Mini Course: Categorical Representation Theory

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Spring 2022

Abstract

Mini course taught by Gurbir Dhillon of Yale, part of the Algebra and Geometry series. Notes are handwritten during lecture, then transcribed later. I will sometimes (in this case often, since a lot of the content is over my head) interject my own thoughts with proofs, examples, explanations of background material, and definitions. As such, any errors found in the text should be assumed to be introduced by me, and this is not necessarily a faithful representation of what occurs in the course. Any comments, concerns, questions, corrections, or communications of any type are encouraged to be directed to my email. Recordings and Gurbir's version of the notes are available on his website. These notes are primarily a documentation of my personal learning journey while following along with the class: There is a lot of material in this document that did not come from the lecture, and some of the lecture material has not been included in these notes. Nevertheless this should provide some non-zero utility for any and all readers. This mini-course turned out to be not so mini.

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CHAPTER 1

I. Overview and Finite Groups

Lecture 1, Feb 3.

We began with a very lengthy discussion of what I believe was the roadmap for the upcoming course. I think this was meant to provide motivation for the more senior members in attendance. I didn't write any of this down, since it may as well have been in Latin to me. One will have to revisit the notes by Gurbir to experience this. We begin, instead, with the motivational theory of finite groups: Let H be a finite group, and k an algebraically closed field with characteristic 0. Consider the category $Rep(H)$, the category of H-representations on k vector spaces. We have the diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d] & GL(W) \arrow[d] \\ \bigoplus_{h\in H} k\cdot h \arrow[r] & End(W) \end{tikzcd}$$

Where the lower left corner is the group algebra $k[H]$. The vertical maps are inclusions, and the horizontal maps are given by a representation/action. This diagram is commutative:

$$\begin{tikzcd} h \arrow[r, mapsto] \arrow[d, mapsto] & \rho(h) \arrow[d, mapsto] \\ 0+\cdots+0+h+\cdots+0 \arrow[r, mapsto] & \rho(h) \end{tikzcd}$$

Given any finite set X, we can consider $Fun(X)$, the k-algebra of functions $X \to k$. We have a vector space isomorphism

$$Fun(X) \cong \bigoplus_{x\in X} k\delta_x$$

where

$$\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & \text{else} \end{cases}$$

is the indicator function. So the group algebra is isomorphic as a vector space, but the algebra structures may be different. We want to determine what multiplication looks like on functions. Given $f : X \to Y$, a map between finite sets, we get two natural maps:

$$f^* : Fun(Y) \to Fun(X)$$

sending

$$\phi \mapsto \phi \circ f$$

and

$$f_* : Fun(X) \to Fun(Y)$$

sending

$$\phi \mapsto f_*\phi$$

where

$$f_*\phi(y) = \sum_{x \in f^{-1}(y)} \phi(x)$$

This is the discretized notion of "integration over the fiber".
Claim:

$$Fun(X) \otimes_k Fun(Y) \cong Fun(X \times Y)$$

Why is this? Define the map

$$\phi \otimes \psi \mapsto \pi_X^* \phi(-) \cdot \pi_Y^* \psi(-)$$

as the pointwise product of functions. Note that, by this definition, we are also sending

$$\delta_x \otimes \delta_y \mapsto \delta_{(x,y)}$$

Exercise: *i) Show $Fun(X) \cong Fun(X)^*$ canonically, as k-vector spaces.*
ii) WRT this duality, f_* and f^* are dual maps.

Proof:
i) $\boxed{\text{unfinished}}$ We have a pairing

$$Fun(X) \otimes Fun(X) \xrightarrow{\sim} Fun(X \times X) \xrightarrow{\Delta^*} Fun(X) \xrightarrow{\pi_*} Fun(\{*\}) \xrightarrow{\sim} k$$

where Δ is the map $x \mapsto (x, x)$ and π sends $x \mapsto *$.

Exercise: The multiplication on the group algebra is

$$Fun(H) \otimes Fun(H) \xrightarrow{\sim} Fun(H \times H) \xrightarrow{m_*} Fun(H)$$

More generally: Given $H \curvearrowright X$, we can define an action $Fun(H) \curvearrowright Fun(X)$.

Exercise: *The usual action is given by*

$$Fun(H) \otimes Fun(X) \xrightarrow{\sim} Fun(H \times X) \xrightarrow{a_*} Fun(X)$$

where $a : H \times X \to X$ is the action map.

So we interpret the multiplication in the group algebra as taking functions on the group and then convolving them. Next time we will discuss involutions, convolutions, and the Hecke algebra . If I can find the time I want to come back and do these exercises, but time is short these days.

CHAPTER 2

II. Hecke algebras of Finite Groups

Lecture 2, Feb 10.

The definitions of inv^G and coinv_G were presented quickly, and were unmotivated for me, so I wanted to try to find out where they come from. First, we define the category $Rep(G)$. Given a group G, the objects of this category are representations (V, ρ) of G, and morphisms are intertwiners between representations, or G-equivariant maps. We can define a functor

$$\mathrm{inv}^G : Rep(G) \to Vect_k$$

sending

$$(V, \rho) \mapsto V^G$$

where $V^G = \{v \in V \mid \rho(g)v = v \ \forall g \in G\}$. Alternatively, it is the largest submodule of V on which G acts trivially. This is a very natural thing to define. This functor sends morphisms

$$(f : (V, \rho) \to (W, \psi)) \mapsto f|_{V^G} : V^G \to W^G$$

To see that the image lands in W^G, we compute

$$\psi(g) f|_{V^G}(v) = f|_{V^G}(\rho(g)v) = f|_{V^G}(v)$$

where the first equality is because f is G-equivariant and the second is because $v \in V^G$. So inv^G is a functor. This is the down to earth definition, which anyone should be comfortable parsing. In fancier language, we can also think of it as a hom-functor: If $\underline{\mathbb{C}}$ is the trivial representation ($g \cdot 1 = 1$), then we send $(V, \rho) \mapsto V^G$. But[1]

$$V^G = Hom_{Rep(G)}(\underline{\mathbb{C}}, (V, \rho))$$

To see this, consider an element of the right hand side. That is, a linear map $f : \underline{\mathbb{C}} \to V$ such that

$$\begin{tikzcd} \underline{\mathbb{C}} \arrow[r, "f"] \arrow[d, "1"'] & V \arrow[d, "\rho_g"] \\ \underline{\mathbb{C}} \arrow[r, "f"] & V \end{tikzcd}$$

[1]One could use any of the three equivalent: $Rep(G) \cong Fun(G)\text{-}Mod \cong K[G]\text{-}Mod$

commutes. Of course, such a map f is determined by the image of $f(1)$. Then the commutativity says that $\rho(g)(f(1)) = f(1)$, so f is determined uniquely by an element of V which is $\rho(g)$ invariant for all g. Hence the equality above. So inv^G is really a hom functor. In this language, what does it do to morphisms? Given an intertwiner $f : (V,\rho) \to (W,\psi)$, we have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle\rho(g)}\downarrow & & \downarrow{\scriptstyle\psi(g)} \\ V & \xrightarrow{f} & W \end{array}$$

and we want a map $\text{inv}^G(f) : Hom_{Rep(G)}(\underline{\mathbb{C}}, (V,\rho)) \to Hom_{Rep(G)}(\underline{\mathbb{C}}, (W,\psi))$. Take some $\varphi \in Hom_{Rep(G)}(\underline{\mathbb{C}}, (V,\rho))$, i.e. a commutative diagram

$$\begin{array}{ccc} \underline{\mathbb{C}} & \xrightarrow{\varphi} & V \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle\rho(g)} \\ \underline{\mathbb{C}} & \xrightarrow{\varphi} & V \end{array}$$

and there is only one thing to do: stick them together

$$\begin{array}{ccccc} \underline{\mathbb{C}} & \xrightarrow{\varphi} & V & \xrightarrow{f} & W \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle\rho(g)} & & \downarrow{\scriptstyle\psi(g)} \\ \underline{\mathbb{C}} & \xrightarrow{\varphi} & V & \xrightarrow{f} & W \end{array}$$

We similarly define $\text{coinv}_G(W) = \underline{\mathbb{C}} \otimes_{Fun(G)} W$, or equivalently $W_G = W/\langle gw - w\rangle$. To see these are equivalent, recall the explicit construction of the tensor algebra: Consider the free abelian group generated by $\mathbb{C} \times W$, equip it with a $Fun(G)$-structure, and quotient by the two-sided ideal generated by all the desired relations the tensor product should hold. Among these is the condition that $(gz, w) \sim (z, gw)$. But because $\mathbb{C}$ carries the trivial representation, we are imposing the relation $(z, w) \sim (z, gw)$, i.e. $w \sim gw$. Then $\mathbb{C}$ just acts as a dummy variable, and we recover $W/\langle gw - w\rangle$.

Basic observation: We can think about $\underline{\mathbb{C}}$ as functions on G which are G-invariant under the action of left or right translation:

$$\underline{\mathbb{C}} \cong Fun(G)^G \cong Fun(G/G) \cong Fun(*)$$

These are all just trivial ways to write down the trivial representation.

Another basic observation: If $G \curvearrowright X$, for X finite, then $Fun(G) \curvearrowright Fun(X)$, and

$$Fun(X)^G \cong Fun(X/G)$$

where X/G is the set of G-orbits. Namely,

$$\begin{tikzcd}
 & Fun(X) & \\
Fun(X)^G \arrow[ur, hook] \arrow[rr, "\sim"] & & Fun(X/G) \arrow[ul, hook', "\pi^*"']
\end{tikzcd}$$

Morally, this means that, in the world of functions, taking invariants is the same as acting on orbits. More informally, quotienting[2] in linear algebra (taking invariants) is the same as quotienting in geometry (consider space of orbits).

Proposition: *inv^G and $coinv_G$ are naturally isomorphic.*

Proof: Suppose there is a vector space on which G acts trivially, $G \curvearrowright W$. There is another characterization of the coinvariants: Given an intertwiner $\varphi : V \to W$, this must factor uniquely through V_G:

$$\begin{tikzcd}
V_G \arrow[r, dashed, "\exists! \bar{\varphi}"] & W \\
V \arrow[u, "q"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

In particular, if we let $W = V^G$, then we have

$$\begin{tikzcd}
V_G \arrow[r, dashed, "\exists! \bar{\varphi}"] & V^G & & \sum_{g \in G} g \cdot v \\
V \arrow[u, "q"] \arrow[ur, "\varphi"'] & & v \arrow[ur, mapsto] &
\end{tikzcd}$$

The map φ is clearly an intertwiner, so it is a candidate for our natural transformation. We want to show it is an isomorphism, i.e. write down an inverse. The horizontal map sends $[v] \mapsto \sum_g g \cdot v$, by the commutativity of the diagram, and is well defined by the universal property. Then define the candidate inverse map as $v \mapsto \frac{1}{|G|}[v]$. Check compositions:

$$[v] \mapsto \sum_g g \cdot v \mapsto \frac{1}{|G|} \left[\sum_g g \cdot v \right] = \frac{1}{|G|} \sum_g [g \cdot v] = \frac{1}{|G|} \sum_g [v] = [v]$$

$$v \mapsto \frac{1}{|G|}[v] \mapsto \frac{1}{|G|} \sum_g g \cdot v = \frac{1}{|G|} \sum_g v = v$$

So the horizontal map is an isomorphism.

[2] I think mistake here? When we take invariants, there is no quotienting. The construction of coinv involves a quotient, not inv. Unless we are referencing the isomorphism. This was something Gurbir said out loud, not wrote down so maybe doesn't matter too much.

To check naturality, we want to verify that the diagram, given a morphism $f : V \to W$,

$$\begin{tikzcd}
V_G \arrow[d, "\mathrm{coinv}_G f"'] \arrow[drr] & & \\
W_G \arrow[drr] & V \arrow[ul, "q_V"'] \arrow[r, "p_V"'] \arrow[d, "f"] & V^G \arrow[d, "\mathrm{inv}^G f"] \\
& W \arrow[ul, "q_W"] \arrow[r, "p_W"'] & W^G
\end{tikzcd}$$

commutes. We only need to verify that the two reader-facing rectangular faces commute, from which it follows that the rectangle in the back also commutes. Specifically, the flat triangles were already shown to commute. The right hand rectangle commutes because $\mathrm{inv}_G f$ is just a restriction map, and similarly the left hand rectangle commutes because $\mathrm{coinv}_G f$ respects the equivalence relation, since coinv_G is a functor. Then we have

$$\begin{aligned}
V \to V_G \to V^G \to W^G \\
= V \to V^G \to W^G \\
= V \to W \to W^G \\
= V \to W \to W_G \to W^G \\
= V \to V_G \to W_G \to W^G
\end{aligned}$$

$V \to V_G$ is surjective, so it is right-cancellable. Thus the back rectangle commutes, as does the entire diagram.

