Internet Appendix: Supplemental Proofs

Notes on the Proof of Lemma 2. Division managers determine $\left(\hat{q}^d_d, \hat{q}^d_{d'}\right)$ in (17). We will focus on two cases: we start with the case where $\gamma_d \geq 0$, and then we consider the case $\gamma_d < 0$. Consider $\tilde{q}^d_d = q_d + \delta$, for $\delta > 0$. Switching to $\tilde{q}^{d-}_d = q_d - \delta$ lowers $\hat{u}_d$ by $2\beta_d a_d \delta$ while leaving the constraint unchanged. Therefore, it must be that $\hat{q}^d_d \leq q_d$. Similarly, switching from $\tilde{q}^d_{d'} = q_{d'} + \delta$, for $\delta > 0$ to $\tilde{q}^{d-}_{d'} = q_{d'} - \delta$ lowers $\hat{u}_d$ by $2\gamma_d a_{d'} \delta$, leaving the constraint unchanged. Therefore, it must also be that $\hat{q}^d_{d'} \leq q_{d'}$. Thus, we can express the Lagrangian as

$$\mathcal{L} \equiv -\hat{u}_d - \lambda\left[g_c - \eta^d\right] - \tau_d\left(\hat{q}^d_d - q_d\right) - \tau_{d'}\left(\hat{q}^d_{d'} - q_{d'}\right) \tag{B1}$$

where $g_c \equiv \ln \frac{q_d}{\hat{q}^d_d} + \ln \frac{q_{d'}}{\hat{q}^d_{d'}}$. Because problem (17) admits corner solutions, we characterize its solution by use of the full Kuhn-Tucker conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \hat{q}^d_d} &= -\frac{\partial \hat{u}_d}{\partial \hat{q}^d_d} - \lambda \frac{\partial g_c}{\partial \hat{q}^d_d} - \tau_d = -\beta_d a_d + \frac{\lambda}{\hat{q}^d_d} - \tau_d = 0, \\
\frac{\partial \mathcal{L}}{\partial \hat{q}^d_{d'}} &= -\frac{\partial \hat{u}_d}{\partial \hat{q}^d_{d'}} - \lambda \frac{\partial g_c}{\partial \hat{q}^d_{d'}} - \tau_{d'} = -\gamma_d a_{d'} + \frac{\lambda}{\hat{q}^d_{d'}} - \tau_{d'} = 0, \\
& \lambda\left(g_c - \eta^d\right) + \tau_d\left(\hat{q}^d_d - q_d\right) + \tau_{d'}\left(\hat{q}^d_{d'} - q_{d'}\right) = 0, \\
\lambda &\geq 0,\ \tau_{d'} \geq 0,\ \tau_d \geq 0,\ \eta^d - g_c \geq 0,\ q_d - \hat{q}^d_d \geq 0,\ q_{d'} - \hat{q}^d_{d'} \geq 0.
\end{aligned} \tag{B2}$$

From the definition of g_c, to satisfy the constraint $\eta^d - g_c \geq 0$ it must be $\hat{q}^d_d > 0$ and $\hat{q}_{d'} > 0$, which implies that $\frac{\partial \mathcal{L}}{\partial \hat{q}^d_d} = \frac{\partial \mathcal{L}}{\partial \hat{q}^d_{d'}} = 0$. Also, $\beta_d a_d > 0$ implies $\lambda > 0$, and thus that $g_c - \eta^d = 0$. In addition, it cannot be that both $\tau_d > 0$ and $\tau_{d'} > 0$ because, if so, then $\hat{q}^d_d = q_d$ and $\hat{q}^d_{d'} = q_{d'}$, which would imply that $g_c = 0 < \eta^d$, which contradicts $\lambda > 0$. This leaves us with three types of solutions: $\tau_d = \tau_{d'} = 0$, $\tau_d > 0 = \tau_{d'}$, and $\tau_d = 0 < \tau_{d'}$.

If $\tau_d = \tau_{d'} = 0$, then we have the case in the main appendix: $\frac{\partial \mathcal{L}}{\partial \hat{q}^d_d} = \frac{\partial \mathcal{L}}{\partial \hat{q}^d_{d'}} = 0$ together imply that $\lambda = \beta_d a_d \hat{q}^d_d$ and $\lambda = \gamma_d a_{d'} \hat{q}^d_{d'}$, giving $\beta_d a_d \hat{q}^d_d = \gamma_d a_{d'} \hat{q}^d_{d'}$. Because $g_c = \eta^d$ implies that $\hat{q}^d_d \hat{q}^d_{d'} = e^{-\eta^d} q_d q_{d'}$, after substitution this implies that $\frac{\beta_d a_d}{\gamma_d a_{d'}}\left(\hat{q}^d_d\right)^2 = e^{-\eta^d} q_d q_{d'}$, or equivalently, $\hat{q}^d_d = \left[e^{-\eta^d} H_d\right]^{\frac{1}{2}} q_d$, where $H_d = \frac{\gamma_d a_{d'} q_{d'}}{\beta_d a_d q_d}$. Similarly, $\hat{q}^d_{d'} = \left[e^{-\eta^d} \frac{1}{H_d}\right]^{\frac{1}{2}} q_{d'}$. In order for this to be feasible, however, it must be that $\hat{q}^d_d \leq q_d$, or equivalently, $H_d \leq e^{\eta^d}$, and $\hat{q}^d_{d'} \leq q_{d'}$, or equivalently, $H_d \geq e^{-\eta^d}$, giving case (ii) when $\gamma_d > 0$. If $\tau_d > 0 = \tau_{d'}$, then $\hat{q}^d_d = q_d$ and, from $g_c = \eta^d$, also $\hat{q}^d_{d'} = e^{-\eta^d} q_{d'}$. Note that $\frac{\partial \mathcal{L}}{\partial \hat{q}^d_{d'}} = 0$ implies that $\lambda = \gamma_d a_{d'} e^{-\eta^d} q_{d'}$ and, from $\frac{\partial \mathcal{L}}{\partial \hat{q}^d_d} = 0$, we have that

$$\tau_d = -\beta_d a_d + \frac{\gamma_d a_{d'} e^{-\eta^d} q_{d'}}{q_d} = \beta_d a_d \left(H_d e^{-\eta^d} - 1\right) > 0, \tag{B3}$$

which requires $H_d > e^{\eta^d}$, giving case (i) when $\gamma_d > 0$. Finally, if $\tau_d = 0 < \tau_{d'}$, then $\hat{q}^d_{d'} = q_{d'}$ and, from $g_c = \eta^d$, also $\hat{q}^d_d = e^{-\eta^d} q_d$. Note that now $\frac{\partial \mathcal{L}}{\partial \hat{q}^d_d} = 0$ implies that $\lambda = \beta_d a_d e^{-\eta^d} q_d$, and, from $\frac{\partial \mathcal{L}}{\partial \hat{q}^d_{d'}} = 0$, we have that

$$\tau_{d'} = -\gamma_d a_{d'} + \frac{\beta_d a_d e^{-\eta^d} q_d}{q_{d'}} = \gamma_d a_{d'} \left(H_d^{-1} e^{-\eta^d} - 1\right) \geq 0, \tag{B4}$$

which requires $0 \leq H_d < e^{-\eta^d}$, giving case (iii) when $\gamma_d \geq 0$.

The case with $\gamma_d < 0$ proceeds similarly, noting that $\hat{q}^d_d \leq q_d$ but $\hat{q}^d_{d'} \geq q_{d'}$. Thus, we can express the Lagrangian as

$$\mathcal{L} \equiv -\hat{u}_d - \lambda\left[g_c - \eta^d\right] - \tau_d\left(\hat{q}^d_d - q_d\right) - \tau_{d'}\left(q_{d'} - \hat{q}^d_{d'}\right) \tag{B5}$$

where $g_c \equiv \ln \frac{q_d}{\hat{q}_d^d} + \ln \frac{q_{d'}}{2q_{d'} - \hat{q}_{d'}^d}$. Again, the full Kuhn-Tucker conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} &= -\frac{\partial \hat{u}_d}{\partial \hat{q}_d^d} - \lambda \frac{\partial g_c}{\partial \hat{q}_d^d} - \tau_d = -\beta_d a_d + \frac{\lambda}{\hat{q}_d^d} - \tau_d = 0, \\
\frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} &= -\frac{\partial \hat{u}_d}{\partial \hat{q}_{d'}^d} - \lambda \frac{\partial g_c}{\partial \hat{q}_{d'}^d} + \tau_{d'} = -\gamma_d a_{d'} - \frac{\lambda}{2q_{d'} - \hat{q}_{d'}^d} + \tau_{d'} = 0, \\
& \lambda\left(g_c - \eta^d\right) + \tau_d\left(\hat{q}_d^d - q_d\right) + \tau_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) = 0, \\
\lambda &\geq 0,\ \tau_{d'} \geq 0,\ \tau_d \geq 0,\ \eta^d - g_c \geq 0,\ q_d - \hat{q}_d^d \geq 0,\ q_{d'} - \hat{q}_{d'}^d \geq 0.
\end{aligned} \tag{B6}$$

From the definition of g_c, to satisfy the constraint $\eta^d - g_c \geq 0$ it must be $\hat{q}_d^d > 0$ and $\hat{q}_{d'} > 0$, which implies that $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = \frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$. Also, $\beta_d a_d > 0$ implies $\lambda > 0$, and thus that $g_c - \eta^d = 0$. In addition, it cannot be that both $\tau_d > 0$ and $\tau_{d'} > 0$ because, if so, then $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = q_{d'}$, which would imply that $g_c = 0 < \eta^d$, which contradicts $\lambda > 0$. This leaves us with three types of solutions: $\tau_d = \tau_{d'} = 0$, $\tau_d > 0 = \tau_{d'}$, and $\tau_d = 0 < \tau_{d'}$.

If $\tau_d = \tau_{d'} = 0$, $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = \frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$ together imply that $\lambda = \beta_d a_d \hat{q}_d^d$ and $\lambda = |\gamma_d|\, a_{d'}\left[2q_{d'} - \hat{q}_{d'}^d\right]$, giving $\beta_d a_d \hat{q}_d^d = |\gamma_d|\, a_{d'}\left[2q_{d'} - \hat{q}_{d'}^d\right]$. Because $g_c = \eta^d$ implies that $\hat{q}_d^d\left[2q_{d'} - \hat{q}_{d'}^d\right] = e^{-\eta^d} q_d q_{d'}$, after substitution this implies that $\frac{\beta_d a_d}{|\gamma_d| a_{d'}}\left(\hat{q}_d^d\right)^2 = e^{-\eta^d} q_d q_{d'}$, or equivalently, $\hat{q}_d^d = \left[e^{-\eta^d} H_d\right]^{\frac{1}{2}} q_d$, where $H_d = \frac{|\gamma_d| a_{d'} q_{d'}}{\beta_d a_d q_d}$. Similarly, $\hat{q}_{d'}^d = \left[2 - \left(e^{-\eta^d} \frac{1}{H_d}\right)^{\frac{1}{2}}\right] q_{d'}$. In order for this to be feasible, however, it must be that $\hat{q}_d^d \leq q_d$, or equivalently, $H_d \leq e^{\eta^d}$, and $\hat{q}_{d'}^d \geq q_{d'}$, or equivalently, $H_d \geq e^{-\eta^d}$, giving case (ii) when $\gamma_d < 0$. Alternatively, if $\tau_d > 0 = \tau_{d'}$, then $\hat{q}_d^d = q_d$ and, from $g_c = \eta^d$, also $\hat{q}_{d'}^d = \left(2 - e^{-\eta^d}\right) q_{d'}$. Note that $\frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$ implies that $\lambda = |\gamma_d|\, a_{d'} e^{-\eta^d} q_{d'}$ and, from $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = 0$, we have that

$$\tau_d = -\beta_d a_d + \frac{|\gamma_d|\, a_{d'}}{q_d} e^{-\eta^d} q_{d'} = \beta_d a_d\left[H_d e^{-\eta^d} - 1\right] > 0, \tag{B7}$$

which requires $H_d > e^{\eta^d}$, giving case (i) when $\gamma_d < 0$. Finally, if $\tau_d = 0 < \tau_{d'}$, then $\hat{q}_{d'}^d = q_{d'}$ and, from $g_c = \eta^d$, also $\hat{q}_d^d = e^{-\eta^d} q_d$. Note that now $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = 0$ implies that $\lambda = \beta_d a_d e^{-\eta^d} q_d$, and, from $\frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$, we have that

$$\tau_{d'} = -\gamma_d a_{d'} + \frac{\beta_d a_d e^{-\eta^d} q_d}{q_{d'}} = |\gamma_d|\, a_{d'}\left(\frac{1}{H_d} e^{-\eta^d} - 1\right) \geq 0, \tag{B8}$$

which requires $0 < H_d \leq e^{-\eta^d}$, giving case (iii) when $\gamma_d < 0$. ■

Proof of Lemma 3. The lemma is shown in two steps. First, we obtain division managers' best response functions, $a_d = \theta_d \beta_d \hat{q}_d^d$. Second, we characterize the Nash equilibrium in terms of $\log(a_d)$ and we apply the contraction mapping theorem, proving uniqueness.

Division manager $d \in \{A, B\}$ chooses effort level a_d to solve (19) by setting

$$\frac{d}{da_d} \hat{u}_d(a, \hat{q}_d^d(a, w)) = \frac{\partial \hat{u}_d}{\partial a_d} + \frac{\partial \hat{u}_d}{\partial \hat{q}_d^d} \frac{\partial \hat{q}_d^d}{\partial a_d} + \frac{\partial \hat{u}_d}{\partial \hat{q}_{d'}^d} \frac{\partial \hat{q}_{d'}^d}{\partial a_d} = \frac{\partial \hat{u}_d}{\partial a_d} = 0, \tag{B9}$$

where the second equality holds by the envelope theorem, as follows. For case (ii) of Lemma 2, we have that $\frac{\partial \hat{u}_d}{\partial \hat{q}_d^d} = \lambda \frac{\partial g}{\partial \hat{q}_d}$ and $\frac{\partial \hat{u}_d}{\partial \hat{q}_{d'}^d} = \lambda \frac{\partial g}{\partial \hat{q}_{d'}}$, giving

$$\frac{\partial \hat{u}_d}{\partial \hat{q}_d^d} \frac{\partial \hat{q}_d^d}{\partial a_d} + \frac{\partial \hat{u}_d}{\partial \hat{q}_{d'}^d} \frac{\partial \hat{q}_{d'}^d}{\partial a_d} = \lambda\left(\frac{\partial g}{\partial \hat{q}_d} \frac{\partial \hat{q}_d^d}{\partial a_d} + \frac{\partial g}{\partial \hat{q}_{d'}} \frac{\partial \hat{q}_{d'}^d}{\partial a_d}\right) = \lambda \frac{dg}{da_d} = 0 \tag{B10}$$

because $g = e^{-\eta^d}$. In cases (i) & (iii), $\hat{q}_d^d$ and $\hat{q}_{d'}^d$ do not depend on a_d, $\frac{\partial \hat{q}_d^d}{\partial a_d} = \frac{\partial \hat{q}_{d'}^d}{\partial a_d} = 0$, so $\frac{d\hat{u}_d}{da_d} = \frac{\partial \hat{u}_d}{\partial a_d} = \beta_d \hat{q}_d^d - \frac{a_d}{\theta_d} = 0$.

