}^{2} M = \bigwedge^{2,0} M \oplus \bigwedge^{1,1} M \oplus \bigwedge^{0,2} M$$

so an element of this space looks like

$$\omega = \sum_{\alpha,\beta} a_{\alpha,\beta} dz_\alpha \wedge dz_\beta + \sum_{\alpha,\beta} b_{\alpha,\beta} dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha,\beta} c_{\alpha,\beta} d\bar{z}_\alpha \wedge d\bar{z}_\beta$$

in local coordinates.

Note that $d|_{\Omega^{p,q}M}$ is a map into a direct sum of $p+q+2$ summands. When X is complex (as opposed to almost complex), the exterior derivative only hits 2 of those components:

Lemma: *If X is a complex manifold, then*

$$Im(d|_{\Omega^{p,q}M}) \subset \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$$

□

In fact this condition is equivalent to integrability. This is not the focus of this document so we will leave out the details, but there are several equivalent statements, one other such being integrability of the distribution $T^{1,0}M$, a statement which will be somewhat familiar to those in Lev's manifolds class this semester.

[9] If working with almost complex, there are no local coordinates, so the above statement is as concrete as it gets

Definition: A complex manifold, M, equipped with a Riemannian metric, g, is called a Hermitian manifold if $g(JX, JY) = g(X, Y)$ for all $X, Y \in TM$. In such a case, the form $h := g - i\omega$ is a Hermitian form on the complex vector spaces (T_xM, J), and g is said to be compatible with the almost complex structure J.
The form ω is yet to be defined, but it is mentioned here because it provides motivation for the name "Hermitian".

Lemma: *Any complex manifold is a Hermitian manifold, i.e. can be equipped with a J-compatible Riemannian metric.*
Proof: Let M be a complex manifold, and regard it as a real manifold, so that it admits a Riemannian metric (partition of unity construction, locally pulling back the euclidean metric then gluing together.). Given any Riemannian metric, g, one can always produce a new, J compatible Riemannian metric by setting

$$g'(-,-) = g(-,-) + g(J-, J-)$$

□

Lemma: *If (M, g) is a Hermitian manifold, there is an associated real-valued $(1,1)$-form, $\omega(X, Y) := g(JX, Y)$, called the fundamental form.*

Proof: To see that $\omega \in \Omega^{1,1}M$, note that $\omega(J-, J-) = \omega(-,-)$:

$$\begin{aligned} \omega(JX, JY) &\equiv g(J^2X, JY) \\ &= g(-X, JY) \\ &\boxed{=} g(-JX, J^2Y) \\ &= g(-JX, -Y) \\ &= g(JX, Y) \\ &= \omega(X, Y) \end{aligned}$$

where the boxed equality follows because g is J-compatible. If $X, Y \in T^{1,0}M$, then

$$\omega(JX, JY) = \omega(iX, iY) = -\omega(X, Y) = \omega(X, Y) \Rightarrow \omega(X, Y) = 0$$

and similarly if $X, Y \in T^{0,1}M$ then

$$\omega(JX, JY) = \omega(-iX, -iY) = -\omega(X, Y) \Rightarrow \omega(X, Y) = 0$$

So the $\Omega^{0,2}M$ and $\Omega^{2,0}M$ components of ω are 0.

□

Definition: Given a Hermitian manifold, (M, g), we say the manifold, (M, ω), is Kahler, or is a Kahler manifold, with associated Kahler form ω, if ω is closed, i.e. $d\omega = 0$.

Remark: Note that, in such a case, (M, ω) is a symplectic manifold, so we may utilize everything we know about symplectic manifolds. In particular, if M is compact, then we must have non-vanishing of all even dimensional cohomology, up to the dimension. So we have a large family of non-examples: S^n where $n \neq 2$. Note that, for n odd, this statement is already completely obvious since complex manifolds must have even dimension. But the even cases are handled with this reasoning. Even more is true: The only spheres which can admit an almost complex structure are S^2, which is integrable, and S^6. The almost complex structure which S^6 is known to have is not integrable, but it hasn't been shown that it can't admit *any* integrable almost complex structure, though it seems to be modern consensus that this is the case.

Remark:

Recall that a $(1,1)$ form means that ω lies only in the middle term of $\bigwedge^2 M$, and thus takes the form

$$\omega = \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j$$

where h_{ij} is a positive definite hermitian matrix.

Example: $\left(\mathbb{C}^n, \sum_{i,j=1}^n dz_i \wedge d\bar{z}_j\right)$. This form comes from the standard Riemannian metric on $\mathbb{R}^{2n}$.

Example: Any Riemann surface is of Kahler type (admits a Kahler metric). This comes from a dimensional argument: Given any Riemann surface, equip it with an arbitrary complex structure. Then its associated $(1,1)$-form must be closed because $H^3_{dR}(M) = 0$.

Example: $\mathbb{C}P^n$ with the Fubini-Study metric.