□

If $G \curvearrowright V$ then $G \curvearrowright V^G$ trivially. By contrast, if $K \leq G$ is a subgroup, you can ask: what naturally acts on V^K? To answer, we will first need the definition of adjoint functors. Consider functors $F : D \to C$ and $G : C \to D$. The pair define an adjunction between C and D if, for every $X \in C$ and $Y \in D$, there is an isomorphism

$$Hom_C(FY, X) \cong Hom_D(Y, GX)$$

which is natural with respect to X and Y. In such a case, F is known as the left adjoint and G is known as the right adjoint.

We have the forgetful functor $Rep(G) \to Rep(K)$, sometimes denoted by $Oblv$, short for Oblivion. This functor admits left and right adjoints (which coincide, for a finite group), given by

$$\begin{aligned}
\mathrm{ind} : Rep(K) \to Rep(G) \\
W \mapsto Fun(G) \otimes_{Fun(K)} W
\end{aligned}$$

This is a way of taking a representation of a subgroup, K, and inducing a representation on the larger group, G. Sometimes this functor is introduced as sending $W \mapsto \oplus_{i=1}^{|G/K|} g_i W$, where $g_1, \ldots, g_n$ are the generators of G/K. Note this definition requires G/K to be finite. In particular, it suffices, but is not necessary, for G to be finite. This VS has an action by G: For every g and g_i, there exists some index $j(i)$ depending on i such that

$$gg_i = g_{j(i)}k_i$$

since $[g_i]$ generate G/K. Then we can just define the action componentwise, applying the action by K where it feels most natural:

$$g \cdot \sum_{i=1}^{|G/K|} g_i v_i := \sum_{i=1}^{|G/K|} g_{j(i)} \rho(k_i) v_i$$

Essentially by construction, this is isomorphic, as a G-representation, to $Hom_K(G, V)$. Here the action of G is just precomposition with L_g. Then by tensor-hom adjunction, we have an equivalence with the definition above. So there are 3 different (but equivalent, when G is finite) ways of defining this functor.

To see that ind and $Oblv$ form an adjoint pair[3], we want to find an isomorphism. Of course, we use the most convenient definition of ind[4], as a hom-functor:

$$Hom_{Rep(K)}\big(Oblv(V,\rho),(W,\psi)\big) \cong Hom_{Rep(G)}\big((V,\rho),\mathrm{ind}(W,\psi)\big)$$

Send

$$\varphi \mapsto \Big(v \mapsto (g \mapsto (\varphi(\rho(g)v))\Big)$$

$g \mapsto f(\rho(g)v)$ is in $\mathrm{ind}(W,\psi) \cong Hom(G,W)$ because f is an intertwiner, and it is easy to check that the commutative diagram which $(v \mapsto (g \mapsto (\varphi(gv)))$ needs to satisfy to live in the RHS is satisfied[5]. We can define its inverse by sending

$$\lambda \mapsto (v \mapsto \lambda(v)(1))$$

and check the various required conditions.

[3] I started out trying to prove this by seeking some general form of tensor-hom, based on the motivation that $Oblv$ is a kind of forgetful-y functor, which when forgetting to set, are often representable, and ind can be defined as a tensor, so maybe the desired isomorphism just comes from tensor-hom. This approach, even if everything I claimed is accurate, presents other problems: $Oblv$ is $Rep(K)$-valued, not Set valued. So it is not representable from the strict definition. But it could be representable under a generalized definition, because $Rep(G)$ is enriched over itself: the VS of a hom-set is also a G-rep. So maybe one can define a functor $Rep(G) \to Rep(K)$ as representable if it is naturally isomorphic to $Hom_{Rep(G)}(A,-)$ considered as a functor $Rep(G) \to Rep(K)$, and hope that tensor-hom still goes through.

[4] When switching between definitions, I believe we are blurring the distinction between duals.

[5] comes down to ρ being a group homomorphism

Lemma: *There is a natural isomorphism of functors $Rep(G) \to Vect_k$*

$$F_1: \qquad Rep(G) \xrightarrow{\quad Oblv \quad} Rep(K) \xrightarrow{\quad \mathrm{inv}^K \quad} Vect_k$$

$$F_2: \qquad Rep(G) \xrightarrow{\qquad\qquad} Vect_k$$

$$W \longmapsto Hom_{Fun(G)}(Fun(G/K), W)$$

Proof:

$$\begin{aligned} F_1 = \mathrm{inv}^K \circ Oblv &= Hom_{Fun(G)}(\mathbb{C}, Oblv(-)) \\ &\cong Hom_{Fun(G)}(\mathrm{ind}\,\mathbb{C}, -) \\ &= Hom_{Fun(G)}(Fun(G) \otimes_{Fun(K)} \mathbb{C}, -) \\ &\cong Hom_{Fun(G)}(\mathbb{C} \otimes_{Fun(K)} Fun(G), -) \\ &= Hom_{Fun(G)}(\mathrm{coinv}_K(Fun(G)), -) \\ &\cong Hom_{Fun(G)}(Fun(G)_K, -) \\ &\cong Hom_{Fun(G)}(Fun(G/K), -) \end{aligned}$$

$\square$

Upshot: W^K carries a canonical right action of $End_{Fun(G)}(Fun(G/K))$: By the previous lemma, we have

$$W^K \cong Hom_{Fun(G)}(Fun(G/K), W)$$

So to act on the right by an endomorphism, we simply precompose. Let's unwind some more:

$$\begin{aligned} End_{Fun(G)}(Fun(G/K)) &\cong Hom_{Fun(G)}(Fun(G/K), Fun(G/K)) \\ &\cong Fun(G/K)^K \\ &\cong Fun(K \setminus G/K) \end{aligned}$$

Informally, consider the action map $Fun(G) \otimes W \to W$ and restrict to W^K: $Fun(G) \otimes W^K \to W$. But because this map is K-invariant, it factors as

$$Fun(G)_{K^r} \otimes W^K \to W$$

Where subscript K^r denotes coinvariants under the right action and superscript K^ℓ denotes invariants under the left action. We can restrict this to a map

$$Fun(G)_{K^r}^{K^\ell} \otimes W^K \to W^K$$

which is isomorphic to
$$Fun(K \setminus G/K) \otimes W^K \to W^K$$

Lemma: *The composition map*

$$End(Fun(G/K)) \otimes End(Fun(G/K)) \to End(Fun(G/K))$$

is isomorphic to
$$Fun(K \setminus G/K) \otimes Fun(K \setminus G/K) \to Fun(K \setminus G/K)$$
thus inducing a product on $Fun(K \setminus G/K)$. Refer to the induced map as c. Note we have an isomorphism

$$Fun(K \setminus G/K) \otimes Fun(K \setminus G/K) \cong Fun\Big((K \setminus G/K) \times (K \setminus G/K)\Big)$$

and if we define[6][7] $K \setminus G \times^K G/K$ to be the double cosets of $G \times G$ modded out by $\Delta K \subset K \times K$, then we have two natural maps

$$\begin{tikzcd}
 & K \setminus G \times^K G/K \arrow[dl, "\pi"'] \arrow[dr, "m"] & \\
(K \setminus G/K) \times (K \setminus G/K) & & K \setminus G/K
\end{tikzcd}$$

which allow us to write down c explicitly:

$$\begin{tikzcd}
c : & Fun(K \setminus G/K) \otimes Fun(K \setminus G/K) \arrow[d, "\sim"'] & \\
 & Fun\Big((K \setminus G/K) \times (K \setminus G/K)\Big) \arrow[d, "\pi^*"'] & \\
 & Fun(K \setminus G \times^K G/K) \arrow[r, "m_*"] & Fun(K \setminus G/K)
\end{tikzcd}$$

Remark: The factor of $\frac{1}{|K|}$ coming from the isomorphism of inv to coinv roughly matches the step of pulling back along π, since we accounted for the size of K in $K \setminus G \times^K G/K$.

Explicitly, given $\phi, \psi \in Fun(K \setminus G/K)$,

$$\phi * \psi := c(\phi, \psi)$$

[6] The notation seems to suggest I can think of this as a fiber product? But there is no map $G \to K$, so I'm not sure what it would be.

[7] I believe the grouping of parenthesis does not matter here. You can either consider $G \times^K G$ and then take double cosets or consider $K \setminus G$ and G/K then take the product over K.

where

$$\phi * \psi(g) = \frac{1}{|K|} \sum_{xy=g} \phi(x)\psi(y)$$

Exercise: *Check that $\mathbb{1}_K$, the indicator function for K, is a unit in $Fun(K \setminus G/K)$. NB: In $Fun(G)$, $\mathbb{1}_K * \mathbb{1}_K = |K|\mathbb{1}_K$.*

Upshot: For any $W \in Rep(G)$, $Fun(K \setminus G/K)$ acts on W^K. The action of $Fun(K \setminus K/K) \cong \mathbb{C}$ factors through it.

Until now, we have considered the trivial action $\underline{\mathbb{C}}$. We can also consider a "twisted Hecke algebra", where we allow for a non-trivial $\mathbb{C}$ representation: Given a one-dimensional representation, $\chi : K \to \mathbb{C}^\times$, we define $W^{K,\chi} = Hom_{k[K]}(\mathbb{C}_\chi, W)$, where $\mathbb{C}_\chi$ is the representation χ. Then from a similar argument[8],

$$Fun(K, \chi \setminus G/K, \chi) \curvearrowright W^{K,\chi}$$

Remark: Inversion on the group G induces an anti-automorphism of algebras

$$Fun(K \setminus G/K) \xrightarrow{i_*} Fun(K \setminus G/K)$$

so we can identify its left and right modules, since a left module is a right anti-module.

Prop:

$$inv^K : Rep(G) \to Fun(K \setminus G/K)\text{-Mod}$$

admits left and right adjoints, (which coincide)

$$Fun(G/K) \otimes_{Fun(K \setminus G/K)} - : Fun(K \setminus G/K)\text{-Mod} \to Rep(G)$$

Proof: This is a special case of the Hom-tensor adjunction for bimodules.

□

Prop: *For any rep W of G, the canonical map*

$$Fun(G/K) \otimes_{Fun(K \setminus G/K)} W^K \to W$$

$$\phi \otimes w \mapsto \frac{1}{|K|} \phi * w$$

is an injection with image the G-isotypic components of W admitting K-invariant vectors[9].

Proof: Claim: By complete reducibility, for any f.d. rep V of G, the tautological map

$$Hom(V, W) \otimes_{End(V)} V \to W$$

[8] $\boxed{\text{define argument of Fun}}$

[9] If W is a finite dimensional representation of a finite group, it is completely reducible; the G-isotypic component of W admitting K-invariant vectors is the direct sum of the irreducible representations containing K-invariant vectors.

is an embedding with image the isotypic components of W for which the corresponding isotypic components for V is nonzero.
By complete reducibility, this is now a linear algebra claim: Given two vector spaces M, N,

$$\begin{tikzcd} Hom(M,N) \otimes_{End(M)} M \arrow[r, "ev"] & N \\ N \otimes_{\mathbb{C}} M^{\vee} \otimes_{End(M)} M \arrow[u, "\sim"] \arrow[r, "\sim"] & N \otimes_{\mathbb{C}} \mathbb{C} \arrow[u, "\sim"'] \end{tikzcd}$$

Slogan: The Hecke algebra knows part of the rep theory of G. There are two extreme cases: $K = e$, the Hecke algebra is $Fun(G)$, and knows everything. When $K = G$, the Hecke algebra is $Fun(G \setminus G/G) \cong \mathbb{C}$, and knows about the G-invariant subspace of W. The tradeoff is the representation theory when $K = e$ is difficult, whereas the representation theory when $K = G$ is easy. You generally want to choose K to service this balancing act.