Thus, the best response functions are $a_d = \theta_d \beta_d \hat{q}_d^d$, where beliefs $\hat{q}_d^d$ are from Lemma 2. If $\gamma_d = 0$, we have that $H_d = 0$, giving $a_d = \theta_d \beta_d e^{-\eta^d} q_d$. If $\gamma_d \neq 0$, the best response depends on the effort by the other division manager, $a_{d'}$. If the other division manager, $d' \neq d$, exerts low effort $a_{d'} < a_{d'}^L \equiv \frac{\theta_d \beta_d^2 e^{-2\eta^d} q_d^2}{|\gamma_d| q_{d'}}$, we have that $H_d < e^{-\eta^d}$ and division manager d holds pessimistic belief as in case (i) of Lemma 2, $\hat{q}_d^d = e^{-\eta^d} q_d$, giving $a_d = a_d^{1*} \equiv \theta_d \beta_d e^{-\eta^d} q_d$. If division manager d' exerts moderate level of effort, $a_{d'}^L \leq a_{d'} < a_{d'}^H \equiv \frac{\theta_d \beta_d^2 e^{\eta^d} q_d^2}{|\gamma_d| q_{d'}}$, division manager d hold beliefs as in case (ii) of Lemma 2; thus $H_d \in \left[e^{-\eta^d}, e^{\eta^d}\right]$ and $a_d = \left[\theta_d^2 \left|\gamma_d\right| a_{d'} \beta_d e^{-\eta^d} q_{d'} q_d\right]^{\frac{1}{3}}$. Finally, if division manager d' exerts a high level of effort, $a_{d'} > a_{d'}^H$, division manager d hold beliefs as in case (iii) of Lemma 2; thus $|H_d| > e^{\eta^d}$ and $a_d = \theta_d \beta_d q_d$. The best response function for DM d is therefore given by

$$a_d^*\left(a_{d'}\right) = \begin{cases} a_d^{1*} \equiv \theta_d \beta_d e^{-\eta^d} q_d & a_{d'} < a_{d'}^L \\ \tilde{a}_d^*\left(a_{d'}\right) \equiv \left[\theta_d^2 \left|\gamma_d\right| a_{d'} \beta_d e^{-\eta^d} q_{d'} q_d\right]^{\frac{1}{3}} & a_{d'}^L \leq a_{d'} \leq a_{d'}^H \\ a_d^{2*} \equiv \theta_d \beta_d q_d & a_{d'} > a_{d'}^H \end{cases}. \tag{B11}$$

A Nash equilibrium is a pair $\{a_A, a_B\}$ such that $a_d = a_d^*\left(a_{d'}\right)$, $d \in \{A, B\}$, $d \neq d'$. Note that $a_d^*\left(a_{d'}\right)$ is a positive, continuous, and increasing function of $a_{d'}$. Expressing the best response in logs, we obtain

$$\ln a_d^*\left(\ln a_{d'}\right) = \begin{cases} \ln \theta_d \beta_d e^{-\eta^d} q_d & \ln a_{d'} < \ln a_{d'}^L \\ \ln \left[\theta_d^2 \left|\gamma_d\right| \beta_d e^{-\eta^d} q_{d'} q_d\right]^{\frac{1}{3}} + \frac{1}{3}\ln\left(a_{d'}\right) & \ln a_{d'}^L \leq \ln a_{d'} \leq \ln a_{d'}^H \\ \ln \theta_d \beta_d q_d & \ln a_{d'} > \ln a_{d'}^H \end{cases}. \tag{B12}$$

Further, note $\frac{d \ln a_d^*}{d \ln a_{d'}} = 0$ for $a_{d'} < a_{d'}^L$ and $a_{d'} > a_{d'}^H$, while $\frac{d \ln a_d^*}{d \ln a_{d'}} = \frac{1}{3}$ for $a_{d'}^L < a_{d'} < a_{d'}^H$. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ so that $F \equiv \left(\ln a_A^*\left(\ln a_B\right), \ln a_B^*\left(\ln a_A\right)\right)'$, and let $d(x, y)$ be the Euclidean distance. For $x, y \in \mathbb{R}^2$, define $\tilde{x}_d \equiv \max\left\{\ln a_d^L, \min\left\{x_d, \ln a_d^H\right\}\right\}$ and $\tilde{y}_d \equiv \max\left\{\ln a_d^L, \min\left\{y_d, \ln a_d^H\right\}\right\}$, we have

$$\begin{aligned} d\left(F(x), F(y)\right) &= \sqrt{\left(\ln a_A^*\left(x_B\right) - \ln a_A^*\left(y_B\right)\right)^2 + \left(\ln a_B^*\left(x_A\right) - \ln a_B^*\left(y_A\right)\right)^2} \\ &= \sqrt{\left(\ln a_A^*\left(\tilde{x}_B\right) - \ln a_A^*\left(\tilde{y}_B\right)\right)^2 + \left(\ln a_B^*\left(\tilde{x}_A\right) - \ln a_B^*\left(\tilde{y}_A\right)\right)^2} \\ &= \sqrt{\left[\frac{1}{3}\left(\tilde{x}_B - \tilde{y}_B\right)\right]^2 + \left[\frac{1}{3}\left(\tilde{x}_A - \tilde{y}_A\right)\right]^2} = \frac{1}{3} d\left(\tilde{x}, \tilde{y}\right) \leq \frac{1}{3} d(x, y), \end{aligned} \tag{B13}$$

which implies that $0 \leq d\left(F(x), F(y)\right) \leq \frac{1}{3} d(x, y)$ for all $x, y \in \mathbb{R}^2$. Thus, F is a contraction mapping and the Nash Equilibrium exists and is unique.

Because the best-response function is constant if d' exerts low effort, $a_{d'} < a_{d'}^L$, and if d' exerts high effort, $a_{d'} > a_{d'}^H$, the Nash Equilibrium is fully determined. All that remains to be determined is the Nash Equilibrium effort for d when $a_{d'}^L \leq a_{d'} \leq a_{d'}^H$. There are three possible cases:

(1) If $a_{d'} = a_{d'}^{1*} > a_{d'}^L$, so that $H_{d'} \leq e^{-\eta^{d'}}$, then

$$a_d = \tilde{a}_d^*\left(a_{d'}^{1*}\right) = \left[\theta_d^2 \theta_{d'} e^{-\left(\eta^d + \eta^{d'}\right)} \left|\gamma_d\right| \beta_{d'} \beta_d q_{d'}^2 q_d\right]^{\frac{1}{3}}; \tag{B14}$$

(2) If $a_{d'} = a_{d'}^{2*} < a_{d'}^H$, so that $H_{d'} \geq e^{\eta^{d'}}$, then

$$a_d = \tilde{a}_d^*\left(a_{d'}^{2*}\right) = \left[\theta_d^2 \theta_{d'} e^{-\eta^d} \left|\gamma_d\right| \beta_{d'} \beta_d q_{d'}^2 q_d\right]^{\frac{1}{3}}; \tag{B15}$$

(3) if $a_{d'}^{1*} < a_{d'} < a_{d'}^{2*}$, so that $H_{d'} \in \left(e^{-\eta^{d'}}, e^{\eta^{d'}}\right)$, then setting $a_d = \tilde{a}_d^*\left(a_{d'}\right)$ and $a_{d'} = \tilde{a}_{d'}^*\left(a_d\right)$, after solving we obtain

$$a_d = \check{a}_d \equiv \left[e^{-\eta^d} \theta_d^2 \beta_d \left|\gamma_d\right|\right]^{\frac{3}{8}} \left[e^{-\eta^{d'}} \theta_{d'}^2 \beta_{d'} \left|\gamma_{d'}\right|\right]^{\frac{1}{8}} \left[q_d q_{d'}\right]^{\frac{1}{2}}. \tag{B16}$$

Comparative statics follow by differentiation. ■

Notes on the Proof of Theorem 1. The proof in the body found the optimal contract when $\gamma_d > 0$. We will show here that the objective is symmetric around zero, completing the proof. Note that, from Lemma 2, $\hat{q}_d^d$ depends on γ_d only through its absolute value, $|\gamma_d|$. Thus, from Lemma 3, equilibrium action $a_d = \beta_d \theta_d \hat{q}_d^d$ also depends on $|\gamma_d|$ only. This implies the first term of the uncertainty discount, $\beta_d a_d \left(q_d - \hat{q}_d^d\right)$, depends only on $|\gamma_d|$. We next show that, if $\gamma_d < 0$, the second term of the uncertainty discount, $\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right)$, is unchanged by offering cross pay, $|\gamma_d|$, rather than relative performance evaluation, $\gamma_d < 0$. From Lemma 2, let $\hat{q}_{d'}^{d+}$ be the belief held by the DM when receiving $|\gamma_d|$ instead of $\gamma_d < 0$. We will show $\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) = \left|\gamma_d\right| a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right)$. Consider in turn cases (i), (ii) and (iii) in Lemma 2.

First, in case (i), if $H_d < e^{-\eta^d}$, then $\hat{q}_{d'}^{d+} = \hat{q}_{d'}^{d} = q_{d'}$, so

$$|\gamma_d|\, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right) = \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d}\right) = 0. \tag{B17}$$

In case (ii), if $H_d \in \left(e^{-\eta^d}, e^{\eta^d}\right)$, then $\hat{q}_{d'}^{d} = \left(2 - \left[e^{-\eta^d} \frac{\beta_d a_d q_d}{|\gamma_d| a_{d'} q_{d'}}\right]^{\frac{1}{2}}\right) q_{d'}$, giving

$$\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d}\right) = \gamma_d a_d \left(\left[\frac{e^{-\eta^d}\beta_d a_d q_d}{|\gamma_d|\, a_{d'} q_{d'}}\right]^{\frac{1}{2}} - 1\right) q_{d'} = |\gamma_d|\, a_{d'} \left(1 - \left[\frac{e^{-\eta^d}\beta_d a_d q_d}{|\gamma_d|\, a_{d'} q_{d'}}\right]^{\frac{1}{2}}\right) q_{d'}. \tag{B18}$$

This implies that replacing γ_d with $|\gamma_d|$, beliefs will remain in case (ii), with $\hat{q}_{d'}^{d+} = \left[e^{-\eta^d} \frac{\beta_d a_d q_d}{|\gamma_d| a_{d'} q_{d'}}\right]^{\frac{1}{2}} q_{d'}$. Thus, we obtain

$$|\gamma_d|\, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right) = \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d}\right). \tag{B19}$$

Finally, in case (iii) with $H_d > e^{\eta^d}$ and $\hat{q}_{d'}^{d} = \left(2 - e^{-\eta^d}\right) q_{d'}$, if HQ replaces γ_d with $|\gamma_d|$, beliefs will be $\hat{q}_{d'}^{d+} = e^{-\eta^d} q_{d'}$ we obtain

$$|\gamma_d|\, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right) = |\gamma_d|\, a_{d'} \left(1 - e^{-\eta^d}\right) q_{d'} = \gamma_d a_{d'} \left(e^{-\eta^d} - 1\right) q_{d'} = \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d}\right). \tag{B20}$$

Therefore, $\hat{\pi}(\gamma_d) = \hat{\pi}(|\gamma_d|)$ and $\hat{\pi}$ is symmetric in γ_d around zero. ■

Proof of Corollary 1. Because the participation constraint (8) binds, HQ payoff, $\hat{\pi}$, now is equal to

$$\sum_{\substack{d,d' \in \{A,B\} \\ d' \neq d}} \left[(1 - \beta_d - \gamma_{d'})\, q_d a_d + \beta_d a_d \hat{q}_d^d + \gamma_d a_{d'} \hat{q}_{d'}^{d} - \frac{a_d^2}{2\theta_d} - \frac{r\sigma^2 \left(\beta_d^2 + 2\beta_d \gamma_d \rho + \gamma_d^2\right)}{2}\right] \tag{B21}$$

where $\{a_A, a_B\}$ are the Nash equilibrium effort levels of Lemma 3.

Different from the case of Theorem 1, because of the presence of the last term, HQ objective function $\hat{\pi}$ admits multiple strict local maxima. The proof therefore proceeds in two steps. First, we consider candidate optimal contracts that induce division managers to hold one of four possible configurations of beliefs (implied by Lemma 2). Specifically, we consider contracts as follows. Case (A): a small exposure to the other division leading to $H_d < e^{-\eta^d}$, case (i) of Lemma 2; Case (B): a moderate positive exposure to the other division, $\gamma_d > 0$ and $H_d \in \left(e^{-\eta^d}, e^{\eta^d}\right)$, within case (ii) of Lemma 2; Case (B'): a moderate negative exposure to the other division, $\gamma_d < 0$ and $H_d \in \left(e^{-\eta^d}, e^{\eta^d}\right)$, also within case (ii) of Lemma 2; Cases (C) and (C'): a large (negative or positive) exposure to the other division, leading to $H_d > e^{\eta^d}$ case (iii) of Lemma 2. Second, we compare payoffs to HQ from optimal contracts in these regions and we determine the globally optimal contract.