Hodge Theory Preliminaries:

If we choose a local orthonormal frame $\{e_1, \ldots, e_n\}$, the Riemannian metric induces a metric on $\bigwedge^k M$[10]. Given a Riemannian metric $\langle -, - \rangle$ on $\bigwedge^k M$, we can define a Hermitian form as $\langle -, - \rangle_H := \langle -, J- \rangle$.

Then we define the adjoint[11] of d:

$$d^\star = -(-1)^{nk} \star d \star : \Omega^k M \to \Omega^{k-1} M$$

and the Laplacian:

$$\Delta = \{d, d^\star\} : \Omega^k M \to \Omega^k M$$

Harmonic analysis is the study of differential forms, ω, satisfying $\Delta\omega = 0$. Such a form is called a harmonic form.

Similarly, given a Kahler manifold, the natural operator on the exterior algebra to consider is no longer d, but the partial operators, ∂ and $\bar{\partial}$, defined by, for all p, q

$$\partial = \pi^{p+1,q} \circ d, \quad \bar{\partial} = \pi^{p,q+1} \circ d$$

where $\pi^{p,q}$ denotes the projection $\Omega^{p+q} M \to \Omega^{p,q} M$. Of course we think of these operators locally as differentiating with respect to z and $\bar{z}$. In this case, we have two different "Laplacians", defined as

$$\Delta^\partial = \{\partial, \partial^\star\}, \quad \Delta^{\bar{\partial}} = \{\bar{\partial}, \bar{\partial}^\star\}$$

where

$$\partial^\star = \star\bar{\partial}\star, \quad \bar{\partial}^\star = \star\partial\star$$

Note that, because X is complex, we have[12] $d^2 = 0 \Rightarrow \partial^2 = \bar{\partial}^2 = 0$. This comes because

$$d^2 = \partial^2 + \bar{\partial}^2 + (\partial\bar{\partial} + \bar{\partial}\partial) = 0$$

And there can be no cancellation because each term individually takes values in different sub-bundles of $\Omega^\bullet$, so each term must be 0. Additionally,

Theorem: *When M is Kahler,*

$$\Delta = 2\Delta^{\bar{\partial}} = 2\Delta^\partial$$

This proof is a real beast. It involves introducing many notions, such as a twisted exterior derivative, lefschetz operator and covariant derivatives, and then a long crunching of symbols resulting in the Kahler identities, so it is the perfect type of proof to skip in a talk, though the result itself is quite useful, as are the Kahler identities you discover along the way.

[10]This is fairly standard, but a construction can be found in Moroianou

[11]To see a more general discussion of the meaning and properties of "adjoint" here, you may google the term "formal adjoint", taking place in the setting of Hermitian bundles.

[12]Leading to the double complex associated to Dolbeault cohomology.

Onto real Hodge theory:
We begin with a real, compact[13] Riemannian manifold, oriented by some volume form dV. We can extend the de Rham cohomology and de Rham theorem to complex coefficients, by considering the complex associated to $\Omega^k_{\mathbb{C}}(M)$ and extending the exterior derivative by complex linearity. This leads to a complex de Rham theorem:

$$H^k_{\text{sing}}(M, \mathbb{C}) \cong H^k_{\text{dR}}(M, \mathbb{C})$$

We denote the subspace of harmonic k-forms as $\mathcal{H}^k(M, \mathbb{C}) \subset \Omega^k_{\mathbb{C}} M$.

Lemma: *$\omega \in \Omega^k_{\mathbb{C}} M$ is harmonic iff it is closed and $d^\star$ closed:*

Proof: $\Leftarrow$: obvious
$\Rightarrow$: If ω is harmonic, then

$$0 = \int_M \langle \Delta\omega, \omega \rangle_H dV = \int_M \langle dd^\star\omega + d^\star d\omega, \omega \rangle_H dV = \int_M |d^\star\omega|^2 + |d\omega|^2 dV$$
$$\Rightarrow d^\star\omega = d\omega = 0$$

□

Corollary: *Giving $\Omega^k_{\mathbb{C}} M$ the natural Hermitian metric $(\omega, \tau) = \int_M \langle \omega, \tau \rangle_H dV$, the following subspaces of $\Omega^k_{\mathbb{C}} M$ are mutually orthogonal:*

$$\mathcal{H}^k(M, \mathbb{C}), \quad d^\star\Omega^{k+1}_{\mathbb{C}} M, \quad d\Omega^{k-1}_{\mathbb{C}} M$$

This statement is quite easy to check pairwise. For example, take $\omega \in \mathcal{H}^k(M, \mathbb{C})$ and $\gamma \in d^\star\Omega^{k+1}_{\mathbb{C}} M$. Then

$$\int_M \langle \omega, \tau \rangle dV = \int_M \langle \omega, d^\star\eta \rangle dV = \int_M \langle d\omega, \eta \rangle dV = 0$$

and the other cases follow similarly. The swapping of $d^\star$ to d across the inner product follows from the more general discussion of formal adjoints which we glossed over in footnote 11.