□

We will work through an extended example. If G is a split, connected, reductive group[10] (for example GL_2), we have

$$T(\mathbb{F}_q) \leftarrow B(\mathbb{F}_q) \rightarrow G(\mathbb{F}_q)$$

where T is a maximal torus and B is a Borel subgroup. In the GL_2 example, T can be chosen to be diagonal matrices, and B upper triangular matrices. Then the left map is "set non-diagonal elements to 0", which is a group homomorphism because multiplication of upper triangular matrices does not mix diagonal with off-diagonal elements, and the right map is inclusion.

Given any character χ of $T(\mathbb{F}_q)$, we can define the functor[11]

$$\begin{aligned} pind_T^G(\mathbb{C}_\chi) &= Fun(G(\mathbb{F}_q)) \otimes_{Fun(B(\mathbb{F}_q))} \mathbb{C}_\chi \\ &\cong Fun(G(\mathbb{F}_q)/B(\mathbb{F}_q), \chi) \end{aligned}$$

consider its endomorphisms:

$$H_\chi{}^{12} := Fun\left(B(\mathbb{F}_q), \chi \setminus G(\mathbb{F}_q)/B(\mathbb{F}_q), \chi\right)$$

And want to know dim H_χ. To state the answer, define the Weyl group: $W = N(T)/T$, where N is the normalizer. This group turns out to be finite.

The Weyl group acts on $T(\mathbb{F}_q)$ by conjugation, and thus acts on $\widehat{T(\mathbb{F}_q)}^{13} := \{\chi : T(\mathbb{F}_q) \to \mathbb{C}^\times\}$ by precomposition with conjugation.

Prop: dim $H_\chi = |\{w \in W \mid w\chi = \chi\}|$.

Proof:
First recall the Bruhat[14] decomposition for GL is a decomposition

$$G = BWB$$

where W is the set of permutation matrices. In general, G is any split, connected, reductive group, B is any Borel subgroup and W is the Weyl group. In particular, any $f \in H_\chi$ is determined by its restrictions to $[w] \in W$.
For any individual orbit, you can ask whether it supports any (B, χ)-invariant functions.

[10] Precisely, G is a functor from Ring to Algebraic Group, with image split, connected, reductive groups.

[11] Standing for "parabolic induction". This is just a particular instance of induction, with the label parabolic because we induce from T to G.

[12] I need some clarification on this definition.

[13] This is known as the Pontryagin dual.

[14] The Bruhat decomposition makes it possible to study the double cosets in terms of a finite dimensional function space, since W is always finite. In particular, to know f completely, the Bruhat decomposition tells me it suffices to check on each coset $[w]$, of which there are finitely many.

Informally, we need that $f(b_1\dot{w}b_2) = \chi(b_1)f(\dot{w})\chi(b_2)$, where $\dot{w}$ denotes a particular choice of representative. What does this imply?

$$\begin{aligned}\chi(t)f(\dot{w}) &= f(t\dot{w}) \\ &= f(\dot{w}\dot{w}^{-1}t\dot{w}) \\ &= f(\dot{w})\chi(\dot{w}^{-1}t\dot{w}) \\ &= f(\dot{w})(w\chi)(t)\end{aligned}$$

So we must have $(w\chi)(t) = \chi(t)$, which immediately implies
$\dim H_\chi \leq |\{w \in W \mid w\chi = \chi\}|$. To see the other inequality, we note the fact[15]: Given a group H and a subgroup K and a character χ of H,

$$Fun(H/K)^{H,\chi} = \begin{cases} 0 & \chi|_K \text{ nonzero} \\ \mathbb{C}\cdot\chi & \text{else}\end{cases}$$

$\boxed{\text{NOT FINISHED}}$

Corollary: *i) If χ is trivial, then dim $H_\chi = |W|$.*
ii) If χ is regular, i.e. $w\chi = \chi \iff w = e$ then dim $H_\chi = 1$.

[15]This almost certainly has an interpretation via associated bundles. Hopefully I can come back and explore that.

CHAPTER 3

III. Hecke Algebras for Principal Series

Lecture 3, Feb 17.

We want to explicitly describe the algebra structure of

$$Fun\left(B(\mathbb{F}_q) \setminus GL_2(\mathbb{F}_q)/B(\mathbb{F}_q)\right)$$

To do this, note we have a natural basis given by indicator functions on double cosets:

$$T_w = \mathbb{1}_{BwB}$$

In this case, the Weyl group is $S_2 \cong \mathbb{Z}_2 \equiv \{1, s\}$. Claim[1] $GL_2/B \cong \mathbb{P}^1$. To see this, we define a flag in $\mathbb{F}_q^n$ is an ascending chain of subspaces, from $\{0\}$ to F_q^n. We will show both sides are isomorphic to the space of flags. For GL_2/B, we see that GL_2 acts on the space of flags in $\mathbb{F}_q^2$ by matrix multiplication. When choosing B, there is an implicit choice of basis, e_1, e_2. Then define the flag $\{0\}, span(e_1), span(e_1, e_2)$. B is the stabilizer of this flag under the action by GL_2. By Orbit–Stabilizer, there is a bijection GL_2/B to the space of flags[2]. For $\mathbb{P}^1$, a choice of flag in this case is a choice of 1-dimensional subspace, since 0 and $\mathbb{F}_q^2$ are fixed. But the space of 1-dimensional sub-spaces is just $\mathbb{P}^1$.

So the basis of our Hecke algebra is just $T_{\mathbb{1}}, T_s$. We claim $T_{\mathbb{1}}$ is the unit, i.e. $T_{\mathbb{1}} T_s = T_s$ and $T_{\mathbb{1}} T_{\mathbb{1}} = T_{\mathbb{1}}$. To see this, we evaluate on a group element, using our calculation of the product from before:

$$\begin{aligned} T_{\mathbb{1}} * T_s([e]) &= \frac{1}{|B|} \sum_{xy=e} T_{\mathbb{1}}(x) T_s(y) \\ &= \frac{1}{|B|} \sum_{x \in G} T_{\mathbb{1}}(x) T_s(x^{-1}) \\ &= \frac{1}{|B|} \sum_{x \in B} T_{\mathbb{1}}(x) T_s(x^{-1}) + \frac{1}{|B|} \sum_{x \notin B} T_{\mathbb{1}}(x) T_s(x^{-1}) \end{aligned}$$

[1] This holds for any field, but we are in the particular case of $\mathbb{F}_q$.
[2] From which we import a smooth structure.

But $x \in B \iff x^{-1} \in B$, so

$$= \frac{1}{|B|}\sum_{x\in B} 1\cdot 0 + \frac{1}{|B|}\sum_{x\notin B} 0\cdot 1 = 0$$

as expected. Further,

$$T_{\mathbb{1}} * T_s([s]) = \frac{1}{|B|}\sum_{xy=s} T_{\mathbb{1}}(x)T_s(y)$$

To evaluate the sum, if $x \in B$, then $y \notin B$, because $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So if $y \in B$, then the product xy would be in B, i.e. upper triangular, and thus could not equal s. Therefore the sum, when $x \in B$, evaluates to $1 \cdot 1$. Thus

$$= \sum_{x\notin B, xy=s} T_{\mathbb{1}}(x)T_s(y) + \frac{1}{|B|}|B|$$

If $x \notin B$, then $T_{\mathbb{1}}(x) = 0$, so the left hand sum is 0.

$$T_{\mathbb{1}} * T_s([s]) = 1$$

which shows that $T_{\mathbb{1}} * T_s = T_s$. A similar calculation will show $T_s * T_1 = T_s$. We can also show $T_1 * T_1 = T_1$:

$$T_{\mathbb{1}} * T_{\mathbb{1}}([e]) = \frac{1}{|B|}\sum_{x\in B} T_{\mathbb{1}}(x)T_{\mathbb{1}}(x^{-1}) + \frac{1}{|B|}\sum_{x\notin B} T_{\mathbb{1}}(x)T_{\mathbb{1}}(x^{-1}) = \frac{1}{|B|}|B| + 0 = 1$$

and

$$T_{\mathbb{1}} * T_{\mathbb{1}}([s]) = \frac{1}{|B|}\sum_{xy=s, x\in B} T_{\mathbb{1}}(x)T_{\mathbb{1}}(y) + \frac{1}{|B|}\sum_{xy=s, x\notin B} T_{\mathbb{1}}(x)T_{\mathbb{1}}(y)$$

As before, when $x \in B$, $y \notin B$, so the left sum is 0 and when $x \notin B$, the right sum is 0. So far we have shown

$*$	$T_{\mathbb{1}}$	T_s
$T_{\mathbb{1}}$	$T_{\mathbb{1}}$	T_s
T_s	T_s	

and we wish to fill in the final spot. To do this, we note that $T_{\mathbb{1}} + T_s = \mathbb{1}_{B\backslash G/B}$, and for any $\phi \in Fun\Big(B(\mathbb{F}_q)/G(\mathbb{F}_2) \setminus B(\mathbb{F}_q)\Big)$, we have

$$\begin{aligned}\mathbb{1}_{B\backslash G/B} * \phi([g]) &\equiv \frac{1}{|B|}\sum_{xy=g} \mathbb{1}_{B\backslash G/B}(x)\phi(y) \\ &= \frac{1}{|B|}\sum_{xy=g} \phi(y) \\ &= \sum_{y\in G/B} \phi(yB)\end{aligned}$$

In particular, letting $\phi = \mathbb{1}_{B\backslash G/B}$, we have

$$\mathbb{1}_{B\backslash G/B}^2 = |G/B|\,\mathbb{1}_{B\backslash G/B}$$

In the case of $GL_2(\mathbb{F}_q)$, we know the cardinality of $|G/B| = q+1$. To see this, recall $GL_2/B \cong \mathbb{P}^1$, and in the latter there are q^2-1 nonzero elements of $\mathbb{F}_q^2$, and $q-1$ scalars to identify the elements with, yielding $\frac{q^2-1}{q-1} = q+1$. Thus

$$T_{\mathbb{1}} + 2T_s + T_s^2 = (q+1)(T_{\mathbb{1}} + T_s)$$

$$T_s^2 = qT_{\mathbb{1}} + (q-1)T_s$$

which completes the square. In this case, if we set $q = 1$, treating q as a parameter for the size of the underlying field $\mathbb{F}_q$, ie it need not be a prime power, then we have $T_s^2 = T_{\mathbb{1}}$. So when we choose the field with one element, we recover the group algebra of $\mathbb{Z}_2$ itself. In this way, we say that the Hecke algebra is a q-deformation of the group algebra of the Weyl group.

An equivalent way to write the equation we discovered is

$$(T_s+1)(T_s-q) = 0$$

To interpret this, we recall $\mathbb{Z}_2$ has two irreps, the trivial representation and the sign representation (nontrivial element acts by -1). In our $q=1$ limit, we had $(T_s+1)(T_s-1) = 0$, and in the full Hecke algebra we had $(T_s+1)(T_s-q) = 0$. This means that the Hecke algebra has two 1 dimensional irreps, also called trivial and sign, where T_1 always acts by 1 and T_s acts by q and -1, respectively, and thus has a decomposition:

$$Fun\Big(B(\mathbb{F}_q)/GL_2(\mathbb{F}_q) \setminus B(\mathbb{F}_q)\Big) \cong \text{triv} \oplus \text{sgn}$$

so the functions on the flag variety also break up into two pieces:

$$Fun\Big(GL_2(\mathbb{F}_q) \setminus B(\mathbb{F}_q)\Big) \cong \pi_1 \oplus \pi_2$$

There's one easy piece to guess, the subrepresentation given by constant functions[3]. So this is a copy of the field, k, but which piece does this correspond to? We note that k is acted on by the Hecke algebra, and we'd like to know what that action is. We already know what T_1 does, so we need to calculate the action of T_s. Taking $k \cdot \mathbb{1}_{B\backslash G/B}$, we have

$$\begin{gathered} T_s * \mathbb{1}_{B\backslash G/B} \\ = \mathbb{1}_{B\backslash G/B} * \mathbb{1}_{B\backslash G/B} - T_1 * \mathbb{1}_{B\backslash G/B} \\ = (q+1)\mathbb{1}_{B\backslash G/B} - 1 \cdot \mathbb{1}_{B\backslash G/B} = q\mathbb{1}_{B\backslash G/B} \end{gathered}$$

[3]or functions which are constant on orbits.