Case (A): If $H_d < e^{-\eta^d}$, have $\hat{q}_d^d = e^{-\eta^d} q_d$ and $\hat{q}_{d'}^{d} = q_{d'}$, which do not depend on γ_d. Similarly, by Lemma 3, $a_d = \beta_d \theta_d e^{-\eta^d} q_d$, which does not depend on γ_d as well. Therefore, setting

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = -r\sigma^2 (\rho\beta_d + \gamma_d) = 0 \tag{B22}$$

gives $\gamma_d = -\rho\beta_d$ and γ_d is set to hedge risk with no effect on incentives. Substituting in $\hat{\pi}$ and differentiating we obtain

$$\frac{\partial \hat{\pi}}{\partial \beta_d} = (1 - 2\beta_d)\, \theta_d q \hat{q}_d^d + \beta_d \theta_d \left(\hat{q}_d^d\right)^2 - r\sigma^2 \beta_d \left(1 - \rho^2\right) \tag{B23}$$

Therefore

$$\beta_d^1 \equiv \frac{1}{1 + \left(1 - \hat{q}_d^d / q\right) + r\sigma^2 \left(1 - \rho^2\right) / \left(\theta q \hat{q}_d^d\right)}. \tag{B24}$$

After substitution, this gives HQ payoff under condition (S)

$$\hat{\pi}^1 \equiv \frac{\left[e^{-\eta}\theta q^2\right]^2}{\left(2 - e^{-\eta}\right) e^{-\eta}\theta q^2 + r\sigma^2 \left(1 - \rho^2\right)}. \tag{B25}$$

Case (B): If $\gamma_d > 0$ and $H_d \in \left(e^{-\eta}, e^{\eta}\right)$, we can express the payoff to HQ as

$$\hat{\pi} = (1 - \beta_A - \gamma_B)\, a_A q_A + (1 - \beta_B - \gamma_A)\, a_B q_B + \hat{u}_A(a_A, \hat{q}^A(a_A, w_A)) + \hat{u}_B(a_B, \hat{q}^B(a_B, w_B)), \tag{B26}$$

where $\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = \min_{\hat{q}^d \in K_d^{\hat{q}}} \hat{u}_d$, with

$$\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = \beta_d a_d \hat{q}_d^d + \gamma_d a_{d'} \hat{q}_{d'}^d - \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) - \frac{a_d^2}{2\theta_d} = -s_d, \tag{B27}$$

and where $\check{a}_d$ is the Nash equilibrium given by (B16). Because $\hat{u}_d$ is strictly concave and the minimum operator is concave, $\hat{u}_d(a_d, \hat{q}^d(a_d, w_d))$ is strictly concave. Therefore, $\hat{\pi}$ is strictly concave as well. Thus, first-order conditions of optimality are sufficient for a local optimum. Similar to the proof of Theorem 1, we have

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -q_d\check{a}_d + (1-\beta_d-\gamma_{d'})\, q_d \frac{\partial \check{a}_d}{\partial \beta_d} + (1-\beta_{d'}-\gamma_d)\, q_{d'} \frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \end{aligned} \tag{B28}$$

In this region, from (B16), we have $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$. Because $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d \hat{q}_{d'}^d$ and $\frac{\partial \hat{u}_d}{\partial \beta_d} = a_d \hat{q}_d^d - r\sigma^2(\beta_d + \rho\gamma_d)$, by applying the envelope theorem to $\hat{u}_d(\check{a}_d, \hat{q}^d)$:

$$\frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} = a_d \hat{q}_d^d - r\sigma^2(\beta_d + \rho\gamma_d) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\beta_d}. \tag{B29}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \beta_d} = 0$ and $\frac{\partial \hat{u}_{d'}}{\partial \check{a}_d} = \gamma_{d'} \hat{q}_d^{d'}$, we obtain

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -a_d\left(q_d - \hat{q}_d^d\right) + (1-\beta_d-\gamma_{d'})\, q_d \frac{3\check{a}_d}{8\beta_d} + (1-\beta_{d'}-\gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8\beta_d} \\ &\quad - r\sigma^2(\beta_d + \rho\gamma_d) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\beta_d}. \end{aligned} \tag{B30}$$

Consider now γ_d. We have that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -q_{d'}\check{a}_{d'} + (1-\beta_d-\gamma_{d'})\, q_d \frac{\partial \check{a}_d}{\partial \gamma_d} + (1-\beta_{d'}-\gamma_d)\, q_{d'} \frac{\partial \check{a}_{d'}}{\partial \gamma_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d}. \end{aligned} \tag{B31}$$

Because $\frac{\partial \hat{u}_d}{\partial \gamma_d} = \check{a}_{d'} \hat{q}_{d'}^d - r\sigma^2(\gamma_d + \rho\beta_d)$, $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d \hat{q}_{d'}^d$, and $\frac{\partial \check{a}_{d'}}{\partial \gamma_d} = \frac{\check{a}_{d'}}{8\gamma_d}$, applying the envelope theorem to $\hat{u}_d(\check{a}_d, \hat{q}^d)$,

$$\frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} = a_{d'} \hat{q}_{d'}^d - r\sigma^2(\gamma_d + \rho\beta_d) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\gamma_d}. \tag{B32}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \gamma_d} = 0$, $\frac{\partial \hat{u}_{d'}}{\partial \check{a}_d} = \gamma_{d'} \hat{q}_d^{d'}$, and $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, we obtain

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) + (1-\beta_d-\gamma_{d'})\, q_d \frac{3\check{a}_d}{8\gamma_d} + (1-\beta_{d'}-\gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8\gamma_d} \\ &\quad - r\sigma^2(\gamma_d + \rho\beta_d) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\gamma_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\gamma_d}. \end{aligned} \tag{B33}$$

Thus, from (B30) and (B33), we obtain the first-order conditions

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -\check{a}_d\left(q_d - \hat{q}_d^d\right) - r\sigma^2(\beta_d + \rho\gamma_d) + \frac{\Delta_d}{\beta_d} = 0, \\ \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) - r\sigma^2(\gamma_d + \rho\beta_d) + \frac{\Delta_d}{\gamma_d} = 0, \end{aligned} \tag{B34}$$

where $\Delta_d \equiv (1-\beta_d-\gamma_{d'})\, q_d \frac{3\check{a}_d}{8} + (1-\beta_{d'}-\gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8} + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8}$, giving

$$\beta_d \check{a}_d\left(q_d - \hat{q}_d^d\right) + r\sigma^2\left(\beta_d^2 + \rho\gamma_d\beta_d\right) = \gamma_d \check{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) + r\sigma^2\left(\gamma_d^2 + \rho\beta_d\gamma_d\right). \tag{B35}$$

By Lemma 2, we have that $\beta_d \check{a}_d \hat{q}_d^d = \gamma_d \check{a}_{d'} \hat{q}_{d'}^d$, which implies that

$$\beta_d \check{a}_d q_d + r\sigma^2 \beta_d^2 = \gamma_d \check{a}_{d'} q_{d'} + r\sigma^2 \gamma_d^2 \tag{B36}$$

We will guess and verify that, due to the symmetry condition (S), it is optimal to implement symmetric effort, $\check{a}_d = \check{a}_{d'} = \check{a}$, and that $q_d = q$, $\eta^d = \eta$, and $\theta_d = \theta$. Define $f(x) \equiv x\check{a}q + r\sigma^2 x^2$. Note $f'(x) = \check{a}q + 2r\sigma^2 x > 0$ for

$x > 0$, so that f is monotonic over positive numbers and $f(\gamma_d) = f(\beta_d)$ if and only if $\gamma_d = \beta_d$. Thus, $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\eta}{2}} q$ and $\check{a}_d = e^{-\frac{\eta}{2}} \theta \beta_d^{\frac{3}{4}} \beta_{d'}^{\frac{1}{4}} q$. In order to optimally implement the same effort, it must be that $\beta_d = \beta_{d'}$, so $\check{a} = e^{-\frac{\eta}{2}} \theta \beta q$. Thus, we obtain the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d} = -\theta \beta_d \hat{q}_d^d \left(q - \hat{q}_d^d\right) + (1 - 2\beta_d)\, q \hat{q}_d^d \frac{\theta}{2} - r\sigma^2 \beta_d (1+\rho) + \frac{\theta \beta_d \left(\hat{q}_d^d\right)^2}{2} = 0. \tag{B37}$$

Therefore

$$\beta_d^2 \equiv \frac{1}{1 + 3\left(1 - \hat{q}_d^d / q\right) + 2r\sigma^2 (1+\rho) / (\theta q \hat{q}_d^d)}. \tag{B38}$$

After substitution, this gives HQ payoff

$$\hat{\pi}^2 \equiv \frac{\theta^2 e^{-\eta} q^4}{\theta e^{-\frac{\eta}{2}} q^2 \left(4 - 3e^{-\frac{\eta}{2}}\right) + 2r\sigma^2 (1+\rho)}. \tag{B39}$$

Because β_d is the same for both divisions, this verifies that a is symmetric. Because HQ objective $\hat{\pi}$ is strictly concave on this region, there is only one solution on this region, so the symmetric solution is the unique solution.

Case (B'): Consider $\gamma_d < 0$ and $H_d \in \left(e^{-\eta}, e^{\eta}\right)$. Following the same process as in case (B) above, we have

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -q_d \check{a}_d + (1 - \beta_d - \gamma_{d'})\, q_d \frac{\partial \check{a}_d}{\partial \beta_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \end{aligned} \tag{B40}$$

Because in this region $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$, we obtain that

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -a_d \left(q_d - \hat{q}_d^d\right) + (1 - \beta_d - \gamma_{d'})\, q_d \frac{3\check{a}_d}{8\beta_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8\beta_d} \\ &\quad - r\sigma^2 (\beta_d + \rho\gamma_d) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\beta_d}. \end{aligned} \tag{B41}$$

Consider now γ_d. We have that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -q_{d'} \check{a}_{d'} + (1 - \beta_d - \gamma_{d'})\, q_d \frac{\partial \check{a}_d}{\partial \gamma_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\partial \check{a}_{d'}}{\partial \gamma_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d}. \end{aligned} \tag{B42}$$

Because $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, $\frac{\partial \check{a}_{d'}}{\partial \gamma_d} = \frac{\check{a}_{d'}}{8\gamma_d}$ and $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d \hat{q}_{d'}^d$, by applying the envelope theorem on $\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} = a_{d'} \hat{q}_{d'}^d - r\sigma^2 (\gamma_d + \rho\beta_d) + \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8}. \tag{B43}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \gamma_d} = 0$, $\frac{\partial \hat{u}_{d'}}{\partial \check{a}_d} = \gamma_{d'} \hat{q}_d^{d'}$, and $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, by applying the envelope theorem on $\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d} = \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\gamma_d}. \tag{B44}$$

Together (B43) and (B44) give that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) + (1 - \beta_d - \gamma_{d'})\, q_d \frac{3\check{a}_d}{8\gamma_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8\gamma_d} \\ &\quad - r\sigma^2 (\gamma_d + \rho\beta_d) + \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\gamma_d}. \end{aligned} \tag{B45}$$

Thus, from (B41) and (B45), we obtain the first-order conditions

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -\check{a}_d \left(q_d - \hat{q}_d^d\right) - r\sigma^2 (\beta_d + \rho\gamma_d) + \frac{\Delta_d}{\beta_d} = 0, \\ \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) - r\sigma^2 (\gamma_d + \rho\beta_d) + \frac{\Delta_d}{\gamma_d} = 0, \end{aligned} \tag{B46}$$

where $\Delta_d \equiv (1-\beta_d-\gamma_{d'})\, q_d \frac{3\check{a}_d}{8} + (1-\beta_{d'}-\gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8} + \gamma_d \hat{q}_{d'}^{d} \frac{\check{a}_{d'}}{8} + \gamma_{d'} \hat{q}_{d}^{d'} \frac{3\check{a}_d}{8}$, giving

$$\beta_d \check{a}_d \left(q_d - \hat{q}_d^d\right) + r\sigma^2\left(\beta_d^2 + \rho\gamma_d\beta_d\right) = \gamma_d \check{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) + r\sigma^2\left(\gamma_d^2 + \rho\beta_d\gamma_d\right). \tag{B47}$$

Again, in this region, $\hat{q}_d^d = \left[e^{-\eta^d} H_d\right]^{\frac{1}{2}} q_d$, and $\hat{q}_{d'}^d = \left(2 - \left[e^{-\eta^d} H_d^{-1}\right]^{\frac{1}{2}}\right) q_{d'}$, where $H_d = \frac{|\gamma_d| a_{d'} q_{d'}}{\beta_d a_d q_d}$. Thus,

$$\gamma_d \check{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) = \gamma_d \check{a}_{d'} q_{d'}\left(e^{-\frac{\eta^d}{2}} H_d^{-\frac{1}{2}} - 1\right) = -\gamma_d \check{a}_{d'} q_{d'} - e^{-\frac{\eta^d}{2}}\left(\beta_d a_d q_d \left|\gamma_d\right| a_{d'} q_{d'}\right)^{\frac{1}{2}}. \tag{B48}$$

Similarly,

$$\beta_d \check{a}_d \hat{q}_d^d = e^{-\frac{\eta^d}{2}}\left(\beta_d \check{a}_d q_d \left|\gamma_d\right| \check{a}_{d'} q_{d'}\right)^{\frac{1}{2}} \tag{B49}$$

Therefore, after substitution, we obtain that (B47) becomes

$$\beta_d \check{a}_d q_d + r\sigma^2\beta_d^2 = \left|\gamma_d\right| \check{a}_{d'} q_{d'} + r\sigma^2\gamma_d^2. \tag{B50}$$

We guess again that HQ optimally implement the same effort from both divisions, $\check{a}_d = \check{a}_{d'}$, which implies that $f(|\gamma_d|) = f(\beta_d)$, where again $f(x) \equiv x\check{a}q + r\sigma^2 x^2$. This implies that $|\gamma_d| = \beta_d$, or equivalently, that $\gamma_d = -\beta_d$, so that $H_d = 1$. Thus, $\hat{q}_d^d = e^{-\frac{\eta}{2}} q$, and $\hat{q}_{d'}^d = \left(2 - e^{-\frac{\eta}{2}}\right) q$. To be consistent with this guess, it must be that $\beta_{d'} = \beta_d$, so that $\check{a}_d = \check{a}_{d'} = e^{-\frac{\eta}{2}} \theta \beta_d q$. Substituting in $\hat{\pi}$ and differentiating we obtain

$$\frac{d\hat{\pi}}{d\beta_d} = -\theta\beta_d \hat{q}_d^d\left(q_d - \hat{q}_d^d\right) - r\sigma^2\beta\left(1+\rho\right) + \frac{1}{2}\left(1-2\beta_d\right)\theta q \hat{q}_d^d + \frac{1}{2}\beta_d\theta\left(\hat{q}_d^d\right)^2 \tag{B51}$$

$$\beta_d^3 \equiv \frac{1}{1 + 3(1 - \hat{q}_d^d/q) + 2r\sigma^2\left(1-\rho\right)/(\theta q \hat{q}_d^d)}. \tag{B52}$$