□

So we may consider the direct sum

$$\mathcal{H}^k(M, \mathbb{C}) \oplus d^\star\Omega^{k+1}_{\mathbb{C}} M, \oplus d\Omega^{k-1}_{\mathbb{C}} M \subset \Omega^k M$$

Theorem (Hodge decomposition): *In the above setting, the direct sum equals the whole space, $\Omega^k_{\mathbb{C}} M$.*

□

Remark: From the above theorem, any form $\omega \in \Omega^k_{\mathbb{C}} M$ can be written as

$$\omega = d\alpha + d^\star\beta + \gamma$$

[13]It's not clear to me how much of the rest of the talk requires compactness. Certainly the Dolbeault and de Rham groups can be defined for non-compact, but in the very first lemma we integrate over M. So for safety we will assume compactness for the rest of the talk.

with $\alpha \in \Omega^{k-1}_{\mathbb{C}} M,\ \beta \in \Omega^{k+1}_{\mathbb{C}} M$ and $\gamma \in \mathcal{H}^k(M, \mathbb{C})$. In particular, if ω is closed, then

$$0 = (d\omega, \beta) = (d(d^*\beta), \beta) = \int_M |d^\star \beta|^2 dV$$

So $d^\star\beta = 0$, so $\omega = d\alpha + \gamma$.

Theorem (Hodge isomorphism): *The natural map*

$$\varphi : \mathcal{H}^k(M, \mathbb{C}) \to H^k_{dR}(M, \mathbb{C})$$

is an isomorphism.

Proof: The kernel is given by the intersection of k-harmonic forms with $k-1$-exact forms, which is 0. It is surjective because given any $[\tau] \in H^k_{dR}(M, \mathbb{C})$, we have

$$\tau = d\alpha + \gamma$$

$$\Rightarrow \varphi(\gamma) = [\gamma] = [\gamma + d\alpha] \equiv [\tau]$$

□

<u>**Hermitian Hodge Theory**</u>:
If M is Hermitian instead of real, still compact, we have a very similar story, but now the splitting of forms must become involved somehow. So now we want to consider some objects indexed by p, q instead of $k = p+q$. We begin by replacing the de Rham cohomology groups with so-called Dolbeault cohomology groups, $H^{p,q}(M, \mathbb{C})$, defined as the cohomology of the complex

$$\dots \xrightarrow{\bar\partial} \Omega^{p,q-1}M \xrightarrow{\bar\partial} \Omega^{p,q}M \xrightarrow{\bar\partial} \Omega^{p,q+1}M \xrightarrow{\bar\partial} \dots$$

Then we again define the space of Harmonic forms as $\mathcal{H}^{p,q}M = \{\omega \in \Omega^{p,q}M \mid \Delta^{\bar\partial}\omega = 0\}$. Essentially the exact same story follows from before, with the appropriate Hodge decomposition and isomorphism, replacing d and $d^\star$ with $\bar\partial$ and $\bar\partial^\star$, and replacing k with p, q where appropriate. But we have slightly more: Before the harmonic cohomology groups were indexed by an integer k. Now we have two integers, and we define the $h^{p,q} := dim_{\mathbb{C}}\mathcal{H}^{p,q}(M, \mathbb{C})$, the Hodge numbers. We have an analogy to Poincare duality:

Theorem (Serre duality): *The spaces* $\mathcal{H}^{p,q}M$ *and* $\mathcal{H}^{m-p,m-q}M$ *are isomorphic, i.e.* $h^{p,q} = h^{m-p,m-q}$.

Proof: Consider the map

$$\star \circ CC : \Omega^{p,q}M \to \Omega^{m-p,m-q}M$$

where CC denotes complex conjugation.

□

So far, all has been more or less analogous to the real Riemannian case.

Of course, because a Hermitian manifold is also Riemannian, we may consider the harmonic k-forms constructed before, $\mathcal{H}^k(M, \mathbb{C})$. But a priori, these groups and hodge numbers may have no relation.

Kahler Hodge Theory:

If in addition, M is Kahler, we have a further decompostion:

$$\mathcal{H}^k M = \bigoplus_{p+q=k} \mathcal{H}^{p,q} M$$

so in fact the cohomology groups decompose exactly as we would hope, mirroring the decomposition of differential forms.

Further, there is additional symmetry between the Hodge and Betti numbers:

Theorem: *If M is compact Kahler, we have*

$$b_k = \sum_{p+q=k} h^{p,q}, \quad h^{p,q} = h^{q,p}, \quad h^{p,p} \geq 1$$

□

Some observations: In particular, this implies some very nice properties of the Hodge diamond: The betti number b_k is the sum of the corresponding row of the hodge diamond, all terms down the middle should be greater than or equal to 1, and the hodge diamond is symmetric about the vertical line.

To see more about implications of the Kahler condition to Hodge theory, one can google the Kahler package, which is a collection of results along these lines (Lefschetz theorem on (1,1) classes, Hard Lefschetz, etc.) One thing I'm also looking forward to studying in the future is formality of Kahler manifolds.