So the Hecke algebra is acting by the trivial representation, so the subrep of constant functions comes from π_1. And what is the other rep? We have an integration map

$$\int : Fun\Big(G(\mathbb{F}_q)/B(\mathbb{F}_q)\Big) \to k$$

$$\phi \mapsto \sum_{g \in G/B} \phi(g)$$

This map is clearly surjective, with a certain kernel, which must be the other representation. It is known as the Steinberg representation, and we may discuss this in more detail later.

Exercise: *Check directly that π_2 as defined above is irreducible.*

We will now discuss the setting of a general G. Now the Weyl group is a Coxeter group, which is a group indexed by simple reflections in the simple roots of G. So any element of W admits multiple minimal reduced expressions

$$w = s_{i_1} \cdot \cdots \cdot s_{i_n}$$

and we define $n := \ell(w)$ as the length of the word, which it turns out to be well defined.

Lemma: *dim $BwB/B = \ell(w)$ (viewed as an algebraic variety).*

Proof: Above my paygrade at the moment.

Corollary: *Given elements $y, w \in W$ such that $\ell(yw) = \ell(y) + \ell(w)$, then*

$$B \setminus ByB \times^B /BwB \to G$$

factors as an isomorphism

$$B \setminus ByB \times^B /BwB \to B \setminus BywB/B$$

Proof: This proof relies on some strategy used in the above proof, so it must necessarily be skipped as well.

Upshot: We start with simple reflections, $T_s, s \in S$, which correspond to

$$\Big(\vee_{s \in S} \mathbb{P}^1 \to G/B \Big)$$

for any w with a reduced expression $w = s_{i_1} \cdot \cdots \cdot s_{i_\ell}$, we get

$$T_w = T_{s_{i_i}} * \cdots * T_{s_{i_\ell}}$$

Theorem: *$Fun(B(\mathbb{F}_q) \setminus G(\mathbb{F}_q)/B(\mathbb{F}_q))$ admits the following presentation:*

$$\Big\langle T_s \mid (T_s + 1)(T_s - q) = 0, T_s T_t T_s T_t \cdots = T_t T_s T_t \ldots \Big\rangle$$

where the second relation has $m_{s,t} \in \{2, 3, 4, 6\}$ terms on each side.

We already saw where the first relation comes from, and the second relations come from the corollary above.

CHAPTER 4

IV. Categorification, function-sheaf dictionary

Lecture 4, Feb 24.

Until now we've studied

$$Fun(G(\mathbb{F}_q)), \quad Fun\Big(B(\mathbb{F}_q) \setminus G(\mathbb{F}_q)/B(\mathbb{F}_q\Big)$$

Today we will discuss several upgrades/generalizations/context changes to this setting. The first is an association called the function sheaf dictionary, replacing functions with certain types of sheaves, and replacing the $\mathbb{F}_q$ points of G with a variety, $G_{\mathbb{F}_q}$:

$$Shv^{et}(G_{\mathbb{F}_q}) \quad Shv^{et}\Big(B_{\mathbb{F}_q} \setminus G_{\mathbb{F}_q}/B_{\mathbb{F}_q}\Big)$$

The next will be a change of base field, which is pretty self explanatory:

$$Shv^{et}(G_{\mathbb{C}}), \quad Shv^{et}(B_{\mathbb{C}} \setminus G_{\mathbb{C}}/B_{\mathbb{C}})$$

and finally via some version of the Riemann-Hilbert correspondence, we arrive at the goal for this part of the course:

$$D-mod(G_{\mathbb{C}}), \quad D-mod(B_{\mathbb{C}} \setminus G_{\mathbb{C}}/B_{\mathbb{C}})$$

So let's get to the first transition with a discussion of some basic categorification. We start with an abelian group A. For concreteness, we can think of

$$A = Fun\Big(B(\mathbb{F}_q \setminus G(\mathbb{F}_q)/B(\mathbb{F}_q)\Big)$$

A categorification of A is a category $\mathcal{C}$ whose Grothendieck group $K_0(\mathcal{C})$ is isomorphic to A. You may require further properties of $\mathcal{C}$ such as abelian, triangulated, etc.

Let's recall the Grothendieck group construction. Given an abelian category, $\mathcal{C}$, let U be the free abelian group generated by isomorphism classes[1] of objects of $\mathcal{C}$. Note that the multiplication in this free abelian group has nothing to do with the direct sum structure on the category. This group has a subgroup $F(U)$ which is generated by elements

[1]Probably we would like to assume $\mathcal{C}$ is small here.

$[A] - [A'] - [A'']$, whenever there is an exact sequence $0 \to A' \to A \to A'' \to 0$. The Grothendieck group $K_0(\mathcal{C})$ is the quotient group $U/F(U)$. Note that we took the free abelian group, so this quotient is actually a group. In particular, for any A_1, A_2 objects of $\mathcal{C}$, there is always an exact sequence

$$0 \to A_1 \to A_1 \oplus A_2 \to A_2 \to 0$$

Thus in $K_0(\mathcal{C})$,

$$[A_1 \oplus A_2] - [A_1] - [A_2] = 0$$

$$[A_1 \oplus A_2] = [A_1] + [A_2]$$

which relates the biproduct on $\mathcal{C}$ to the formal multiplication law in the Grothendieck group.

Now we denote $Vect_k^{\heartsuit}$ as the abelian category of vector spaces over k, equipped with a direct sum and tensor product[2], and $Vect_k^{\heartsuit,f.d.}$ as the full[3] subcategory of finite dimensional vector spaces. The standard example is the claim that finite dimensional vector spaces categorify the integers. Precisely this means

Lemma:

$$K_0(Vect_K^{\heartsuit,f.d.}) \cong \mathbb{Z}$$

Proof:
Two finite dimensional vector spaces over K are isomorphic iff they have the same dimension, so the equivalence classes in U are determined uniquely by an integer, $[W] = [V] \iff \dim W = \dim V$. Define a group homomorphism:

$$\psi : K_0(Vect_K^{\heartsuit,f.d.}) \to \mathbb{Z}$$

$$[V] \mapsto \dim V$$

This map is easily seen to be well-defined on the generators, and we extend it by imposing linearity. Thus it has the property

$$\psi([V] - [W]) = \dim V - \dim W$$

which is clearly both injective and surjective.

$\square$

For a more involved example, one can try

[2] I don't really understand what the heart means. I think has something to do with a subcategory of a triangulated/derived categorie

[3] A full subcategory is one which you may drop some of the objects, but for any two objects you do have in the subcategory, you must have all of their morphisms. In this case $Vect_k^{\heartsuit,f.d.}$ is clearly a full subcat of $Vect_k^{\heartsuit}$, defined as expected.

Exercise:

$$K_0(FinAbGrp) \cong F_{\{\mathbb{Z}_p\}}$$

where $F_{\{\mathbb{Z}_p\}}$ is the free abelian group generated by $\mathbb{Z}_p$ for p a prime.

Often the properties of the thing being categorified come from properties of the category, and sometimes things are better understood in this light.

From the above example, we know how to add and multiply integers. The addition descends from the direct sum of vector spaces and multiplication descends from the tensor product.

Some notation: We denote $Vect$ without a heart and without k as the triangulated category of chain complexes of vector spaces. So this may contain elements such as

$$\cdots \to k^{\oplus \mathbb{R}} \to k^{\oplus \mathbb{N}} \to k^{\mathbb{N}} \to \ldots$$

So the chain complexes may extend to infinity in both directions, and the actual VS's appearing in each index may be arbitrarily large. We want to restrict this in both ways to end up with something nice: Denote $Vect^{f.d.}$ as the full subcategory of chain complexes with only finitely many nonzero cohomology groups, and with each cohomology group finite dimensional.

In particular, there may be chain complexes in $Vect^{f.d.}$ which have infinitely large terms, but that should be undetectable in the cohomology of the chain complex:l

Example:

$$\bigoplus_{n \in \mathbb{Z}} \left(k^{\oplus \mathbb{R}^{\mathbb{R}}} \to k^{\oplus \mathbb{R}^{\mathbb{R}}} \right) [-n]$$

with the identity map, but cohomology is isomorphic to the 0 vector space.

By a similar claim,

$$K_0(Vect^{f.d.}) \cong \mathbb{Z}$$

where instead of sending a VS to its dimension, we send a chain complex to its Euler characteristic. In particular,

$$\chi(k[0]) = 1, \quad \chi(k[1]) = -1, \quad \chi(k[-1]) = -1, \ldots$$

We can also direct sum and tensor product chain complexes. For the direct sum this is just done termwise, with the direct sum of differentials. Because we are just adding up the differentials, we get $H^i(C \oplus D) \cong H^i(C) \oplus H^i(D)$, so

$$\chi(C \oplus D) = \chi(C) + \chi(D)$$

To tensor product two chain complexes we end up with a bicomplex which consists of all pairwise tensor products, then we direct sum by the total degree:

$$(C \otimes D)^i = \bigoplus_{j+k=i} C^j \otimes D^k$$

and the differential obeys the Leibniz rule:

$$d(v^j \otimes w^k) = d(v^j) \otimes w^k + (-1)^j (v^j \otimes d(w^k))$$

for $v^j \in C^j, w^k \in D^k$. Then

$$H^i(C \otimes D) \cong \bigoplus_{j+k=i} H^j(C) \otimes H^k(D)$$

so the cycles are spanned by simple tensors of cycles. As a result,

$$\begin{aligned} \chi(C \otimes D) &= \sum_i (-1)^i \sum_{j+k=i} \dim H^j(C) \otimes H^k(D) \\ &= \sum_{j,k} (-1)^j \dim H^j(C) \cdot (-1)^k \dim H^k(D) \\ &= \chi(C) \cdot \chi(D) \end{aligned}$$

i.e. we have a ring isomorphism

$$\chi : K_0(Vect^{f.d.}) \to \mathbb{Z}$$

Function-sheaf dictionary Take I (rough pass):
If S is a topological space, then a function on S is an S-family of numbers, in a trivial way. So from our previous discussion, this should categorify to S-families of vector spaces, which is roughly what a sheaf on S is.

Idea of a sheaf of k vector spaces on S: $\mathcal{F}$ on S associates to any open $V \subset S \to \mathcal{F}(U)$, a k-vector space, and for any $V \subset U \subset S$ a map

$$res : \mathcal{F}(U) \to \mathcal{F}(V)$$

When dealing with a topological space[4], we may denote the restriction map with the usual restriction of functions notation. Further, $\mathcal{F}$ must satisfy a gluing property: Given an open cover of V, $V = \cup_\alpha U_\alpha$, and sections $s_\alpha \in \mathcal{F}(U_\alpha)$ which agree on intersections:

$$s_\alpha \Big|_{U_\alpha \cap U_\beta} = s_\beta \Big|_{U_\alpha \cap U_\beta}$$

there exists a global section which restricts to it on each open:

$$\exists s \in \mathcal{F}(U), \text{ s.t. } s\Big|_{U_\alpha} = s_\alpha$$

[4]instead of something more general, like a sheaf on a site

equivalently, that $\mathcal{F}(V) \to \prod_\alpha \mathcal{F}(U_\alpha)$ given by restriction to each component is an equalizer for the maps

$$\prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_\alpha \mathcal{F}(V_\alpha \cap V_\beta)$$

An equalizer for the diagram above is an object and a morphism $eq : X \to \prod_\alpha \mathcal{F}(U_\alpha)$ which makes the diagram commute, and satisfies the expected universal property: For any $\varphi : W \to \prod_\alpha \mathcal{F}(U_\alpha)$ which makes the diagram

$$\begin{tikzcd} X \arrow[r, "eq"] & \prod_\alpha \mathcal{F}(U_\alpha) \arrow[r, shift left] \arrow[r, shift right] & \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta) \\ W \arrow[u, dashed, "\exists!"] \arrow[ur, "\varphi"] & & \end{tikzcd}$$

commute, there exists a unique morphism $W \to X$.