After substitution, this gives HQ payoff

$$\hat{\pi}^3 \equiv \frac{\theta^2 e^{-\eta} q^4}{\theta e^{-\frac{\eta}{2}} q^2\left(4 - 3e^{-\frac{\eta}{2}}\right) + 2r\sigma^2\left(1-\rho\right)}, \tag{B53}$$

which verifies the guess that HQ optimally implements symmetric effort. Comparing $\hat{\pi}^2$ and $\hat{\pi}^3$, observe that they differ only for the final term in the denominator. Thus, $\hat{\pi}^3 \gtreqless \hat{\pi}^2$ as $\rho \gtreqless 0$, and

$$\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\} = \frac{\theta^2 e^{-\eta} q^4}{\theta e^{-\frac{\eta}{2}} q^2\left(4 - 3e^{-\frac{\eta}{2}}\right) + 2r\sigma^2\left(1-|\rho|\right)}. \tag{B54}$$

Case (C): If $\gamma_d > e^{\eta}\beta_d$, we have that $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = e^{-\eta} q_{d'}$, so

$$\frac{\partial\hat{\pi}}{\partial\gamma_d} = -a_{d'} q_{d'}\left(1 - e^{-\eta}\right) - r\sigma^2\left(\rho\beta_d + \gamma_d\right) < 0, \tag{B55}$$

and setting $\gamma_d > e^{\eta}\beta_d$ is not optimal. Similarly, if $\gamma_d < -e^{\eta}\beta_d$, we have that $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = \left(2 - e^{-\eta}\right) q$

$$\frac{\partial\hat{\pi}}{\partial\gamma_d} = a_{d'} q_{d'}\left(1 - e^{-\eta}\right) + r\sigma^2\left(|\gamma_d| - \rho\beta_d\right) > 0 \tag{B56}$$

and setting $\gamma_d < -e^{\eta}\beta_d$ is not optimal. Thus, $H_d \leq e^{\eta}$.

The second and final step is to compare $\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\}$ and $\hat{\pi}^1$. Let

$$f\left(\eta\right) \equiv 2\left(1 - e^{-\frac{\eta}{2}}\right)^2 \theta q^2 + r\sigma^2\left(1-|\rho|\right)\left[e^{\eta}\left(1+|\rho|\right) - 2\right], \tag{B57}$$

so that $\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\} > \hat{\pi}^1$ if and only if $f > 0$. Note $f\left(0\right) = -r\sigma^2\left(1-|\rho|\right)^2 < 0$,

$$f'\left(\eta\right) = 2\left(1 - e^{-\frac{\eta}{2}}\right) e^{-\frac{\eta}{2}} \theta q^2 + r\sigma^2 e^{\eta}\left(1-\rho^2\right) > 0 \tag{B58}$$

and $\lim_{\eta\to\infty} f\left(\eta\right) = +\infty$, which implies there is a unique $\bar{\eta}$ such that $\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\} > \hat{\pi}^1$ if and only if $\eta > \bar{\eta}$. Thus, for $\eta \leq \bar{\eta}$ the optimal contract is in Case (A), with $\beta_d = \beta_d^1$ and $\gamma_d = -\rho\beta_d$. For $\eta > \bar{\eta}$ the optimal contract is in Case (B) for $\rho < 0$, with $\beta_d = \beta_d^2$ and $|\gamma_d| = \beta_d$, but in Case (B') for $\rho > 0$, with $\beta_d = \beta_d^3$ and $|\gamma_d| = \beta_d$.

Finally, note that the first term of f, $2\left(1 - e^{-\frac{\eta}{2}}\right)^2 \theta q^2$, is strictly positive. Because $f\left(\bar{\eta}\right) = 0$, it must be that

$r\sigma^2(1-|\rho|)\left[e^{\bar\eta}(1+|\rho|)-2\right]<0$. This implies that $\frac{\partial f}{\partial r}=\sigma^2(1-|\rho|)\left[e^{\eta}(1+|\rho|)-2\right]<0$ in a neighborhood of $\bar\eta$. By the implicit function theorem, we obtain that $\frac{d\bar\eta}{dr}=-\frac{\frac{\partial f}{\partial r}}{f'(\eta)}>0$, and $\bar\eta$ is increasing in r. For $\rho\neq 0$, define $\eta_\rho\equiv-\ln(|\rho|)$ and note that

$$f(\eta_\rho)=2\left(1-\sqrt{|\rho|}\right)^2\theta q^2+r\sigma^2\frac{(1-|\rho|)^2}{|\rho|}>0 \tag{B59}$$

which implies that $\bar\eta<\eta_\rho$. Finally, note that $\frac{\partial f}{\partial|\rho|}=2r\sigma^2(1-e^{\eta}|\rho|)>0$ because $\eta<\eta_\rho$, so $\frac{d\bar\eta}{d|\rho|}<0$. ■

Proof of Theorem 3. We guess and verify that HQ has positive exposure to both divisions, $\phi_d=1-\beta_d-\gamma_d>0$, and that beliefs are as in case (ii) of Lemma 4, $H_d^{HQ}\in\left(e^{-\eta^{HQ}},e^{\eta^{HQ}}\right)$. Because (8) binds and $r=0$, HQ payoff $\hat\pi$ is equal to

$$\sum_{\substack{d,d'\in\{A,B\}\\ d\neq d'}}\left[a_dq_d-(1-\beta_d-\gamma_{d'})\,a_d\left(q_d-\hat q_d^{HQ}\right)-\beta_da_d\left(q_d-\hat q_d^d\right)-\gamma_da_{d'}\left(q_{d'}-\hat q_{d'}^d\right)\right], \tag{B60}$$

where $\hat q^d=(\hat q_d^d,\hat q_{d'}^d)$ are division manager beliefs from Lemma 2, a_d are the Nash equilibrium effort levels from Lemma 3, and $\hat q^{HQ}=(\hat q_d^{HQ},\hat q_{d'}^{HQ})$ are HQ beliefs from Lemma 4. The proof is in two steps and is similar to the proof of Theorem 1. First, we show that $\gamma_d<0$ is suboptimal; then we find the optimal contract for $\gamma_d\geq 0$.

Similar to Theorem 1, switching from γ_d to $|\gamma_d|$ does not affect $\hat q_d^d$, and thus does not affect a_d and $\beta_da_d\left(q_d-\hat q_{d'}^d\right)$. Letting again $\hat q_{d'}^{d+}$ be the belief held by a division manager when receiving $|\gamma_d|$ instead of $\gamma_d<0$, we have that $\gamma_da_{d'}\left(q_{d'}-\hat q_{d'}^d\right)=|\gamma_d|\,a_{d'}\left(q_{d'}-\hat q_{d'}^{d+}\right)$ for all $\gamma_d<0$. This implies that

$$(1-\beta_{d'}-|\gamma_d|)\,a_{d'}\left(q_{d'}-\hat q_{d'}^{HQ}\right)<(1-\beta_{d'}-\gamma_d)\,a_{d'}\left(q_{d'}-\hat q_{d'}^{HQ}\right) \tag{B61}$$

for $\gamma_d<0$ because $\hat q_{d'}^{HQ}<q_{d'}$, and thus that setting $\gamma_d<0$ is dominated by offering its absolute value, $|\gamma_d|$.

Because HQ strictly prefers offering $|\gamma_d|>0$ to all $\gamma_d<0$, it is sufficient to consider $\gamma_d\geq 0$. If HQ sets $0\leq\gamma_d<e^{-\eta}\frac{\beta_da_dq_d}{a_{d'}q_{d'}}$, division managers beliefs are in case (i) of Lemma 2, with $\hat q_d^d=e^{-\eta}q_d$ and $\hat q_{d'}^d=q_d$, giving $a_d=\beta_d\theta_de^{-\eta}q_d$. Further, $\frac{\partial\hat\pi}{\partial\gamma_d}=a_{d'}\left(\hat q_{d'}^d-\hat q_{d'}^{HQ}\right)>0$ because $\hat q_{d'}^{HQ}\in\left(e^{-\eta^{HQ}}q_d,q_d\right)$, so setting $\gamma_d<e^{-\eta}\frac{\beta_da_dq_d}{a_{d'}q_{d'}}$ is not optimal. Alternatively, if HQ sets $\gamma_d>e^{\eta}\frac{\beta_da_dq_d}{a_{d'}q_{d'}}$, division manager beliefs are in case (iii) of Lemma 2, with $\hat q_d^d=q_d$ and $\hat q_{d'}^d=e^{-\eta}q_{d'}$, giving $a_d=\beta_d\theta_dq_d$. Thus, $\frac{\partial\hat\pi}{\partial\gamma_d}=-a_{d'}\left(\hat q_{d'}^{HQ}-\hat q_{d'}^d\right)<0$ because $\hat q_{d'}^{HQ}\in\left(e^{-\eta^{HQ}}q_d,q_d\right)$ and $\eta^{HQ}<\eta$, so setting $\gamma_d>e^{\eta}\frac{\beta_da_dq_d}{a_{d'}q_{d'}}$ is not optimal. Thus, HQ sets $e^{-\eta}\frac{\beta_da_dq_d}{a_{d'}q_{d'}}\leq\gamma_d\leq e^{\eta}\frac{\beta_da_dq_d}{a_{d'}q_{d'}}$ and induce beliefs that are in case (ii) of Lemma 2, with $H_d\in\left(e^{-\eta},e^{\eta}\right)$.

Similar to the proof of Theorem 1, we can express HQ's objective as

$$\hat\pi=\phi_A\check a_A\hat q_A^{HQ}+\phi_B\check a_B\hat q_B^{HQ}+\hat u_A(a_A,\hat q^A(a_A,w_A))+\hat u_B(a_B,\hat q^B(a_B,w_B)), \tag{B62}$$

where $\phi_d=1-\beta_d-\gamma_{d'}$, $\hat u_d(\check a_d,\hat q^d)=\min_{\hat q^d\in K_d^{\hat q}}\hat u_d$, with $\hat u_d=\beta_d\check a_d\hat q_d^d+\gamma_d\check a_{d'}\hat q_{d'}^d-\frac{\check a_d^2}{2\theta_d}=0$, and $\check a_d$ is the Nash equilibrium of division managers given by (B16) in the proof of Lemma 3. Consider first

$$\begin{aligned}\frac{d\hat\pi}{d\beta_d}&=-\hat q_d^{HQ}\check a_d+\phi_d\check a_d\frac{\partial\hat q_d^{HQ}}{\partial\beta_d}+\phi_{d'}\check a_{d'}\frac{\partial\hat q_{d'}^{HQ}}{\partial\beta_d}+\phi_d\hat q_d^{HQ}\frac{\partial\check a_d}{\partial\beta_d}+\phi_{d'}\hat q_{d'}^{HQ}\frac{\partial\check a_{d'}}{\partial\beta_d}\\&\quad+\frac{d\hat u_d(\check a_d,\hat q^d(\check a_d,w_d))}{d\beta_d}+\frac{d\hat u_{d'}(\check a_{d'},\hat q^{d'}(\check a_{d'},w_{d'}))}{d\beta_d}.\end{aligned} \tag{B63}$$

Because $\hat q^{HQ}$ solves (22), from the envelope theorem $\phi_d\check a_d\frac{\partial\hat q_d^{HQ}}{\partial\beta_d}+\phi_{d'}\check a_{d'}\frac{\partial\hat q_{d'}^{HQ}}{\partial\beta_d}=0$, which, together with $\frac{d\hat u_d(\check a_d,\hat q^d(\check a_d,w_d))}{d\beta_d}=a_d\hat q_d^d+\gamma_d\hat q_{d'}^d\frac{\check a_{d'}}{8\beta_d}$ and $\frac{d\hat u_{d'}(\check a_{d'},\hat q^{d'}(\check a_{d'},w_{d'}))}{d\beta_d}=\gamma_{d'}\hat q_d^{d'}\frac{3\check a_d}{8\beta_d}$ from the proof of Theorem 1, gives

$$\frac{d\hat\pi}{d\beta_d}=-\check a_d\left(\hat q_d^{HQ}-\hat q_d^d\right)+\phi_d\hat q_d^{HQ}\frac{3a_d}{8\beta_d}+\phi_{d'}\hat q_{d'}^{HQ}\frac{a_{d'}}{8\beta_d}+\gamma_d\hat q_{d'}^d\frac{\check a_{d'}}{8\beta_d}+\gamma_{d'}\hat q_d^{d'}\frac{3\check a_d}{8\beta_d}. \tag{B64}$$

Consider now γ_d. Applying again the envelope theorem on $\hat\pi\left(\hat q^{HQ}\right)$, we obtain

$$\begin{aligned}\frac{d\hat\pi}{d\gamma_d}&=-\hat q_{d'}^{HQ}\check a_{d'}+\phi_d\hat q_d^{HQ}\frac{\partial\check a_d}{\partial\gamma_d}+\phi_{d'}\hat q_{d'}^{HQ}\frac{\partial\check a_{d'}}{\partial\gamma_d}\\&\quad+\frac{d\hat u_d(\check a_d,\hat q^d(\check a_d,w_d))}{d\gamma_d}+\frac{d\hat u_{d'}(\check a_{d'},\hat q^{d'}(\check a_{d'},w_{d'}))}{d\gamma_d}.\end{aligned} \tag{B65}$$

Substituting $\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(a_d,w_d))}{d\gamma_d} = a_{d'}\hat{q}^d_{d'} + \gamma_d\hat{q}^d_{d'}\frac{\check{a}_{d'}}{8\gamma_d}$ and $\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\gamma_d} = \gamma_{d'}\hat{q}^{d'}_d\frac{3\check{a}_d}{8\gamma_d}$ from the proof of Theorem 1,

$$\frac{d\hat{\pi}}{d\gamma_d} = -\check{a}_{d'}\left(\hat{q}^{HQ}_{d'} - \hat{q}^d_{d'}\right) + \phi_d\hat{q}^{HQ}_d\frac{3\check{a}_d}{8\gamma_d} + \phi_{d'}\hat{q}^{HQ}_{d'}\frac{\check{a}_{d'}}{8\gamma_d} + \gamma_d\hat{q}^d_{d'}\frac{\check{a}_{d'}}{8\gamma_d} + \gamma_{d'}\hat{q}^{d'}_d\frac{3\check{a}_d}{8\gamma_d}. \quad \text{(B66)}$$