The statement "$\mathcal{F}(V)$ equalizes the diagram above for all open covers $V = \cup_\alpha U_\alpha$" is equivalent to a presheaf $\mathcal{F}$ being a sheaf, for sheaves valued in a more general category. If $\mathcal{F}$ happens to be a sheaf valued in an abelian category, something further is true. From the above, $\mathcal{F}(V)$ will equalize the diagram, but also the diagram will be exact, in a sense that we will describe. First let's see why equalizing is equivalent to the gluing condition: If $\mathcal{F}(V)$ (and the map)is an equalizer for the diagram, then we can take the singleton $\{*\} \to \prod_\alpha \mathcal{F}(U_\alpha)$ such that the diagram commutes. The equalizer diagram commuting implies that

$$\{*\} \to \prod_\alpha \mathcal{F}(U_\alpha) \xrightarrow{\alpha} \prod_\alpha \mathcal{F}(U_\alpha \cap U_\beta)$$

is equal to

$$\{*\} \to \prod_\alpha \mathcal{F}(U_\alpha) \xrightarrow{\beta} \prod_\alpha \mathcal{F}(U_\alpha \cap U_\beta)$$

The singleton condition is vacuous, so we see that choosing a map $\{*\}$ is equivalent to choosing a section on each U_α which agrees on intersections. By the universal property, this map must factor

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "res"] & \prod_\alpha \mathcal{F}(U_\alpha) \\ \{*\} \arrow[u] \arrow[ur] & \end{tikzcd}$$

which implies that any element in the image of $\{*\}$ has a preimage in $\mathcal{F}(V)$ under the restriction map. In other words, there is some element, given by the image of $\{*\}$ under the vertical map, which restricts to the chosen sections on each open, i.e. they have been glued together, so equalizing $\Rightarrow$ sheaf condition, and the global section itself is unique because the induced map from the universal property is unique. This is already taking up quite some time/space, so maybe I will leave the other direction as an exercise for the reader.

There is also a notion of exactness of the diagram

$$\mathcal{F}(V) \longrightarrow \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

when working with an abelian category. I think the right thing to say is that the diagram considering the difference of the parallel morphisms

$$0 \longrightarrow \mathcal{F}(V) \xrightarrow{res} \prod_\alpha \mathcal{F}(U_\alpha) \xrightarrow{\alpha-\beta} \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact. The claim is that exactness here is also equivalent to the sheaf condition. This one is easier to see than before. We only need to check that

$$Ker(\alpha - \beta) = Im(res)$$

is equivalent to the sheaf condition. This is clear because an element of the kernel is exactly a choice of section on each U_α which agrees on intersections, and it being in the image of *res* implies it can be glued together to a global section. This establishes a bijection between global sections and sections which agree on overlaps, as desired.

Examples: Let S be a real smooth manifold. There are three examples of sheaves on S:

$\underline{\mathbb{R}}$, the sheaf of locally constant real-valued functions on S: For an open set $U \subset S$, $\underline{\mathbb{R}}(U)$ is the set of locally constant functions on U.
If U is connected, this is just constant functions on U, which is isomorphic to $\mathbb{R}$. One may ask why to consider locally constant instead of just constant. It is because the presheaf of globally constant functions is not a sheaf. For example let S be the disjoint union of two circles, and consider the presheaf of real valued constant functions on S, $\mathcal{F}$. Choose a section on the first circle to be the constant function with value 1 and a section on the second circle to be the section with constant value 0. These sections agree on overlap (the overlap is empty), and thus they must glue together if $\mathcal{F}$ is to be a sheaf. But an element of $\mathcal{F}(S)$ is supposed to be a constant function on S. Therefore it cannot restrict to the appropriate sections because it would have to take two different values in S, and thus is not constant[5]. The sheafification of the constant presheaf is the sheaf of locally constant functions.

$C^0(\mathbb{R})$, the sheaf of continuous functions on S, similarly defined.

$C^\infty(\mathbb{R})$, the sheaf of smooth functions on S.

Because each sheaf actually has functions as its sections, the restriction maps are just given by restriction of functions.

[5]Note we did not have to necessarily choose the space S to be a disconnected space. This fails to be a sheaf even for connected spaces, like $\mathbb{R}^n$. This is because we can just choose two open sets which are disjoint in $\mathbb{R}^n$, then make the same argument as above.

Given a topological space S and a field k, there is a category of sheaves on vector spaces over K, whose objects are sheaves on S and the morphisms are natural transformations between these sheaves. To unravel this definition, note that the "assignment" that a sheaf makes of vector spaces to open sets can be packaged as the data of a contravariant[6] functor $\mathcal{F} : Op(S) \to Vect_k$, where $Op(S)$ is the category whose objects consist of open sets of S and morphisms are given by inclusions. In fact many would take this as the definition of a sheaf of vector spaces over k on S. Then of course a morphism between two functors is just a natural transformation. Denote this category as $Shv_k^{\heartsuit}(S)$.

Given a continuous map $f : S_1 \to S_2$, we can move sheaves on each space to a sheaf on the other space:

$$f_* : Shv_K^{\heartsuit}(S_1) \to Shv_k^{\heartsuit}(S_2)$$

defined by, for any open $U \subset S_2$, $f_*(\mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$, which type checks because $f^{-1}(U)$ is an open subset of S_1, since f is cts.

Example: Given $s \in S$, there is an inclusion $i : \{*\} \to S$, so we can push forward:

$$i_* : Shv_k^{\heartsuit}(\{*\}) \to Shv_k^{\heartsuit}(S)$$

Proposition: *There is an isomorphism of categories*

$$Shv_k^{\heartsuit}(\{*\}) \cong Vect_k^{\heartsuit}$$

For this, we will need a lemma. We'll state and prove the general version, but we need only a specific case of this.
Lemma: *For any sheaf over a topological space X, valued in a category $\mathcal{C}$, $\mathcal{F}(\emptyset)$ is a terminal object.*
First let's note that when $\mathcal{C} = Vect_k$, this implies that for any sheaf, $\mathcal{F}(\emptyset) = \{0\}$, the 0 vector space.
Proof: Because $\mathcal{F}$ is a sheaf, it satisfies the equalizer condition (note since we are not assuming an abelian category, we cannot use the exactness condition): For every open cover of an open set $U = \cup_\alpha U_\alpha$,

$$\mathcal{F}(U) \longrightarrow \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

Letting $U = \emptyset$ and α having indexing set $\emptyset$, we recall that the empty product results in a terminal object, which is unique up to unique isomorphism. Denote the terminal object in $\mathcal{C}$ as T. Then the equalizer becomes

$$\mathcal{F}(\emptyset) \longrightarrow T \rightrightarrows T$$

and note that the parallel morphisms must be the identity, by the definition of a terminal object. Also because T is terminal, for any other object in $\mathcal{C}$, X, there is a unique morphism

[6]Contravariant because for every inclusion $U \to V$, the restriction map goes in the other direction, $res : \mathcal{F}(V) \to \mathcal{F}(U)$

to T:

$$\begin{tikzcd}
\mathcal{F}(\varnothing) \arrow[r] & T \arrow[r, shift left] \arrow[r, shift right] & T \\
X \arrow[ur] & &
\end{tikzcd}$$

and note that the diagram trivially commutes. Thus by the universal property of the equalizer, there is a unique map

$$\begin{tikzcd}
\mathcal{F}(\varnothing) \arrow[r] & T \arrow[r, shift left] \arrow[r, shift right] & T \\
X \arrow[u, dashed, "\exists!"] \arrow[ur] & &
\end{tikzcd}$$

making the diagram commute. So every object X has a unique morphism to $\mathcal{F}(\varnothing)$ which makes the diagram commute. But note that *any* morphism $X \to \mathcal{F}(\varnothing)$ makes the above diagram commute, and thus must be equal to $\exists!$. So the "makes the diagram commute" condition is vacuous, and thus every $X \in \mathcal{C}$ has a unique morphism to $\mathcal{F}(\varnothing)$, so $\mathcal{F}(\varnothing)$ is the terminal object in $\mathcal{C}$.

$\square$

Proof of the proposition:
Define the functor

$$\psi : Shv_k^{\heartsuit}(\{*\}) \to Vect_k^{\heartsuit}$$

by sending

$$\mathcal{F} \mapsto \mathcal{F}(\{*\})$$

and for a natural transformation $h : \mathcal{F}_1 \to \mathcal{F}_2$,

$$h \mapsto \Big(h_{\{*\}} : \mathcal{F}_1(\{*\}) \to \mathcal{F}_2(\{*\}) \Big)$$

and define a functor $\varphi : Vect_k^{\heartsuit} \to Shv_k^{\heartsuit}(\{*\})$ by sending

$$V \mapsto \varphi(V)$$

where $\varphi(V)(\{*\}) = V$ and $\varphi(V)(\varnothing) = \{0\}$. We also need to say what φ does to morphisms. Luckily there is only one

$$(\varnothing \to \{*\}) \mapsto \Big(0 : \varphi(V)(\{*\}) \equiv V \to \varphi(V)(\varnothing) = 0 \Big)$$

Then we need to say what φ does to morphisms:

$$(V_1 \to V_2) \mapsto \ell : \varphi(V_1) \to \varphi(V_2)$$

where ℓ is the natural transformation:

$$\ell_{\{*\}} : \varphi(V_1)(\{*\}) = V_1 \to \varphi(V_2)(\{*\}) = V_2 \text{ so define } := V_1 \to V_2$$

$$\ell_\varnothing : \varphi(V_1)(\varnothing) \to \varphi(V_2)(\varnothing) := 0 \to 0$$

Note that we did not need the lemma we proved for this part. Now we just check compositions:

$$V \mapsto \varphi(V) \mapsto \varphi(V)(\{*\}) \equiv V$$

and

$$\mathcal{F} \mapsto \mathcal{F}(\{*\}) \mapsto \varphi(\mathcal{F}(\{*\}))$$

where $\varphi(\mathcal{F}(\{*\}))(\{*\}) = \mathcal{F}(\{*\})$, and $\varphi(\mathcal{F}(\{*\}))(\varnothing) = \{0\}$, and we want to check that $\mathcal{F} = \varphi(\mathcal{F}(\{*\}))$ as sheaves. They agree on $\{*\}$, and by the lemma they agree on $\varnothing$, so they are equal on objects. But these are functors, so we need to check that they agree on morphisms too. There is only one:

$$\varphi(\mathcal{F}(\{*\}))(\varnothing \to \{*\}) : \varphi(\mathcal{F}(\{*\}))(\{*\}) \to \varphi(\mathcal{F}(\{*\}))(\varnothing)$$

$$= \mathcal{F}(\{*\}) \to 0 = \mathcal{F}(\varnothing \to \{*\})$$

So $\psi \circ \varphi$ and $\varphi \circ \psi$ agree with the appropriate identity functor on objects. We need to also check they agree on morphisms: for a natural transformation

$$(\mathcal{F} \to \mathcal{H}) \mapsto \left(h_{\{*\}} : \mathcal{F}(\{*\}) \to \mathcal{H}(\{*\})\right) \mapsto \ell : \psi(\mathcal{F}(\{*\})) \to \psi(\mathcal{F}(\{*\}))$$

and we want to check that $\ell = \mathcal{F} \to \mathcal{H}$. We have

$$\ell_{\{*\}} = \psi($$

Now we can return to the example and ask what is the push forward map under inclusion of a point? It sends (we can consider its domain as $Vect_k$ due to the previous proposition)

$$V \mapsto i_* V(U) = \begin{cases} V & * \in U \\ 0 & \text{else} \end{cases}$$

known as the skyscraper sheaf on S over $*$, a sheaf which is "supported over a single point".

The skyscraper sheaf is our analog of an indicator function from the function world. To make this precise, note that the stalk over every point is 0 except for $\{*\}$, where the stalk is V.

We denote[7] $Shv_k(S)$ as the unbounded derived[8] category of $Shv_k^\heartsuit(S)$.

[7] At a certain point Gurbir started using *Sh* instead of *Shv*. I think this was just a clerical mistake, but if there is some content to this then I missed it.

[8] i don't really feel comfortable with derived cats yet so I won't be doing much outside of what's being said in the lectures with derived/triangulated things.

Fact: The functor $f_* : Shv_k^\heartsuit(S_1) \to Shv_k^\heartsuit(S_2)$ prolongs to a right derived functor

$$Rf_* =: f_* : Shv_k(S_1) \to Shv_k(S_2)$$

Fact: If S_1, S_2 reasonable, and $S_1 \to S_2$ is a proper map, then

$$f_* : Shv_k(S_1) \to Shv_k(S_2)$$

admits a right adjoint[9]

$$f^! : Shv_k(S_2) \to Shv_k(S_1)$$

Example: $i : \{*\} \to S$. Then

$$i_* : Vect_k \to Shv_k(S)$$

is an adjunction, with adjoint $i^!$. At the level of the abelian categories, we also have the adjunction

$$i_* : Vect_k^\heartsuit \to Shv_k^\heartsuit(S)$$

also with adjoint pair $i^!$, where, denoting $i(*) \equiv s$,

$$i^!(\mathcal{F}) = \{\text{sections of } \mathcal{F} \text{ supported at } s\}$$

i.e. for any $U \ni s$, consider the exact sequence,

$$0 \longrightarrow K_U \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}(U \setminus s)$$

where K_U is the thing that makes this exact, i.e. sections on U which vanish on $U \setminus s$, so that

$$i^! \mathcal{F} \cong \varinjlim_{U \ res} K_U$$

and the full adjunction comes by prolonging the abelian category adjunction.

We're going to use this example later so let's prove it[10]:

Proposition: *If $i : \{*\} \to S$ is an inclusion, then i_* and $i^!$ defined above are an adjoint pair between $Shv_k^\heartsuit(\{*\})$ and $Shv_k^\heartsuit(S)$.*

Proof:
We want to show, for every $\mathcal{F} \in Shv_k^\heartsuit(\{*\})$ and $\mathcal{G} \in Shv_k^\heartsuit(S)$,

$$Hom_{Shv_k^\heartsuit(S)}(i_*\mathcal{F}, \mathcal{G}) \cong Hom_{Shv_k^\heartsuit(\{*\})}(\mathcal{F}, i^!\mathcal{G})$$

[9] Is there a precise description of this adjoint?

[10] I can't prove the more general case of a map $S_1 \to S_2$ because idk how to work with $i^!$. Maybe the approach there is to show that i_* admits a right adjoint through adjoint functor theorem and denote $i^!$ as such.

Given a sheaf morphism $h : i_*\mathcal{F} \to \mathcal{G}$, we want to define a linear map (from our previous lemma) from $\mathcal{F}(*) \to i^!\mathcal{G}(*)$. From the sheaf morphism we are given for every $U \in Op(S)$, a linear map

$$h_U : i_*\mathcal{F}(U) \to \mathcal{G}(U)$$

satisfying the naturality condition. In particular, for every $U \ni s$, we have that

$$h_{U\setminus\{s\}} = 0$$

since $i_*\mathcal{F}(U \setminus \{s\}) = 0$, so the image of this map actually lands in $K_U \subset \mathcal{G}(U)$:

$$h_U : i_*\mathcal{F}(U) \equiv \mathcal{F}(s) \to ker\Big(\mathcal{G}(U) \to \mathcal{G}(U \setminus \{s\})\Big)$$

To see this, note that for any $v \in F(s)$, by the naturality of h,

$$res_{U,U\setminus\{s\}}(h_U(v)) = h_{U\setminus\{s\}}(res_{U,U\setminus\{s\}}(v)) = 0$$

Then taking the limit over all such U yields the morphism $\mathcal{F} \to i^!\mathcal{G}$. In summary,

$$(h : i_*\mathcal{F} \to \mathcal{G}) \mapsto \left(\varinjlim_U h_U : \mathcal{F}(s) \to Ker\Big(\mathcal{G}(U) \to \mathcal{G}(U \setminus \{s\}\Big)\right)$$

The above is technically an abuse of notation. For a direct limit, you are supposed to write the objects in the argument of the $\varinjlim$, but here it is more important to keep track of the maps.

prove this is an isomorphism. Injectivity may be easy due to universal property of limit. .

$\square$

Remark: You can replace $\{*\} \to S$ by any closed $Z \subset S$, and the above example applies mutatis mutandis.

CHAPTER 5

V. Function-sheaf Dictionary Continued

Lecture 5, March 3.

Recall from last time, given a point in S, we had an adjoint pair $i_*, i^!$ between $Shv(\{*\})$ and $Shv(S)$. In particular, given $\mathcal{F} \in Shv_k^{\heartsuit}(S)$, we can associate to it the collection of vector spaces

$$i_s^!\mathcal{F}, s \in S$$

and this is the promised family of vector spaces over S we were looking for in the last lecture.

Example: If S is an oriented, smooth real manifold, the constant sheaf $\underline{\mathbb{C}}$ is associated to the family

$$i_s^!\mathbb{C}_s \cong \mathbb{C}[-\dim S]$$

This cohomological shift is very related to the shift in Poincare duality.

Recall for a map of finite sets $f : X \to Y$, we defined two operations $f_* : Fun(X) \to Fun(Y)$ and $f^! : Fun(Y) \to Fun(X)$, given by integration over fibers and pullback, respectively. For a map of topological spaces $f : X \to Y$, we defined the analogous maps $f_* : Shv(X) \to Shv(Y)$ and $f^! : Shv(Y) \to Shv(X)$. Recall that these maps aren't adjoint in general. It holds under certain conditions, and we proved one special case of that last time.

Basic properties: Given $x \in X$,

$$i_x^! f^! \mathcal{F} \cong i_{f(x)}^! \mathcal{F}$$

i.e. pullback does what it should on shriek stalks. This is because we have the commutative diagram

$$\begin{tikzcd} x \arrow[r, "i_x"] \arrow[d, "f"] & X \arrow[d, "f"] \\ f(x) \arrow[r, "i_{f(x)}"] & Y \end{tikzcd}$$

inducing the diagram[1]

$$\begin{tikzcd} Shv_k^\heartsuit(\{x\}) & Shv_k^\heartsuit(X) \arrow[l, "i_x^!"'] \\ Shv_k^\heartsuit(\{f(x)\}) \arrow[u, "f^!"] & Shv_k^\heartsuit(Y) \arrow[l, "i_{f(x)}^!"'] \arrow[u, "f^!"] \end{tikzcd}$$

Example: For projection $\pi : X \to \{*\}$, we have

$$\pi^!\mathbb{C} =: \omega_X$$

is called the dualizing sheaf. There is the basic property

$$i_x^! \omega_X \cong \mathbb{C}$$

Remark: If X is a smooth oriented real manifold, then

$$\mathbb{C} \cong \omega_X[-\dim_X]$$

For the pushforward, recall that for finite sets, $\pi : X \to \{*\}$

$$\pi_*\mathcal{F} = \text{derived global sections of } \mathcal{F}$$

Example 1: If $\mathcal{F} = \mathbb{C}$, then $\pi_*\mathcal{F} = H^*(X, \mathbb{C})$ e.g. for H^0, we get $\prod_{\pi_0(X)} \mathbb{C}$.

Example 2: I $\mathcal{F} = \omega_X$, then $\pi_*\mathcal{F} \cong H^*(X, \mathbb{C})^\vee$, known as the Borel-Moore homology.

Exercise: Deduce for a smooth, oriented X, Poincare duality.

Fact: Given $f : X \to Y$, the following diagram commutes, when considering certain types of sheaves, which we will discuss later, known as constructible:

$$\begin{tikzcd} Shv(F_y) \arrow[d, "f_*"] & Shv(X) \arrow[l, "i^!"'] \arrow[d, "f_*"] \\ Shv(\{y\}) & Shv(Y) \arrow[l, "i_y^!"'] \end{tikzcd}$$

i.e. the shriek-fibers of $f_*\mathcal{F}$ store the cohomology of $\mathcal{F}$ along the fibers of the map f.

Remark: In general

$$\begin{tikzcd} X \times_y Z \arrow[d, "\bar{f}"] & X \arrow[l, "\bar{i}"'] \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y \end{tikzcd}$$

[1]Here we are suppressing the left vertical map, since it is trivial.

induces

$$\begin{tikzcd} Shv(X \times_y Z) \arrow[d, "\bar{f}_*"] & Shv(X) \arrow[l, "\bar{i}^!"'] \arrow[d, "f_*"] \\ Shv(Z) & Shv(Y) \arrow[l, "i^!"'] \end{tikzcd}$$

known as the base change for shriek pullbacks.

There is a notion of multiplication of "constructible" sheaves: Given $\mathcal{F}, \mathcal{G} \in Shv^{con}(X)$, one can define $\mathcal{F} \otimes^! \mathcal{G}$, and

$$i_x^!(\mathcal{F} \otimes^! \mathcal{G}) \cong (i_x^! \mathcal{F}) \otimes (i_x^! \mathcal{G})$$

Observation: The direct sum of two sheaves has the property $i_x^!(\mathcal{F} \oplus \mathcal{G}) \cong i_x^! \mathcal{F} \oplus i_x^! \mathcal{G}$.

We are beginning to see a function sheaf dictionary: For functions we could add, multiply, push forward and pull back, and we have just discussed how to extend these operations to sheaves.

The finite sets we were previously looking at were things of the form $Fun(G(\mathbb{F}_q))$. If we want to promote this to a sheaf, we don't want to consider $Shv(G(\mathbb{F}_q))$, since this is discrete so there is no geometry. Instead we want to consider $Shv(G_{\mathbb{F}_q})$.

So we need a notion of constant sheaves, etc. for varieties over fields which are not $\mathbb{C}$. The answer is going to be to use the etale sheaves, and associated constructible derived category of etale sheaves: $Shv_{et}^{con}(X_{\mathbb{F}_q})$.

Consider a local system[2] on $\mathbb{C}^*$. Choose some point and consider the monodromy about the origin. This is equivalent[3] to a representation of $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$, i.e. you have one interesting monodromy corresponding to winding around the puncture one time counterclockwise.

Example: Consider the Mobius strip with monodromy -1 going around the loop once, with the local system $z^{1/2}$, defined[4]as follows: Let $f(z) = z^2$, and consider the push forward $f_*\mathbb{C}$. This sheaf splits as a direct sum $f_*\mathbb{C} \cong \mathbb{C} \oplus z^{1/2}$. This is locally but not globally constant, and can be trivialized on any analytic ball, but cannot be trivialized on any nonempty Zariski open set.

So the Zariski topology is not good for studying local systems. In this case, $f^!(z^{1/2})$ is trivial, however.

[2]A local system is just a locally constant sheaf.

[3]The correct statement is in terms of an equivalence of categories between the cat of local systems and the cat of representations.

[4]This definition is slightly opaque, but it was the best I could find with a quick google search.

i.e. if we allow ourselves to talk about triviality on certain (etale) covers, we can get somewhere.

Etale sheaves are roughly sheaves with sections who have good descent properties, i.e. you can glue sections. More specifically, given an etale cover[5] $U \to X$, we have an exact sequence

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_X U)$$

Exercise: Think about the square root sheaf in the context of the double cover.

Then constructible means that the sheaf lies in the full triangulated subcategory generated by local sheaves (local in the etale topology) under !-pullback and π_*-pushforward.

So we don't know exactly what constructible etale sheaves are, but we've got some idea of what the category $Shv_{et}^{con}(X_{\mathbb{F}_q})$ looks like. So we can ask about decategorification

$$Shv_{et}^{con}(X_{\mathbb{F}_q}) \to Fun(X(\mathbb{F}_q))$$

where the right hand side is a finite set. From the function-sheaf dictionary discussion, the map should be

$$\mathcal{F} \mapsto \left(x \mapsto \chi(i_x^! \mathcal{F})\right)$$

and we can ask that the decategorification interacts nicely with the operations we constructed on both sides. For addition and multiplication, this does hold, due to the properties we stated without proof of direct sum and multiplication. Similarly the fact we showed about pullback on shriek stalks matches the function theoretic pullback. However for $*$-pushforward there is a problem:

Example: $X = \mathbb{P}^1_{\mathbb{F}_q} \to \{*\}$. Question: Does

$$\chi(\pi_* \mathcal{F}) = \sum_{x \in \mathbb{P}^1} \chi(i_x^! \mathcal{F})$$

the term on the left comes from using the sheaf pushforward and the right comes from the function push forward, so we want these to be equal. Take $\mathcal{F} = \mathbb{Q}_X$, the constant sheaf.

Then the LHS is $\pi_* \mathbb{Q}_\ell = R\Gamma_{et}(\mathbb{P}^1_{\mathbb{F}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell[-2](1)^{hFr}$. I'm not really sure what this stuff means, not even sure I transcribed it corretly, but the upshot is that on the LHS we have 2 1-dimensional vector spaces, whereas on the RHS we have $q+1$ 1-dimensional VS's, so they cannot be equal.