Thus, from (B64) and (B66) we obtain the first-order conditions

$$\frac{d\hat{\pi}}{d\beta_d} = -\check{a}_d\left(\hat{q}^{HQ}_d - \hat{q}^d_d\right) + \frac{\Delta_d}{\beta_d} = 0, \quad \frac{d\hat{\pi}}{d\gamma_d} = -\check{a}_{d'}\left(\hat{q}^{HQ}_{d'} - \hat{q}^d_{d'}\right) + \frac{\Delta_d}{\gamma_d} = 0, \quad \text{(B67)}$$

where $\Delta_d \equiv \phi_d\hat{q}^{HQ}_d\frac{3\check{a}_d}{8} + \phi_{d'}\hat{q}^{HQ}_{d'}\frac{\check{a}_{d'}}{8} + \gamma_d\hat{q}^d_{d'}\frac{\check{a}_{d'}}{8} + \gamma_{d'}\hat{q}^{d'}_d\frac{3\check{a}_d}{8}$, giving

$$\beta_d\check{a}_d\left(\hat{q}^{HQ}_d - \hat{q}^d_d\right) = \gamma_{d'}\check{a}_{d'}\left(\hat{q}^{HQ}_{d'} - \hat{q}^d_{d'}\right). \quad \text{(B68)}$$

Because, from Lemma 2, $\beta_d\check{a}_d\hat{q}^d_d = \gamma_d\check{a}_{d'}\hat{q}^d_{d'}$, we have that (B68) implies $\beta_d\check{a}_d\hat{q}^{HQ}_d = \gamma_d\check{a}_{d'}\hat{q}^{HQ}_{d'}$. Because $H^{HQ}_d \in \left(e^{-\eta^{HQ}}, e^{\eta^{HQ}}\right)$, from Lemma 4, $\phi_d a_d\hat{q}^{HQ}_d = \phi_{d'}a_{d'}\hat{q}^{HQ}_{d'}$. Thus, $\frac{a_{d'}\hat{q}^{HQ}_{d'}}{a_d\hat{q}^{HQ}_d} = \frac{\beta_d}{\gamma_d} = \frac{\phi_d}{\phi_{d'}}$. Define m_d such that $\beta_d = m_d\phi_d$, so $\gamma_d = m_d\phi_{d'}$, which implies $\phi_d = 1-\beta_d-\gamma_{d'} = \frac{1}{1+m_d+m_{d'}}$, and thus $\beta_d = \gamma_d = \frac{m_d}{1+m_d+m_{d'}}$. Substituting in $\gamma_d = \beta_d$ into $\check{a}$ from Lemma 3, we have $\check{a}_d = \left(\theta^3_d\theta_{d'}\right)^{\frac{1}{4}} e^{-\frac{\eta}{2}}(\beta^3_d\beta_{d'})^{\frac{1}{4}}(q_dq_{d'})^{\frac{1}{2}}$. Substituting into HQ objective, we obtain

$$\hat{\pi} = (\theta_A\theta_B)^{\frac{1}{2}}q_Aq_B(\beta_A\beta_B)^{\frac{1}{2}}\left[2e^{-\frac{\eta^{HQ}}{2}}e^{-\frac{\eta}{2}}(1-\beta_A-\beta_B) + \frac{3}{2}e^{-\eta}(\beta_A+\beta_B)\right]. \quad \text{(B69)}$$

Differentiating, we obtain the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d} = (\theta_A\theta_B)^{\frac{1}{2}}q_dq_{d'}(\beta_d^{-1}\beta_{d'})^{\frac{1}{2}}\left[e^{-\frac{\eta^{HQ}}{2}}(1-3\beta_d-\beta_{d'})e^{-\frac{\eta}{2}} + \frac{3}{4}e^{-\eta}(3\beta_d+\beta_{d'})\right] = 0, \quad \text{(B70)}$$

giving

$$e^{\frac{1}{2}(\eta-\eta^{HQ})} + 3\left(\frac{3}{4} - e^{\frac{1}{2}(\eta-\eta^{HQ})}\right)\beta_d + \left(\frac{3}{4} - e^{\frac{1}{2}(\eta-\eta^{HQ})}\right)\beta_{d'} = 0. \quad \text{(B71)}$$

Because this holds for both divisions, after solving we obtain

$$\beta_A = \beta_B = \frac{1}{4-3e^{\frac{1}{2}(\eta^{HQ}-\eta)}} = \frac{1}{1+3(1-\hat{q}^d_d/\hat{q}^{HQ}_d)} = \gamma_d, \quad \text{(B72)}$$

giving (25). Note $\beta < \frac{1}{2}$ because $\eta > \eta^{HQ} + 2\ln\frac{3}{2}$ and $H^{HQ}_d = \hat{H}_d \in \left(e^{-\eta^{HQ}}, e^{\eta^{HQ}}\right)$. This implies that $\check{a}_d = \frac{\left(\theta^3_d\theta_{d'}\right)^{\frac{1}{4}}e^{-\frac{\eta}{2}}(q_dq_{d'})^{\frac{1}{2}}}{4-3e^{\frac{1}{2}(\eta^{HQ}-\eta)}}$, and thus that $\hat{q}^d_d = e^{-\frac{\eta}{2}}q_d\hat{H}_d^{\frac{1}{2}}$ and $\hat{q}^{HQ}_d = e^{-\frac{\eta^{HQ}}{2}}q_d\hat{H}_d^{\frac{1}{2}}$. ■

Proof of Theorem 4. Because the participation constraint (8) binds, we can express HQ's payoff as

$$\hat{\pi} = \phi_Aa_A\hat{q}^{HQ}_A + \phi_Ba_B\hat{q}^{HQ}_B + \hat{u}_A(a_A,\hat{q}^A(a_A,w_A)) + \hat{u}_B(a_B,\hat{q}^B(a_B,w_B)), \quad \text{(B73)}$$

where $\phi_d = 1-\beta_d-\gamma_{d'}$ and $\hat{u}_d(a_d,\hat{q}^d(a_d,w_d)) = \min_{\hat{q}^d\in K^{\hat{q}}_d}\hat{u}_d$, with

$$\hat{u}_d(a_d,\hat{q}^d(a_d,w_d)) = \beta_da_d\hat{q}^d_d + \gamma_da_{d'}\hat{q}^d_{d'} - \frac{r\sigma^2}{2}\left(\beta^2_d + 2\rho\beta_d\gamma_d + \gamma^2_d\right) - \frac{a^2_d}{2\theta_d} = -s_d, \quad \text{(B74)}$$

where $\hat{q}^d$ is from Lemma 2, a_d is from Lemma 3, and $\hat{q}^{HQ}$ is from Lemma 4. Different from Theorem 3, and similar to Theorem 2, because of division manager risk aversion, HQ objective function π admits again multiple local maxima. The proof proceeds again in two steps. First, we consider candidate optimal contracts that induce division managers to hold one of four possible configurations of beliefs (implied by Lemma 2) in the same four cases examined in the proof of Corollary 1, Cases (A), (B), (B'), (C), and (C'). Second, we compare payoffs to HQ from optimal contracts in these regions and we determine the globally optimal contract.

Case (A): If $H_d < e^{-\eta}$, have $\hat{q}^d_d = e^{-\eta}q_d$ and $\hat{q}^d_{d'} = q_{d'}$, which do not depend on γ_d. Similarly, by Lemma 3, $a_d = \beta_d\theta_de^{-\eta}q_d$, which implies that both a_d and $a_{d'}$ do not depend on γ_d. Therefore,

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\hat{q}^{HQ}_{d'}a_{d'} + \phi_da_d\frac{\partial\hat{q}^{HQ}_d}{\partial\gamma_d} + \phi_{d'}a_{d'}\frac{\partial\hat{q}^{HQ}_{d'}}{\partial\gamma_d} + \phi_d\hat{q}^{HQ}_d\frac{\partial a_d}{\partial\gamma_d} + \phi_{d'}\hat{q}^{HQ}_{d'}\frac{\partial a_{d'}}{\partial\gamma_d} \\ &\quad + \frac{d\hat{u}_d(a_d,\hat{q}^d(a_d,w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(a_{d'},\hat{q}^{d'}(a_{d'},w_{d'}))}{d\gamma_d},\end{aligned} \quad \text{(B75)}$$

where, by the envelope theorem on $\hat{\pi}$, we have $\phi_d a_d \frac{\partial \hat{q}_d^{HQ}}{\partial \gamma_d} + \phi_{d'} a_{d'} \frac{\partial \hat{q}_{d'}^{HQ}}{\partial \gamma_d} = 0$. In addition, on this region, $\frac{\partial a_d}{\partial \gamma_d} = \frac{\partial a_{d'}}{\partial \gamma_d} = 0$, which implies that $\frac{d\hat{u}_d(a_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} = \frac{\partial \hat{u}}{\partial \gamma_d} = a_{d'} \hat{q}_{d'}^d - r\sigma^2 (\rho\beta_d + \gamma_d)$ and $\frac{d\hat{u}_{d'}(a_{d'}, \hat{q}^{d'}(a_{d'}, w_{d'}))}{d\gamma_d} = 0$. Thus,

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = a_{d'} \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) - r\sigma^2 (\rho\beta_d + \gamma_d). \tag{B76}$$

Because HQ has long exposure to the symmetric divisions, $\hat{q}_d^{HQ} = \hat{q}_{d'}^{HQ} = e^{-\frac{\eta^{HQ}}{2}} q$. Thus, $\frac{\partial \hat{\pi}}{\partial \gamma_d} = 0$ if and only if $\gamma = -M\beta$, where $M \equiv \rho - \bar{\rho}$ and $\bar{\rho} \equiv \frac{\theta \hat{q}_d^d}{r\sigma^2} \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) = \frac{e^{-\eta}\theta q^2}{r\sigma^2} \left(1 - e^{-\frac{\eta^{HQ}}{2}} \right)$. Similarly,

$$\frac{d\hat{\pi}}{d\beta_d} = \hat{q}_d^d \hat{q}_d^{HQ} \theta (1 - 2\beta_d) - M\beta_d \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) \hat{q}_d^d \theta + \beta_d \theta \left(\hat{q}_d^d \right)^2 - r\sigma^2 \beta_d (1 - \rho M). \tag{B77}$$

Note $1 - \rho M = 1 - \rho^2 + \rho\bar{\rho}$ and $1 - 2\rho M + M^2 = 1 - \rho^2 + \bar{\rho}^2$, so $1 - \rho M = 1 - 2\rho M + M^2 + \bar{\rho}(\rho - \bar{\rho})$. Also, $r\sigma^2 \bar{\rho} (\rho - \bar{\rho}) = \theta \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) \hat{q}_d^d (\rho - \bar{\rho})$. Thus, we obtain the first-order condition

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= \hat{q}_d^d \hat{q}_d^{HQ} \theta (1 - 2\beta_d) + \beta_d \theta \left(\hat{q}_d^d \right)^2 \\ &\quad -2M\beta_d \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) \hat{q}_d^d \theta - r\sigma^2 \beta_d \left(1 - 2\rho M + M^2 \right) = 0, \end{aligned} \tag{B78}$$

which implies

$$\beta_d^4 \equiv \frac{1}{1 + 2(\rho - \bar{\rho}) \left(\frac{\hat{q}_{d'}^d}{\hat{q}_{d'}^{HQ}} - 1 \right) + \left(1 - \frac{\hat{q}_d^d}{\hat{q}_d^{HQ}} \right) + \frac{r\sigma^2 \left(1 - \rho^2 + \bar{\rho}^2 \right)}{\theta \hat{q}_d^{HQ} \hat{q}_d^d}}. \tag{B79}$$

After substitution, this gives HQ payoff

$$\hat{\pi}^4 \equiv \frac{e^{-\left(\eta^{HQ} + 2\eta \right)} \theta^2 q^4}{\left(2M + 2(1 - M) e^{-\frac{\eta^{HQ}}{2}} - e^{-\eta} \right) e^{-\eta} \theta q^2 + r\sigma^2 (1 - 2\rho M + M^2)}. \tag{B80}$$

Case (B): If $\gamma_d > 0$ and $H_d \in \left(e^{-\eta}, e^{\eta} \right)$, as in the proof of Theorem 3, applying the envelope theorem on $\hat{\pi}\left(\hat{q}^{HQ} \right)$, we have

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -\hat{q}_d^{HQ} \check{a}_d + (1 - \beta_d - \gamma_{d'}) \hat{q}_d^{HQ} \frac{\partial \check{a}_d}{\partial \beta_d} + (1 - \beta_{d'} - \gamma_d) \hat{q}_{d'}^{HQ} \frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \end{aligned} \tag{B81}$$

Because in this region $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$, we have

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -a_d \left(\hat{q}_d^{HQ} - \hat{q}_d^d \right) + (1 - \beta_d - \gamma_{d'}) \hat{q}_d^{HQ} \frac{3\check{a}_d}{8\beta_d} + (1 - \beta_{d'} - \gamma_d) \hat{q}_{d'}^{HQ} \frac{\check{a}_{d'}}{8\beta_d} \\ &\quad -r\sigma^2 (\beta_d + \rho\gamma_d) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\beta_d}. \end{aligned} \tag{B82}$$

Consider now γ_d. Applying again the envelope theorem on $\hat{\pi}\left(\hat{q}^{HQ} \right)$, we have

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -\hat{q}_{d'}^{HQ} \check{a}_{d'} + (1 - \beta_d - \gamma_{d'}) \hat{q}_d^{HQ} \frac{\partial \check{a}_d}{\partial \gamma_d} + (1 - \beta_{d'} - \gamma_d) \hat{q}_{d'}^{HQ} \frac{\partial \check{a}_{d'}}{\partial \gamma_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d}. \end{aligned} \tag{B83}$$