Recall that $|\mathbb{P}^1_{\mathbb{F}_q}| = q+1$ coming from an $\mathbb{A}^0$ which has 1 point and $\mathbb{A}^1$ which has q points. Similarly, $H^* \mathbb{P}^1_{\mathbb{F}_q} = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell[-2]$, where the left term represents the class of the

[5] I don't think we defined an etale cover and the wiki page is too hard for me to understand since I don't know what a scheme is, but on stack exchange someone said an etale cover btw complex projective varieties for example, is a regular surjective morphism which is a covering map in the usual sense when the two varieties are given the complex topology.

whole thing and the right thing represents the class of the point. The cohomology is determined by the cellular decomposition, so the idea is we want an alternative decategorification that remembers more about the geometry of the cohomology classes.

So for X over $\mathbb{F}_q$, and a sheaf $\mathcal{F}$ on X, you have the Frobenius map

$$Fr : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$$

and the pullback of $\mathcal{F}$ will be Frobenius-equivariant, essentially by definition.

Analogy: Given a normal covering $\widetilde{C} \to C$ of top spaces, i.e. $Aut_C(\widetilde{C})$ acts simply transitively on fibers of $\widetilde{C}$ over C. Then we have that local systems on C are equivalent to $Aut_C(\widetilde{C})$ equivariant local systems on $\widetilde{C}$.

Example: Choosing $Spec\overline{\mathbb{F}}_q \to Spec\mathbb{F}_q$ then local systems on $Spec\mathbb{F}_q$ are equivalent to $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-equivariant local systems on $Spec\overline{\mathbb{F}}_q \cong Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ representations on $\mathbb{Q}_\ell$ vector spaces.

CHAPTER 6

VI. Function-sheaf Dictionary Finished, Categorical Group Algebra

Lecture 6, March 10.

Recapping last time: If X is a variety over $\mathbb{C}$, we discussed the category $Shv(X)$, and the pullback and push forward on sheaves as a generalization of the operations for finite sets.

There is a map $Shv^b_{f.d.}(X) \to Fun(X_{\mathbb{C}})$, and really on the left side we mean sheaves all of whose !-stalks have finitely many cohomologies each finite dimensional, so dualizing sheaves, constant sheaves, and things you can construct out of these. The map is defined as

$$\mathcal{F} \mapsto \Big(x \mapsto \chi(i_x^! \mathcal{F})\Big)$$

e.g.

$$\omega_X \mapsto \mathbb{1}_X$$

We denote this map as dimension, and this map commutes with the operations plus, times, and pullback on sheaves and functions, but the push forward had some severe problems. In particular because $X_{\mathbb{C}}$ can be typically non-finite, so we can't sum over fibers. To get to finitely many points, we pass from working with complex algebraic varieties to varieties over $\mathbb{F}_q$:

$$Shv^{con}(X_{\mathbb{C}}) \to Shv^{con}_{et}(X_{\mathbb{F}_q}).$$

though I believe we did not rigorously define these categories, and the arrow above is an association, not a map or functor.

Now, is it true that if we have a map between two $\mathbb{F}_q$ varieties, X and Y, then there a diagram

$$\begin{array}{ccc}
Shv^{con}_{et}(X_{\mathbb{F}_q}) & \xrightarrow{dim} & Fun(X(\mathbb{F}_q)) \\
\big\downarrow \pi_* & & \big\downarrow \pi_* \\
Shv^{con}_{et}(Y_{\mathbb{F}_q}) & \xrightarrow{dim} & Fun(Y(\mathbb{F}_q))
\end{array}$$

The good news is that now, after moving to $\mathbb{F}_q$ varieties, the map on the right is actually defined. The bad news is that the answer is still no.

Example[1]**:** Take $\mathbb{P}^1_{\mathbb{F}_q} \to Spec\ \mathbb{F}_q$, and the dualizing sheaf on $\mathbb{P}^1_{\mathbb{F}_q}$. Then following the diagram by first going to the right, we have $\omega \mapsto \mathbb{1}_{\mathbb{P}^1_{\mathbb{F}_q}} \mapsto \sum_{x \in \mathbb{P}^1(\mathbb{F}_q)} = q+1$. If we start by going down, we have $\omega \mapsto \pi_* \omega \in Shv_{et}(Spec\ \mathbb{F}_q)$. But what is this category exactly? A wrong but reasonable guess is, in complex geometry, sheaves over a point is equivalent to vector spaces. We may guess that $Spec\ \mathbb{F}_q$ is a point, so probably we would get Vect again, but this is not true. What is true is that if you take k a separably closed, then

$$Shv_{et}(Spec\ k) \cong Vect$$

for example choosing $k = \mathbb{C}$. Why is this? An etale cover $X \to Spec\ k$ is just a finite union of copies of $Spec\ k$, and a connected cover is just $Spec\ k$.

For a general field F, an etale cover of F, $X \to F$ is a disjoint union $\sqcup Spec\ F_i \to F$, but F_i does not necessarily have to be isomorphic to F. All we can say is that $F \to F_i$ is a finite separable extension. Gurbir gave some idea as to why this was true, but I didn't follow it, so one has to consult his notes to find out why.

We are supposed to think of separable extensions as being analogous to covering spaces in topology. In particular, the fundamental group of the base space is analogous to the Galois group of the extension. So the correct statement is that

$$Shv_{et}(Spec\ F) \cong Rep^{cont}(Gal(F^S/F))$$

where continuous means that the action of the Galois group factors through the Galois group of all finite extensions.

Informal explanation: we have $Spec\ F^S \to Spec\ F$ where the left term is actually a point, so the universal cover is contractible, i.e.[2] $Spec\ F \cong K(\pi, 1) \cong B(Gal(F^S/F))$, the classifying space.

For us, we were considering

$$Shv_{et}(Spec\ \mathbb{F}_q) \cong Rep(Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)) \cong Rep(\hat{\mathbb{Z}})$$

[1]Some weirdness happening here. The set underlying the scheme $Spec\ \mathbb{F}_q$ is just the one point space, since fields only have two ideals, 0 and the whole field, and spectrum only takes prime ideals, which must be proper. So the dualizing sheaf makes sense because we are pulling back from the inclusion of a point. But a scheme is not just a set, it is also locally ringed. There are two open sets of a one point space, $\emptyset$ and the point. In an analogue of the lemma we proved a while ago about sheaves, we know that 0 must be assigned to the trivial ring. The point is assigned the field itself. So this is a point which comes equipped with the data of the field which it was defined over. So not totally trivial, still a bit mysterious to me about its role.

[2]I'm familiar with $K(G, n)$ where G is a group, but here we have a map π, which I'm assuming is the projection map. From wiki, the definition is as follows: Given a map $a : G \to G'$, $K(a, n) = \{[f] \mid f : K(G, n) \to K(G', n), H_n(f) = a\}$.

i.e. for each $n \in \mathbb{Z}^{\geq 1}$, there is a unique extension of degree n, and the Galois group of the extension is $\mathbb{Z}_n$, generated by $(x \mapsto x^q)$, the Frobenius map. For the full thing:

$$\begin{aligned} Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q) &\cong \varprojlim_n Gal(\mathbb{F}_{q^n}/\mathbb{F}_q) \\ &\cong \varprojlim \mathbb{Z}/n\mathbb{Z} \cong \hat{\mathbb{Z}} \cong \prod_\ell \mathbb{Z}_\ell \end{aligned}$$

where ℓ are primes, and the final isomorphism follows from CRT.

So $\hat{\mathbb{Z}}$ is topologically generated by the Frobenius element 1, and so $\hat{\mathbb{Z}}$ is roughly like $\mathbb{Z}$, and so $Spec(\mathbb{F}_q) \cong B\mathbb{Z} \cong S^1$.

So by construction we have the diagram:

$$\begin{tikzcd} Shv_{et}(Spec\ \overline{\mathbb{F}}_q) \arrow[r, "\cong"] & Vect \\ Shv_{et}(Spec\ \mathbb{F}_q) \arrow[u, "i^!"] \arrow[r, "\cong"] & Rep(\hat{\mathbb{Z}}) \arrow[u, "Oblv"'] \end{tikzcd}$$

Now let's apply this to the case we were studying:

$$\begin{tikzcd} Shv_{et}^{con}\left(\mathbb{P}^1_{\mathbb{F}_q} \times_{Spec\ \mathbb{F}_q} Spec\ \overline{\mathbb{F}}_q\right) \cong \mathbb{P}^1_{\overline{\mathbb{F}}_q} \arrow[d, "\pi_*"] & & Shv_{et}^{con}(\mathbb{P}^1_{\mathbb{F}_q}) \arrow[ll, "i^!"'] \arrow[d, "\pi_*"] \\ Shv_{et}(Spec\ \overline{\mathbb{F}}_q) & & Shv_{et}(Spec\ \mathbb{F}_q) \cong Rep(\hat{\mathbb{Z}}) \arrow[ll, "i^!"'] \end{tikzcd}$$

So the underlying vector space of $\pi_*(\omega_{\mathbb{P}^1}) = H^*_{et}(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \omega) \cong H^*_{et}(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell[2]) \cong \mathbb{Q}_\ell[2] \oplus \mathbb{Q}_\ell$, in degree -2 and 0, respectively, so its dimension is 2, not $q+1$.

So to recap: We wanted the two circuits in our diagram to be equal, but just checking dimensions: on one side, we have the number of points of $\mathbb{P}^1(\mathbb{F}_q) = q+1$, corresponding to the cell decomposition. On the other side, we have $H^*_{et}(\mathbb{P}^1_{\overline{\mathbb{F}}_q}) = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell[-2]$, and these are both contributing one, whereas to get them to match, we want the first term to contribute q, which corresponds to the class of all of $\mathbb{P}^1$.

So our dimension map is not getting the job done. To fix this, the idea is that the dimension of a VS is the trace of its identity morphism. Instead, we could consider taking the trace of some other endomorphism. One obvious candidate is the monodromy around the loop, the Frobenius map. So we take

$$Shv_{et}^{con}(X_{\mathbb{F}_q}) \to Fun(X(\mathbb{F}_q))$$

$$\mathcal{F} \mapsto \left\{ (x : Spec\, \mathbb{F}_q \to X) \mapsto Tr(Fr, i_x^! \mathcal{F}) \right\}$$

and denote this map by FF, because of something in French.

Theorem (Lefschetz fixed pt formula): *Given $\pi : X_{\mathbb{F}_q} \to Y_{\mathbb{F}_q}$, the following diagram commutes:*

$$\begin{tikzcd}
Shv_{et}^{con}(X_{\mathbb{F}_q}) \arrow[r, "FF"] \arrow[d, "\pi_*"] & Fun(X(\mathbb{F}_q)) \arrow[d, "\pi_*"] \\
Shv_{et}^{con}(Y_{\mathbb{F}_q}) \arrow[r, "FF"] & Fun(Y(\mathbb{F}_q))
\end{tikzcd}$$

Example: If we take $Tr(Fr, H^*(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) \cong \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell[-2] = H^0 \oplus H^2$. On H^0, Fr acts as identity (points go to points), so the trace contributes a 1. On H^2: Let's recall that if M, N are compact, oriented n-folds, and $f : M \to N$ is a smooth map, then the elements of the top dimensional cohomology correspond to points on the manifold, by Ponicare Duality. $f^* : H^n(N) \to H^n(M)$ sends

$$n \mapsto \sum f^{-1}(n)$$

so f^* calculates the degree of the map.

For us, we have $\mathbb{P}^1_{\overline{\mathbb{F}}_q} \to \mathbb{P}^1_{\overline{\mathbb{F}}_q}$ sends $[x : y] \mapsto [x^q : y^q]$ this is a degree q map, so

$$Tr(Fr, H^2(\mathbb{P}^1_{\overline{\mathbb{F}}_q})) = q$$

so we have fixed our problem, or rather seen an instance of the problem being fixed, and the claim is that this is what we should do in general. FF is the "right" operation for the function-sheaf dictionary, since it behaves nicely with all operations, including $*$-pushforward.