Because in this region $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \gamma_d} = \frac{\check{a}_{d'}}{8\gamma_d}$, we have that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'} \left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d \right) + (1 - \beta_d - \gamma_{d'}) \hat{q}_d^{HQ} \frac{3\check{a}_d}{8\gamma_d} + (1 - \beta_{d'} - \gamma_d) \hat{q}_{d'}^{HQ} \frac{\check{a}_{d'}}{8\gamma_d} \\ &\quad -r\sigma^2 (\gamma_d + \rho\beta_d) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\gamma_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\gamma_d}. \end{aligned} \tag{B84}$$

Thus, from (B82) and (B84) we obtain the first-order conditions

$$\frac{d\hat{\pi}}{d\beta_d} = -a_d\left(\hat{q}_d^{HQ}-\hat{q}_d^d\right)-r\sigma^2\left(\beta_d+\rho\gamma_d\right)+\frac{\Delta_d}{\beta_d}=0 \tag{B85}$$
$$\frac{d\hat{\pi}}{d\gamma_d} = -a_{d'}\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^d\right)-r\sigma^2\left(\rho\beta_d+\gamma_d\right)+\frac{\Delta_d}{\gamma_d}=0,$$

where $\Delta_d=\phi_d\hat{q}_d^{HQ}\frac{3a_d}{8}+\phi_{d'}\hat{q}_{d'}^{HQ}\frac{a_{d'}}{8}+\gamma_d\hat{q}_{d'}^d\frac{a_{d'}}{8}+\gamma_{d'}\hat{q}_d^{d'}\frac{3a_d}{8}$, giving

$$\beta_d a_d\left(\hat{q}_d^{HQ}-\hat{q}_d^d\right)+r\sigma^2\left(\beta_d^2+\rho\gamma_d\beta_d\right)=\gamma_d a_{d'}\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^d\right)+r\sigma^2\left(\rho\gamma_d\beta_d+\gamma_d^2\right) \tag{B86}$$

From Lemma 2, we have $\beta_d a_d\hat{q}_d^d=\gamma_d a_{d'}\hat{q}_{d'}^d$. Also, because $\phi_d>0$ and HQ has beliefs as in case (ii) of Lemma 4, with $\phi_d a_d\hat{q}_d^{HQ}=\phi_{d'}a_{d'}\hat{q}_{d'}^{HQ}$, we have

$$\beta_d a_d\hat{q}_d^{HQ}+r\sigma^2\beta_d^2=\gamma_d\frac{\phi_d}{\phi_{d'}}a_d\hat{q}_d^{HQ}+r\sigma^2\gamma_d^2. \tag{B87}$$

We now show that $\phi_A=\phi_B$. Suppose to the contrary that $\phi_A>\phi_B$. Because (B87) holds for both divisions, $\beta_A>\gamma_A$ but $\beta_B<\gamma_B$. This would imply, however, that $\phi_A=1-\beta_A-\gamma_B<1-\beta_B-\gamma_A=\phi_B$, which is a contradiction. Similarly, $\phi_A<\phi_B$ would also imply a contradiction. Thus, $\phi_A=\phi_B$. Further, this implies

$$\left(a_d\hat{q}_d^{HQ}+r\sigma^2\left(\beta_d+\gamma_d\right)\right)\left(\beta_d-\gamma_d\right)=0. \tag{B88}$$

Since the first term is strictly positive, $\beta_d=\gamma_d$. Further, because the divisions are symmetric, the first-order conditions are symmetric, which implies the existence of a symmetric solution, $\beta_A=\beta_B$. Because the problem is strictly concave on this region, this must be the unique solution. Thus, $a_A=a_B=e^{-\frac{\eta}{2}}\theta\beta q$. Also, $\hat{q}_d^{HQ}=\hat{q}_{d'}^{HQ}=e^{-\frac{\eta^{HQ}}{2}}q$ and $\hat{q}_d^d=\hat{q}_{d'}^d=e^{-\frac{\eta}{2}}q$, so $\Delta_d=(1-2\beta)e^{-\frac{\eta^{HQ}}{2}}q\frac{e^{-\frac{\eta}{2}}\theta\beta q}{2}+\beta e^{-\frac{\eta}{2}}q\frac{e^{-\frac{\eta}{2}}\theta\beta q}{2}$, which gives the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d}=\frac{1}{2}\theta\hat{q}_d^d\hat{q}_d^{HQ}-2\beta\theta\hat{q}_d^d\hat{q}_d^{HQ}+\frac{3}{2}\theta\beta\left(\hat{q}_d^d\right)^2-r\sigma^2\beta\left(1+\rho\right)=0. \tag{B89}$$

and thus

$$\beta_d^5\equiv\frac{1}{1+3\left(1-\hat{q}_d^d/\hat{q}_d^{HQ}\right)+\frac{2r\sigma^2(1+\rho)}{\theta\hat{q}_d^{HQ}\hat{q}_d^d}}=\hat{\beta}. \tag{B90}$$

After substitution, this gives HQ payoff

$$\hat{\pi}^5\equiv\frac{\theta^2q^4e^{-\left(\eta^{HQ}+\eta\right)}}{\theta q^2\left(4e^{-\frac{\left(\eta^{HQ}+\eta\right)}{2}}-3e^{-\eta}\right)+2r\sigma^2\left(1+\rho\right)}. \tag{B91}$$

Theorem 3 showed that $\gamma_d>0$ is optimal when $r=0$. Similarly, $\gamma_d>0$ when $\rho=0$. Further, for $\rho<0$, granting $\gamma_d<0$ results in a larger risk premium, $\frac{r\sigma^2}{2}\left(\beta_d^2+2\rho\beta_d+\gamma_d^2\right)$, than setting $\gamma_d>0$. Thus, $\gamma_d=\beta_d$ dominates all $\gamma_d<0$ with $H_d\in\left(e^{-\eta},e^{\eta}\right)$ for all $\rho\le 0$. Note that $\hat{\pi}^5\ge\hat{\pi}^4$ if and only if $g_L\ge 0$, where

$$g_L \equiv \left(2M+2\left(1-M\right)e^{-\frac{\eta^{HQ}}{2}}+2e^{-\eta}-4e^{-\frac{\left(\eta^{HQ}+\eta\right)}{2}}\right)e^{-\eta}\theta q^2 \tag{B92}$$
$$+r\sigma^2\left(1-2\rho M+M^2-2e^{-\eta}\left(1+\rho\right)\right).$$

and note that $g_L|_{\eta=\eta^{HQ}=0}=-r\sigma^2\left(1+\rho\right)^2<0$, which implies that $\hat{\pi}^4>\hat{\pi}^5$ for $\eta=\eta^{HQ}=0$. Note also that $\frac{\partial g_L}{\partial M}=2\left(1-e^{-\frac{\eta^{HQ}}{2}}\right)e^{-\eta}\theta q^2+2r\sigma^2\left(M-\rho\right)=0$, because $M\equiv\rho-\bar{\rho}$ and $\bar{\rho}\equiv\frac{e^{-\eta}\theta q^2}{r\sigma^2}\left(1-e^{-\frac{\eta^{HQ}}{2}}\right)$, and thus that $\frac{\partial g_L}{\partial\eta}=-g_L+2\left(e^{-\frac{\left(\eta^{HQ}+\eta\right)}{2}}-e^{-\eta}\right)e^{-\eta}\theta q^2+r\sigma^2\left(1-2\rho M+M^2\right)>0$ for all $g_L<0$. This implies that, for a given η^{HQ}, there is a unique $\hat{\eta}$ so that $g_L\left(\hat{\eta},\eta^{HQ}\right)=0$, and for all $\eta>\hat{\eta}$, it is $g_L>0$ and thus $\hat{\pi}^5>\hat{\pi}^4$.

Consider now η^{HQ}. Note first that $\frac{\partial g_L}{\partial\eta^{HQ}}=\left(2e^{-\frac{\eta}{2}}-\left(1-M\right)\right)e^{-\frac{\eta^{HQ}}{2}}e^{-\eta}\theta q^2>0$ for $\eta<\eta'\equiv-2\ln\frac{1}{2}\left(1-M\right)$. Substituting η' in g_L, we obtain

$$g_L|_{\eta=\eta'}\equiv\frac{\left(1+M\right)^2\left(1-M\right)^2}{8}\theta q^2+r\sigma^2\left(1-2\rho M+M^2-\frac{\left(1-M\right)^2}{4}\left(1+\rho\right)\right)>0, \tag{B93}$$

where the inequality is obtained by noting that $h(\rho) \equiv 1-2\rho M+M^2-\frac{(1-M)^2}{4}(1+\rho)$ is linear in ρ for any given M, thus achieving its minimum at an endpoint. Because $h(1)=\frac{1}{2}(1-M)^2>0$ and $h(-1)=(1+M)^2>0$, $h(\rho)>0$ for all $\rho \in [-1,1]$, so $g_L|_{\eta=\eta'}>0$. Thus, in the neighborhood of $g_L=0$, $\eta<\eta'$, so $\frac{\partial g_L}{\partial \eta^{HQ}}>0$. Thus, there is a unique $\hat{\eta}^{HQ}$ (allowing for the possibility that $\hat{\eta}^{HQ}=0$) such that $\hat{\pi}^5>\hat{\pi}^4$ for $\eta^{HQ}>\hat{\eta}^{HQ}$.

Case (B'): Consider $\gamma_d<0$ with $H_d \in \left(e^{-\eta}, e^{\eta}\right)$.

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -\hat{q}_d^{HQ}\check{a}_d+(1-\beta_d-\gamma_{d'})\,\hat{q}_d^{HQ}\frac{\partial \check{a}_d}{\partial \beta_d}+(1-\beta_{d'}-\gamma_d)\,\hat{q}_{d'}^{HQ}\frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad +\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\beta_d}+\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\beta_d}.\end{aligned} \tag{B94}$$

Because $\frac{\partial \check{a}_d}{\partial \beta_d}=\frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d}=\frac{\check{a}_{d'}}{8\beta_d}$, we have that

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -a_d\left(\hat{q}_d^{HQ}-\hat{q}_d^d\right)+(1-\beta_d-\gamma_{d'})\,\hat{q}_d^{HQ}\frac{3\check{a}_d}{8\beta_d}+(1-\beta_{d'}-\gamma_d)\,\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8\beta_d} \\ &\quad -r\sigma^2(\beta_d+\rho\gamma_d)+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\beta_d}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\beta_d}.\end{aligned} \tag{B95}$$

Consider now γ_d. We have that

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\hat{q}_{d'}^{HQ}\check{a}_{d'}+(1-\beta_d-\gamma_{d'})\,\hat{q}_d^{HQ}\frac{\partial \check{a}_d}{\partial \gamma_d}+(1-\beta_{d'}-\gamma_d)\,\hat{q}_{d'}^{HQ}\frac{\partial \check{a}_{d'}}{\partial \gamma_d} \\ &\quad +\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\gamma_d}+\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\gamma_d}.\end{aligned} \tag{B96}$$

Because $\frac{\partial \check{a}_d}{\partial \gamma_d}=\frac{3\check{a}_d}{8\gamma_d}$, $\frac{\partial \check{a}_{d'}}{\partial \gamma_d}=\frac{\check{a}_{d'}}{8\gamma_d}$, we obtain

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^d\right)+(1-\beta_d-\gamma_{d'})\,\hat{q}_d^{HQ}\frac{3\check{a}_d}{8\gamma_d}+(1-\beta_{d'}-\gamma_d)\,\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8\gamma_d} \\ &\quad -r\sigma^2(\gamma_d+\rho\beta_d)+\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\gamma_d}.\end{aligned} \tag{B97}$$

From (B95) and (B97) we obtain the first-order conditions

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -\check{a}_d\left(\hat{q}_d^{HQ}-\hat{q}_d^d\right)-r\sigma^2(\beta_d+\rho\gamma_d)+\frac{\Delta_d}{\beta_d}=0, \\ \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^d\right)-r\sigma^2(\gamma_d+\rho\beta_d)+\frac{\Delta_d}{\gamma_d}=0,\end{aligned} \tag{B98}$$

where $\Delta_d \equiv \phi_d\hat{q}_d^{HQ}\frac{3\check{a}_d}{8}+\phi_{d'}\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8}+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8}$, giving

$$\beta_d\check{a}_d\left(\hat{q}_d^{HQ}-\hat{q}_d^d\right)+r\sigma^2\left(\beta_d^2+\rho\gamma_d\beta_d\right)=\gamma_d\check{a}_{d'}\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^d\right)+r\sigma^2\left(\gamma_d^2+\rho\beta_d\gamma_d\right). \tag{B99}$$

Because the first-order conditions are symmetric, there exists a symmetric solution: $\beta_A=\beta_B=\beta$ and $\gamma_A=\gamma_B=\gamma$. Thus, $a_d=a=e^{-\frac{\eta}{2}}\theta\beta^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}q$. This also implies that $\phi_A=\phi_B$, so $\hat{q}_d^{HQ}=e^{-\frac{\eta^{HQ}}{2}}q$. Also, $H_d=\frac{|\gamma|}{\beta}$, so $\hat{q}_d^d=e^{-\frac{\eta}{2}}\frac{|\gamma|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}q$ and $\hat{q}_{d'}^d=(2-e^{-\frac{\eta}{2}}\frac{\beta^{\frac{1}{2}}}{|\gamma|^{\frac{1}{2}}})q$. Thus, $\beta a\hat{q}_d^d=e^{-\frac{\eta}{2}}\beta^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}aq$ and

$$\gamma a\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^d\right)=\gamma ae^{-\frac{\eta^{HQ}}{2}}q-2\gamma aq-e^{-\frac{\eta}{2}}\beta^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}aq, \tag{B100}$$

which implies that

$$\beta ae^{-\frac{\eta^{HQ}}{2}}q+r\sigma^2\beta^2=|\gamma|\left(2-e^{-\frac{\eta^{HQ}}{2}}\right)aq+r\sigma^2\gamma^2 \tag{B101}$$

Because $\frac{\gamma}{\beta}\in\left(-e^{\eta},-e^{-\eta}\right)$, there exists $\hat{\xi}\in\left(e^{-\eta},e^{\eta}\right)$ such that $\gamma=-\hat{\xi}\beta$. Substituting in $a=e^{-\frac{\eta}{2}}\theta\beta\hat{\xi}^{\frac{1}{2}}q$, (B101) is equivalent to $f\left(\hat{\xi}\right)=0$, where

$$f\left(\hat{\xi}\right)\equiv\left[\left(2e^{\frac{\eta^{HQ}}{2}}-1\right)\hat{\xi}-1\right]e^{-\frac{\eta^{HQ}}{2}}e^{-\frac{\eta}{2}}\hat{\xi}^{\frac{1}{2}}\theta q^2+r\sigma^2\left(\hat{\xi}^2-1\right)=0. \tag{B102}$$