Now let's see some examples of this dictionary:

Example: The first will be the group algebra of the rational points[3] of a split, reductive group, $G(\mathbb{F}_q)$. So let $G_{\mathbb{F}_q}$ be any algebraic group over $\mathbb{F}_q$.
Of course, the rational points $G(\mathbb{F}_q)$ must form a finite group. Then we consider the group algebra

$$\overline{\mathbb{Q}}_\ell[(G(\mathbb{F}_q)] \cong Fun(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$$

[3] I think we've used this term a couple times already, but a k-rational point of a variety X is a point which belongs to $k^n \cap X$, as opposed to an algebraic closure $K \supset k$. In particular if k is algebraically closed, there is no difference between X and its k-rational points, denoted as $X(k)$. If X is an algebraic group then is $X(k)$ a group? An algebraic group? It's not obvious to me that multiplication should keep me in the base field and not send me out into the algebraic closure, but if the multiplication rule is not "weird" then it does seem like this should be the case, but if you're handed something completely arbitrary i'm not sure how to conclude that. Nevertheless it appears the answer is yes.

and multiplication, comultiplication, and the antipode map came from push forwards in geometry. The multiplication (and the isomorphism) we discussed in the first lecture, it came from pushing forward along the multiplication map on $G(\mathbb{F}_q)$.
For comultiplication, we always can just send $g \mapsto g \otimes g$ in the group algebra. On functions we also just push forward along the inclusion into the diagonal $\Delta : G(\mathbb{F}_q) \to G(\mathbb{F}_q) \times G(\mathbb{F}_q)$. In the group algebra, this corresponds to the notion that we can always tensor reps.
For the antipode map, in the group algebra we send $g \mapsto g^{-1}$, so on functions we push forward along inversion. So the group algebra is in fact a Hopf algebra.

In fact, the same formulas make sense for sheaves. Explicitly,

$$Shv_{et}^{con}(G_{\mathbb{F}_q})$$

is a monoidal (and even Hopf) triangulated category.

To understand the multiplication (convolution), we examine the diagram:

$$\begin{tikzcd}
Shv_{et}^{con}(G_{\mathbb{F}_q}) \otimes Shv_{et}^{con}(G_{\mathbb{F}_q}) \arrow[r] \arrow[d, "\cong"'] & Shv_{et}^{con}(G_{\mathbb{F}_q}) \\
Shv_{et}^{con}(G_{\mathbb{F}_q} \times G_{\mathbb{F}_q}) \arrow[ur, "m_*"] &
\end{tikzcd}$$

Of course, this does not actually make sense: what does it even mean to take the tensor product of two categories? Something very close is true, though:

$$\begin{tikzcd}
Shv_{et}^{con}(G_{\mathbb{F}_q}) \times Shv_{et}^{con}(G_{\mathbb{F}_q}) \arrow[r] \arrow[d] & Shv_{et}^{con}(G_{\mathbb{F}_q}) \\
Shv_{et}^{con}(G_{\mathbb{F}_q} \times G_{\mathbb{F}_q}) \arrow[ur, "m_*"] &
\end{tikzcd}$$

given by

$$\begin{tikzcd}
(\mathcal{F}, \mathcal{G}) \arrow[r, mapsto] \arrow[d, mapsto] & m_*\Big(\pi_1^!(\mathcal{F}) \otimes^! \pi_2^!(\mathcal{G})\Big) \\
\pi_1^!(\mathcal{F}) \otimes^! \pi_2^!(\mathcal{G}) \arrow[ur, mapsto] &
\end{tikzcd}$$

where π_i represents the projection onto the ith component.

We could have written this definition of the convolution down before discussing the function-sheaf dictionary, but now this has more motivation.

Example: In the group algebra, recall that the multiplication of two elements from the group itself is just equal to their product in the group. In the language of functions, this means that

$$m_*(\mathbb{1}_g \boxtimes \mathbb{1}_h) = \mathbb{1}_{gh}$$

this is an identity in $Fun(G(\mathbb{F}_q))$, and the above are delta functions, those supported at a single point.

In the categorical group algebra, we associate

$$g \in G(\mathbb{F}_q) \rightsquigarrow \delta_g$$

where δ_g is a skyscraper sheaf at g.

So skyscraper sheaves correspond to the group elements in the group algebra. In particular, δ_e is the monoidal unit.

We think of a general element of

$$Shv_{et}^{con}(G_{\mathbb{F}_q})$$

as a "linear combination of δ-sheaves". Of course for functions, this is literally true. For sheaves this doesn't make sense, but we think of a sheaf as a continuous collection of vector spaces parameterized by the space (the shriek stalks), and somehow the sheaf is made by gluing these skyscrapers together.

Remark: The same definition makes sense for any field.

CHAPTER 7

VII. Artin comparison, introduction to D-modules

Lecture 7, March 17.

Previously, we finished discussing the function-sheaf dictionary, and we discussed the key example of going between

$$Fun(G(\mathbb{F}_q)) \rightsquigarrow Shv^{con}_{et}(G_{\mathbb{F}_q})$$

and because functions formed a Hopf algebra under convolution, diagonals, and inversion, the category of sheaves had the same structure as well.

And we remarked that the statement above works for any field. Today we want to consider $k = \mathbb{C}$.

Outline for today: We want to start with $X_{\mathbb{C}}$ a complex variety, and consider $Shv^{con}(X_{\mathbb{C}})$, not etale. So sheaves with respect to the analytic topology[1] on the analytification $X^{an}_{\mathbb{C}}$.

I hadn't heard of analytification before so let's take a detour to discuss: The goal of analytification is to define a functor from smooth[2] complex varieties to complex manifolds. On objects, given a smooth complex variety $X_{\mathbb{C}}$, we need a complex manifold X^{an}. As a set, we define $X^{an} := X_{\mathbb{C}}$. Now we need to equip it with a smooth complex structure, starting with a topology.
In the case that $X_{\mathbb{C}}$ is affine, we have $X_{\mathbb{C}} \cong Z(P) \subset \mathbb{A}^n$ for some n, with P an ideal in $\mathbb{C}[z_1, \dots, z_n]$. Equip $\mathbb{A}^n = \mathbb{C}^n$ with the Euclidean topology, and let X^{an} inherit the subspace topology.
But what if we chose a different isomorphism and different ideal $X_{\mathbb{C}} \to Z(P')$?

Lemma: *X^{an} is well defined (up to isomorphism) under choice of isomorphism and ideal.*

[1] Generated by open balls

[2] We could drop the smooth requirement and we would get a functor into more general spaces known as complex analytic spaces. As may be expected, requiring smoothness of $X_{\mathbb{C}}$ guarantees that X^{an} will have no singularities.

Proof: Such a choice would induce an isomorphism of affine varieties, $\varphi : Z(P) \to Z(P')$

$$\begin{tikzcd} X_{\mathbb{C}} \arrow[r, "\cong"] \arrow[d, "\cong"'] & Z(P) \subset \mathbb{A}^n \arrow[dl, "\varphi"] \\ Z(P') \subset \mathbb{A}^m & \end{tikzcd}$$

We want an isomorphism of topological spaces equipped with the Euclidean topology. Take a Euclidean closed set $V \subset Z(P')$. This means there is some W open in $\mathbb{A}^m$ such that $V = W \cap Z(P')$. Then

$$\varphi^{-1}(V) = \varphi^{-1}\big(W \cap Z(P')\big) = \varphi^{-1}(W) \cap \varphi^{-1}(Z(P')) = \varphi^{-1}(W) \cap Z(P)$$

But the RHS is Zariski closed, since φ is continuous in the Zariski topology. Closed in Zariski implies closed in the Euclidean topology, so $\varphi^{-1}(V)$ is Euclidean closed, so φ is Euclidean continuous. Same argument for φ^{-1}. Thus φ is a homeomorphism wrt the Euclidean topology.

$\square$

Any variety X is covered by affine varieties, and on overlaps $X_i \cap X_j$, the spaces X_i^{an} and X_j^{an} are homeomorphic, by the above lemma. Thus this assigns a global topology on the whole variety X.

Now we need a smooth structure, and here it will become necessary that $X_{\mathbb{C}}$ is smooth. In introductory differential geometry, people usually meet smooth manifolds by defining an atlas with smooth transition functions. It is equivalent to specifying a sheaf of smooth functions, making X into a ringed space.

In particular for any $p \in X_{\mathbb{C}}$, say of dimension d, there is a neighborhood $U \ni p$ with $U \cong Z(f_1, \ldots, f_{n-d}) \subset \mathbb{A}^n$ for some n and f_i such that the rank of the $n - d \times n$ complex matrix

$$\left(\frac{\partial f_i}{\partial x_j}(\varphi(p)) \right)_{ij}$$

is $n - d$. Reorder the basis so that the upper left square $n - d \times n - d$ is invertible. Then by the implicit function theorem for holomorphic functions, there exist Euclidean open $V_{\varphi(p)}$ of $\varphi(p)$ in $\mathbb{A}^n$, a Euclidean open $W \subset \mathbb{C}^d$, and holomorphic functions $w_1, \ldots, w_{n-d} : W \to \mathbb{C}$ inducing a homeomorphism

$$\psi : W \to V_{\varphi(p)} \cap Z(f_1, \ldots, f_{n-d})$$

sending

$$(w_1(z), \ldots, w_{n-d}(z), z_{n-d+1}, \ldots, z_n)$$

Now we can define the sheaf. Let V' be an open neighborhood of p contained in $\varphi^{-1}V_{\varphi(p)}$. Define a function $f : V' \to \mathbb{C}$ to be holomorphic iff $\psi^{-1}V' \to V' \to \mathbb{C}$ is holomorphic.

One can check that this does truly define a sheaf, and it is well defined under the many choices we made, and this is in fact true, making X^{an} into a complex manifold.

If $f : X \to Y$ is a morphism of complex varieties, then it is a rational function without poles, which must be holomorphic, so there is not much to do here. Thus we have defined the functor, which completes this discussion of analytification.

So today we seek to establish the fully faithful embeddings of categories:

$$Shv_{et}^{con}(X_{\mathbb{C}}) \hookrightarrow Shv^{con}(X_{\mathbb{C}}^{an}) \hookrightarrow D-mod(X_{\mathbb{C}})$$

The first functor is known as Artin comparison. In essence, recall that all of this etale topology business came up because we needed to talk about sheaves over varieties which were not over the complex numbers. Thus in the case of a complex variety, there should be a way to sidestep all this machinery, and that's what Artin comparison does. The second functor is known as the Riemann-Hilbert correspondence, which will be a more involved analogue of associating to a system of ODE's the monodromy of its solutions.

Theorem (Artin): *Fix a prime ℓ. There is a canonical fully faithful embedding*

$$Shv_{et}^{con}(X_{\mathbb{C}}) \to Shv^{con}\left(X_{\mathbb{C}}^{an}, \overline{\mathbb{Q}}_\ell - \text{coefficients}\right)$$

Example: Let's look at local systems on both sides. On the RHS, if we fix an n, then rank n local systems are classified by homomorphisms $\pi_1(X) \to GL_n(\overline{\mathbb{Q}}_\ell)$.
On the LHS, local systems correspond to continuous homomorphisms $\pi_1^{et}(X) \to GL_n(\overline{\mathbb{Q}}_\ell)$. Explicitly, $\pi_1^{et}(X) \cong$ the profinite completion of $\pi_1(X)$ wrt finite quotients. From this, we can say they are in fact classified by continuous homomorphisms $\pi_1(X) \to GL_n(\overline{\mathbb{Q}}_\ell)$.

Example: Let $X = \mathbb{G}_m$, with $n = 1$. Then a monodromy of an etale local system has to lie in $\overline{\mathbb{Z}}_\ell^\times$, whereas the monodromy of any local system lies in $\overline{\mathbb{Q}}_\ell^\times$.

Example: $\omega_X \mapsto 1$, constant sheaves go to constant sheaves, and Artin comparison commutes with integrating and pulling back sheaves, π_* and $\pi^!$. For more details on Artin comparison, see the notes or book of J.S. Milne on Etale cohomology.

Now we want to get to algebraic D-modules. Idea: If you have a sheaf of complex vector spaces on $X_{\mathbb{C}}$, then you can scale your local sections by locally constant functions, since they form a vector space. If you have a quasi-coherent sheaf on $X_{\mathbb{C}}$, then you can scale local sections by any polynomial functions, instead of locally constant. Then a D-module on $X_{\mathbb{C}}$ is a thing where you can scale local sections by polynomial functions, and you can differentiate sections by polynomial vector fields.