Note $f\left(e^{-\eta}\right)<0<f\left(1\right)=2\left[e^{\frac{\eta^{HQ}}{2}}-1\right]e^{-\frac{\eta^{HQ}+\eta}{2}}\theta q^{2}$ and $f'>0$, so $\hat{\xi}\in\left(e^{-\eta},1\right)$ for $\eta^{HQ}>0$, but $\hat{\xi}=1$ if $\eta^{HQ}=0$. Comparative statics on $\hat{\xi}$ follow because $max\left\{\frac{\partial f}{\partial r},\frac{\partial f}{\partial\sigma^{2}},\frac{\partial f}{\partial\eta}\right\}<0<\min\left\{\frac{\partial f}{\partial\theta},\frac{\partial f}{\partial q},\frac{\partial f}{\partial\eta^{HQ}}\right\}$. Further, $\frac{\partial\hat{\pi}}{\partial\beta}=0$ iff

$$\beta=\frac{1}{1+\left(\frac{\hat{q}_{d'}^{d}}{\hat{q}_{d'}^{HQ}}-1\right)\hat{\xi}+2(1-\frac{\hat{q}_{d}^{d}}{\hat{q}_{d}^{HQ}})+\frac{2r\sigma^{2}\left(1-\rho\hat{\xi}\right)}{\theta\hat{q}_{d}^{HQ}\hat{q}_{d}^{d}}};\quad\gamma=-\hat{\xi}\beta<0. \tag{B103}$$

After substitution in $\hat{\pi}$, we have

$$\hat{\pi}^{6}\equiv\frac{e^{-(\eta^{HQ}+\eta)}\hat{\xi}\theta^{2}q^{4}}{2e^{-\frac{\eta^{HQ}}{2}}\left[1+\left(2e^{\frac{\eta^{HQ}}{2}}-1\right)\hat{\xi}\right]e^{-\frac{\eta}{2}}\hat{\xi}^{\frac{1}{2}}\theta q^{2}-3e^{-\eta}\hat{\xi}q^{2}\theta+r\sigma^{2}\left(1-2\rho\hat{\xi}+\hat{\xi}^{2}\right)}. \tag{B104}$$

Note $\hat{\pi}^{6}\geq\hat{\pi}^{4}$ if and only if $g_{S}\geq0$, where

$$\begin{aligned} g_{S}\equiv&\left(2M+2\left(1-M\right)e^{-\frac{\eta^{HQ}}{2}}+2e^{-\eta}\right)\theta q^{2}+e^{\eta}r\sigma^{2}\left(1-2\rho M+M^{2}\right)\\ &-2e^{-\frac{\eta^{HQ}}{2}}\left[1+\left(2e^{\frac{\eta^{HQ}}{2}}-1\right)\hat{\xi}\right]e^{-\frac{\eta}{2}}\hat{\xi}^{-\frac{1}{2}}\theta q^{2}-r\sigma^{2}\frac{\left(1-2\rho\hat{\xi}+\hat{\xi}^{2}\right)}{\hat{\xi}}, \end{aligned} \tag{B105}$$

with

$$\frac{\partial g_{S}}{\partial\eta}=e^{\eta}r\sigma^{2}\left(1-\rho^{2}+(\rho-M)^{2}\right)+\{e^{-\frac{\eta^{HQ}}{2}}e^{-\frac{\eta}{2}}\left[1+\left(2e^{\frac{\eta^{HQ}}{2}}-1\right)\hat{\xi}\right]\hat{\xi}^{-\frac{1}{2}}-2e^{-\eta}\}\theta q^{2}. \tag{B106}$$

Note that $\left[1+\left(2e^{\frac{\eta^{HQ}}{2}}-1\right)\hat{\xi}\right]\hat{\xi}^{-\frac{1}{2}}$ is increasing and larger than 2 for $\hat{\xi}\in\left(e^{-\eta},1\right)$, so $\frac{\partial g_{S}}{\partial\eta}>0$. Also, $\frac{\partial g_{S}}{\partial\eta^{HQ}}=-\left(1-M\right)e^{-\frac{\eta^{HQ}}{2}}\theta q^{2}+\left(1-\hat{\xi}\right)e^{-\frac{\eta^{HQ}}{2}}e^{-\frac{\eta}{2}}\hat{\xi}^{-\frac{1}{2}}\theta q^{2}$. Because $M<e^{-\eta}<\hat{\xi}$, we have that $\frac{\partial g_{S}}{\partial\eta^{HQ}}<0$. Defining $\hat{\eta}$, $\hat{\eta}_{1}^{HQ}$ so that $g_{S}\left(\hat{\eta},\hat{\eta}_{1}^{HQ}\right)=0$, part (ii)(b) of Theorem 4 is proven.

Case (C): If $\gamma_{d}>e^{\eta}\beta_{d}$, $\frac{\partial a_{d}}{\partial\gamma_{d}}=0$, so $\frac{\partial a_{d'}}{\partial\gamma_{d}}=0$, and thus $\frac{\partial\hat{\pi}}{\partial\gamma_{d}}=-a_{d'}\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^{d}\right)-r\sigma^{2}\left(\rho\beta+\gamma\right)<0$, so $\gamma\leq e^{\eta}\beta$. Similarly, for Case (C'), if $\gamma_{d}<-e^{\eta}\beta_{d}$, $\frac{\partial a_{d}}{\partial\gamma_{d}}=\frac{\partial a_{d'}}{\partial\gamma_{d}}=0$, so $\frac{d\hat{\pi}}{d\gamma_{d}}=-a_{d'}\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^{d}\right)-r\sigma^{2}\left(\rho\beta_{d}+\gamma_{d}\right)$. Because $\phi_{d'}>0>\gamma_{d}$, $\hat{q}_{d'}^{HQ}<q_{d'}<\hat{q}_{d'}^{d}$. Also, $\rho\in(-1,1)$. Thus, $\frac{d\hat{\pi}}{d\gamma_{d}}>0$ for $\gamma_{d}<-e^{\eta}\beta_{d}$, so it must be that $\gamma_{d}\geq-e^{\eta}\beta_{d}$. Therefore, Cases (C) and (C') are suboptimal.

All that remains to be shown is part (ii)(b) of Theorem 4, by showing that $\hat{\pi}^{5}\geq\hat{\pi}^{6}$ when η^{HQ} is large enough. Note $\hat{\pi}^{5}\geq\hat{\pi}^{6}$ if and only if $g_{E}\geq0$, where

$$\begin{aligned} g_{E}\equiv&\ 2e^{-\frac{\eta^{HQ}}{2}}\left[1+\left(2e^{\frac{\eta^{HQ}}{2}}-1\right)\hat{\xi}\right]e^{-\frac{\eta}{2}}\hat{\xi}^{-\frac{1}{2}}\theta q^{2}+r\sigma^{2}\left(1-2\rho\hat{\xi}+\hat{\xi}^{2}\right)/\hat{\xi}\\ &-4e^{-\frac{(\eta^{HQ}+\eta)}{2}}\theta q^{2}-2r\sigma^{2}\left(1+\rho\right). \end{aligned} \tag{B107}$$

Note $\frac{\partial g_{E}}{\partial\hat{\xi}}=\frac{f(\hat{\xi})}{\hat{\xi}^{2}}=0$. Note that

$$\frac{\partial g_{E}}{\partial\eta^{HQ}}=\left[-\left(1-\hat{\xi}\right)\hat{\xi}^{-\frac{1}{2}}+2\right]e^{-\frac{(\eta^{HQ}+\eta)}{2}}\theta q^{2}\geq0 \tag{B108}$$

if and only if $\hat{\xi}\geq3-2\sqrt{2}$. Recall $\hat{\xi}$ is strictly decreasing in η^{HQ}. This implies that g_{E} an inverse U-shaped function of η^{HQ} and that there is a unique $\tilde{\eta}^{HQ}$, defined by $\hat{\xi}(\tilde{\eta}^{HQ})=3-2\sqrt{2}$, such that $\frac{\partial g_{E}}{\partial\eta^{HQ}}>0$ for $\eta^{HQ}<\tilde{\eta}^{HQ}$ and $\frac{\partial g_{E}}{\partial\eta^{HQ}}<0$ for $\eta^{HQ}>\tilde{\eta}^{HQ}$. Next, we will show that $g_{E}>0$ for all $\eta^{HQ}\geq\tilde{\eta}^{HQ}$ and, thus, for all $\hat{\xi}\leq3-2\sqrt{2}$. Note that, from (B102), we can express (B107) as

$$g_{E}=4e^{-\frac{\eta^{HQ}}{2}}e^{-\frac{\eta}{2}}\left(\hat{\xi}^{-\frac{1}{2}}-1\right)\theta q^{2}+\frac{f\left(\hat{\xi}\right)}{\hat{\xi}}+2r\sigma^{2}\left[\frac{1}{\hat{\xi}}-2\rho-1\right]. \tag{B109}$$

The first term is positive because $\hat{\xi}<1$, the second term is zero, and the third term is positive for all $\frac{1}{\hat{\xi}}>3$, which is satisfied for $\hat{\xi}\leq3-2\sqrt{2}<\frac{1}{3}$. This implies that $g_{E}(\eta^{HQ})>0$ for all $\eta^{HQ}\geq\tilde{\eta}^{HQ}$. Thus, if $g_{E}\left(0\right)\geq0$, $g_{E}>0$ for all $\eta^{HQ}>0$, and thus define $\hat{\eta}_{2}^{HQ}\equiv0$; otherwise, if $g_{E}\left(0\right)<0$, there is a unique $\hat{\eta}_{2}^{HQ}$ such that $g_{E}(\hat{\eta}_{2}^{HQ})=0$, with

$\hat{\eta}_2^{HQ} < \tilde{\eta}^{HQ}$, completing the proof of Theorem 4. ■

Proof of Lemma 5. First, the division manager is indifferent between contracts $(\beta, 0, \psi)$ and $(\beta, 0, -\psi)$, so we restrict attention to $\psi \geq 0$ (if $\psi < 0$, the proof follows by substituting in $|\psi|$ for ψ). Next, the division manager is indifferent between contracts $(\beta, 0, \psi)$ and $(\beta, \psi, 0)$. Note granting $(\beta, 0, \psi)$ gives HQ $\Pi = \min_{q^{HQ} \in K^{HQ}} \pi$, where

$$\pi = (1-\beta)\, a_A q_A^{HQ} + \mu q_B^{HQ} - \psi \mu q_C^{HQ} - s \tag{B110}$$

Conversely, granting $(\beta, \psi, 0)$ gives HQ $\Pi = \min_{q^{HQ} \in K^{HQ}} \pi$, where

$$\pi = (1-\beta)\, a_A q_A^{HQ} + (1-\psi)\, \mu q_B^{HQ} - s \tag{B111}$$

Note that the s and a_A will be the same with both contracts. Thus, it is better to grant $(\beta, \psi, 0)$ than $(\beta, 0, \psi)$ iff

$$\min_{q^{HQ} \in K^{HQ}} (1-\beta)\, a_A q_A^{HQ} + (1-\psi)\, \mu q_B^{HQ} \geq \min_{q^{HQ} \in K^{HQ}} (1-\beta)\, a_A q_A^{HQ} + \mu q_B^{HQ} - \psi \mu q_C^{HQ} \tag{B112}$$

Let the solution to the left-hand side be $\hat{q}^{HQ}$ and the solution to the right-hand side be $\tilde{q}^{HQ}$. Because $\psi > 0$, we know that $\tilde{q}_C^{HQ} \geq q_C$. First, suppose that $\psi \leq 1$. This implies that, by the definition of the minimum,

$$\begin{aligned}
(1-\beta)\, a_A \tilde{q}_A^{HQ} + \mu \tilde{q}_B^{HQ} - \psi \mu \tilde{q}_C^{HQ} &\leq (1-\beta)\, a_A \hat{q}_A^{HQ} + \mu \hat{q}_B^{HQ} - \psi \mu q_C \\
&\leq (1-\beta)\, a_A \hat{q}_A^{HQ} + \mu \hat{q}_B^{HQ} - \psi \mu \hat{q}_B^{HQ} \\
&= \min_{q^{HQ} \in K^{HQ}} (1-\beta)\, a_A q_A^{HQ} + (1-\psi)\, \mu q_B^{HQ}
\end{aligned} \tag{B113}$$

Because $(1-\psi) \geq 0$, $\hat{q}_B^{HQ} \leq q_B$, so the second line follows by monotonicity. Therefore, for $\psi < 1$, it is better to pay with B than with C. If $\psi > 1$, by symmetry of the set, because $\left(\hat{q}_A^{HQ}, \hat{q}_B^{HQ}, q_C\right) \in K^{HQ}$ and $q_B = q_C$, then $\left(\hat{q}_A^{HQ}, q_B, \hat{q}_B^{HQ}\right) \in K^{HQ}$. Thus,

$$\begin{aligned}
(1-\beta)\, a_A \tilde{q}_A^{HQ} + \mu \tilde{q}_B^{HQ} - \psi \mu \tilde{q}_C^{HQ} &\leq (1-\beta)\, a_A \hat{q}_A^{HQ} + \mu q_B - \psi \mu \hat{q}_B^{HQ} \\
&\leq (1-\beta)\, a_A \hat{q}_A^{HQ} + \mu_B \hat{q}_B^{HQ} - \psi \mu_B \hat{q}_B^{HQ} \\
&= \min_{q^{HQ} \in K^{HQ}} (1-\beta)\, a_A q_A^{HQ} + (1-\psi)\, \mu_B q_B^{HQ}
\end{aligned} \tag{B114}$$

The second line holds by monotonicity, because $\psi > 1$, so HQ has a negative exposure to B, and thus $\hat{q}_B^{HQ} \geq q_B$. ■

Proof of Theorem 5. Note that the participation constraint binds, so we can express the objective as

$$\Pi = (1-\beta)\, a_A \hat{q}_A^{HQ} + (1-\gamma)\, \mu \hat{q}_B^{HQ} - \psi \mu \hat{q}_C^{HQ} + \hat{U} \tag{B115}$$

where $\hat{q}^{HQ}$ solves $\min_{q^{HQ} \in K^{HQ}} \pi$ and $\hat{u} = \beta a_A q_A^A + \gamma \mu q_B^A + \psi \mu q_C^A - \frac{r\sigma^2}{2}\left(\beta^2 + \gamma^2 + \psi^2\right) - \frac{a_A^2}{2\theta_A}$ and $\hat{U} = \min_{q^A \in K^A} \hat{u}$. Applying the envelope theorem.

$$\frac{d\Pi}{d\beta} = -a_A \hat{q}_A^{HQ} + \frac{\partial \hat{U}}{\partial \beta} + \frac{\partial \Pi}{\partial a_A} \frac{\partial a_A}{\partial \beta} \tag{B116}$$

Because $\frac{\partial U}{\partial \beta} = a_A \hat{q}_A^A - r\sigma^2 \beta$, we can express this as

$$\frac{d\Pi}{d\beta} = -a_A \left(\hat{q}_A^{HQ} - \hat{q}_A^A\right) - r\sigma^2 \beta + \frac{\partial \Pi}{\partial a_A} \frac{\partial a_A}{\partial \beta} \tag{B117}$$

Similarly, because $\frac{\partial U}{\partial \gamma} = \mu \hat{q}_B^A - r\sigma^2 \gamma$ and $\frac{\partial U}{\partial \psi} = \mu \hat{q}_C^A - r\sigma^2 \psi$ we can express

$$\frac{d\Pi}{d\gamma} = -\mu \left(\hat{q}_B^{HQ} - \hat{q}_B^A\right) - r\sigma^2 \gamma + \frac{\partial \Pi}{\partial a_A} \frac{\partial a_A}{\partial \gamma} \tag{B118}$$

and

$$\frac{d\Pi}{d\psi} = -\mu \left(\hat{q}_C^{HQ} - \hat{q}_C^A\right) - r\sigma^2 \psi + \frac{\partial \Pi}{\partial a_A} \frac{\partial a_A}{\partial \psi}$$

Next, note that HQ will set $\psi \neq 0$ only if $\frac{\partial a_A}{\partial \psi} > 0$. Suppose to the contrary that $\frac{\partial a_A}{\partial \psi} = 0$, or equivalently, that either $\hat{q}_C^A = q_C$ or $\hat{q}_C^A = e^{-\eta^A} q_C$. For $\psi > 0$, HQ is negative exposed to the source of risk but the DM is positively exposed, $\hat{q}_C^{HQ} \geq q_C \geq \hat{q}_C^A$, so $\frac{d\Pi}{d\psi} < 0$ for such $\psi > 0$. Similarly, for $\psi < 0$, $\hat{q}_C^{HQ} \leq q_C \leq \hat{q}_C^A$, so $\frac{d\Pi}{d\psi} > 0$ for such $\psi < 0$. Thus, either DM will have interior beliefs toward the external risk, so that $\frac{\partial a_A}{\partial \psi} > 0$, or HQ will grant him no exposure, $\psi = 0$.

Thus, we have shown that there are three possible regions. First, it could be possible for HQ to grant only division-based pay. Second, it could be possible for HQ to grant only division-based pay and the internal risk, but not the external risk. By Lemma 5, we can exclude the corresponding region of granting only division-based pay and the external risk for all $\eta^{HQ} > 0$, and it is WLOG optimal to grant the internal risk rather than the external even when $\eta^{HQ} = 0$ (the visionary HQ is indifferent). Finally, we can consider when HQ grants division-based pay, the internal risk, and the external risk. As we showed in the previous paragraph, it must grant sufficiently of this to induce interior beliefs, so that $\frac{da_A}{d\psi} > 0$. Note that, similar to Lemma 2, on this region, interior beliefs satisfy $\lambda = \beta a_A \hat{q}_A^A = \gamma \mu_B \hat{q}_B^A = \psi \mu_C \hat{q}_C^A$ which implies $\lambda = e^{-\frac{\eta^A}{3}} \left[\beta a_A q_A \gamma \mu_B q_B \psi \mu_C q_C\right]^{\frac{1}{3}}$, so $\hat{q}_A^A = e^{-\frac{\eta^A}{3}} \beta_A^{-\frac{2}{3}} a_A^{-\frac{2}{3}} \left[q_A \gamma \mu_B q_B \psi \mu_C q_C\right]^{\frac{1}{3}}$, $\hat{q}_B^A = e^{-\frac{\eta^A}{3}} \gamma^{-\frac{2}{3}} \mu_B^{-\frac{2}{3}} \left[\beta a_A q_A q_B \psi \mu_C q_C\right]^{\frac{1}{3}}$, and $\hat{q}_C^A = e^{-\frac{\eta^A}{3}} \psi^{-\frac{2}{3}} \mu_C^{-\frac{2}{3}} \left[\beta a_A q_A \gamma \mu_B q_B q_C\right]^{\frac{1}{3}}$. Substituting into $a_A = \beta_A \theta_A \hat{q}_A^A$, this implies $a_A = e^{-\frac{\eta^A}{5}} \theta_A^{\frac{3}{5}} \beta^{\frac{1}{5}} q_A^{\frac{1}{5}} \gamma^{\frac{1}{5}} \mu_B^{\frac{1}{5}} q_B^{\frac{1}{5}} \psi^{\frac{1}{5}} \mu_C^{\frac{1}{5}} q_C^{\frac{1}{5}}$, so $\frac{\partial a_A}{\partial \beta} = \frac{a_A}{5\beta}$, $\frac{\partial a_A}{\partial \gamma} = \frac{a_A}{5\gamma}$, and $\frac{\partial a_A}{\partial \psi} = \frac{a_A}{5\psi}$.

Suppose to the contrary that HQ used all three, setting $\beta > 0$, $\gamma > 0$, and $\psi > 0$. Substituting in for $\frac{\partial a_A}{\partial \beta}$, $\frac{\partial a_A}{\partial \gamma}$, and $\frac{\partial a_A}{\partial \psi}$, the FOCs simplify to

$$\beta a_A \left(\hat{q}_A^{HQ} - \hat{q}_A^A\right) + r\sigma^2\beta^2 = \frac{\partial \Pi}{\partial a_A}\frac{a_A}{5} \tag{B119}$$

$$\gamma \mu \left(\hat{q}_B^{HQ} - \hat{q}_B^A\right) + r\sigma^2\gamma^2 = \frac{\partial \Pi}{\partial a_A}\frac{a_A}{5} \tag{B120}$$

$$\psi \mu \left(\hat{q}_C^{HQ} - \hat{q}_C^A\right) + r\sigma^2\psi^2 = \frac{\partial \Pi}{\partial a_A}\frac{a_A}{5} \tag{B121}$$

This would imply that

$$\begin{aligned} \beta a_A \left(\hat{q}_A^{HQ} - \hat{q}_A^A\right) + r\sigma^2\beta^2 &= \gamma \mu \left(\hat{q}_B^{HQ} - \hat{q}_B^A\right) + r\sigma^2\gamma^2 \\ &= \psi \mu \left(\hat{q}_C^{HQ} - \hat{q}_C^A\right) + r\sigma^2\psi^2 \end{aligned} \tag{B122}$$

Also, because the DM has interior beliefs, $\beta a_A \hat{q}_A^A = \gamma \mu \hat{q}_B^A = \psi \mu \hat{q}_C^A$, so this implifies to

$$\beta a_A \hat{q}_A^{HQ} + r\sigma^2\beta^2 = \gamma \mu \hat{q}_B^{HQ} + r\sigma^2\gamma^2 = \psi \mu \hat{q}_C^{HQ} + r\sigma^2\psi^2 \tag{B123}$$

When η^{HQ} is big enough, the HQ will have interior beliefs toward asset A and B, unless $\beta = 1$ and $\gamma = 1$. This implies that $(1-\beta)\, a_A \hat{q}_A^{HQ} = (1-\gamma)\, \mu \hat{q}_B^{HQ}$, so $\mu \hat{q}_B^{HQ} = \frac{1-\beta}{1-\gamma} a_A \hat{q}_A^{HQ}$. Similarly to the proof of Corollary 1, $\beta = \gamma$.

Suppose that it is optimal to set $\psi > 0$ (symmetric arguments for $\psi < 0$ will hold). For this to be optimal, it must be that $\hat{q}_C^A < q_C$, or equivalently, $\psi \mu q_C > \gamma \mu \hat{q}_B^A$. Note $\hat{q}_B^A > e^{-\eta^A} q_B = e^{-\eta^A} q_C$ because $q_B = q_C$. Thus, for the DM to have interior beliefs, it must be that $\psi > \beta e^{-\eta^A}$. Back to HQ's FOC, this implies that

$$\psi \mu \hat{q}_C^{HQ} + r\sigma^2\psi^2 > \beta e^{-\eta^A} \mu \hat{q}_C^{HQ} + r\sigma^2 \left(\beta e^{-\eta^A}\right)^2 \geq \beta e^{-\eta^A} \mu q_B + r\sigma^2 \left(\beta e^{-\eta^A}\right)^2 \tag{B124}$$

The second inequality holds because HQ has a negative exposure to the external risk, so $\hat{q}_C^{HQ} \geq q_C$ and $q_C = q_B$. If HQ has interior beliefs toward all three sources of risk, $\hat{q}_B^{HQ} = e^{-\frac{\eta^{HQ}}{3}} (1-\gamma)^{-\frac{2}{3}} \mu_B^{-\frac{2}{3}} \left[(1-\beta)\, a_A q_A q_B \psi \mu_C q_C\right]^{\frac{1}{3}}$, which is decreasing in η^{HQ}. Thus, $\exists \bar{\eta}^{HQ}$ such that for all $\eta^{HQ} > \bar{\eta}^{HQ}$

$$\beta e^{-\eta^A} \mu q_B + e^{-2\eta^A} r\sigma^2\beta^2 > \beta \mu \hat{q}_B^{HQ} + r\sigma^2\beta^2 \tag{B125}$$

Thus, when η^{HQ} is large enough, $\frac{d\Pi}{d\psi} < 0$ for all $\psi > \beta e^{-\eta^A}$, so it is optimal to set $\psi = 0$. ∎

Proof of Theorem 7. Because there are synergies, the output of each division is increasing in the effort of both division managers. That is, the drift of division A is $(a_A + \zeta a_B)\, q_A$, and the drift of division B is $(a_B + \zeta a_A)\, q_B$. For simplicity, we will assume that HQ is uncertainty neutral, though HQ uncertainty only makes equity more attractive. Because the participation constraint binds, the payoff to HQ will be

$$\pi = (1 - \beta_A - \gamma_B)(a_A + \zeta a_B)\, q_A + (1 - \beta_B - \gamma_A)(a_B + \zeta a_A)\, q_B + \hat{U}_A + \hat{U}_B \tag{B126}$$

Division manager d has payoff $\hat{U}_d = \min_{q^d \in K^d} \hat{u}_d$, where

$$\hat{u}_d = \beta_d (a_d + \zeta a_{d'})\, \hat{q}_d^d + \gamma_d (a_{d'} + \zeta a_d)\, \hat{q}_{d'}^d - \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) - \frac{a_d^2}{2\theta_d} \tag{B127}$$

Beliefs are as solved in Lemma 2. We are also assuming both divisions have the same η. Let $H_d = \frac{|\gamma_d|\left(a_{d'}+\zeta a_d\right)q_{d'}}{\beta_d\left(a_d+\zeta a_{d'}\right)q_d}$. If $H_d \leq e^{-\eta}$, $\hat{q}_d^d = e^{-\eta}q_d$ and $\hat{q}_{d'}^d = q_{d'}$. Because we are assuming symmetry, these are the same. Conversely, if $\gamma_d > 0$ and $H_d \in \left[e^{-\eta}, e^{\eta}\right]$, then $\hat{q}_d^d = \left(e^{-\eta}H_d\right)^{\frac{1}{2}} q_d$ and $\hat{q}_{d'}^d = \left(e^{-\eta}\frac{1}{H_d}\right)^{\frac{1}{2}} q_{d'}$. Finally, if $\gamma_d < 0$ and $H_d \in \left[e^{-\eta}, e^{\eta}\right]$, $\hat{q}_d^d = \left(e^{-\eta}H_d\right)^{\frac{1}{2}} q_d$ but $\hat{q}_{d'}^d = \left[2 - \left(e^{-\eta}\frac{1}{H_d}\right)^{\frac{1}{2}}\right] q_{d'}$. Given beliefs, note $\frac{\partial u_d}{\partial a_d} = \beta_d \hat{q}_d^d + \gamma_d \zeta \hat{q}_{d'}^d - \frac{a_d}{\theta_d}$, so

$$a_d = \theta_d \left[\beta_d \hat{q}_d^d + \gamma_d \zeta \hat{q}_{d'}^d\right] \tag{B128}$$

There are three types of contracts that might arise in equilibrium: interior beliefs with long exposure, $H_d \in \left[e^{-\eta}, e^{\eta}\right]$ with $\gamma_d > 0$, interior beliefs with short exposure, $H_d \in \left[e^{-\eta}, e^{\eta}\right]$ and $\gamma_d < 0$, and corner beliefs, $H_d \leq e^{-\eta}$. First, we will show that the optimal contract when $H_d \in \left[e^{-\eta}, e^{\eta}\right]$ with $\gamma_d > 0$ is equity. Then, we will show that the optimal contract will be on this region when ζ is large enough.

First, note that the optimal contract is symmetric and induces symmetric effort, $a_A = a_B$. Suppose to the contrary that the HQ gives different contracts to the different division managers. By symmetry, the HQ receives the same payoff by trading the contract between the two managers. Note that the objective, as the minimum of strictly concave functions, is strictly concave. Thus, HQ receives a strictly higher payoff by giving both division managers the average of the two contracts, so it cannot be optimal to give different contracts to the two division managers. Because the HQ grants the same contract to the two division managers, in equilibrium, they will exert symmetric effort: $a_d = a_{d'}$.

When $H_d \in \left[e^{-\eta}, e^{\eta}\right]$ and $\gamma_d > 0$, from Lemma 2 and because $a_d = a_{d'}$, $\hat{q}_d^d = e^{-\frac{\eta}{2}}\gamma_d^{\frac{1}{2}} q_{d'}^{\frac{1}{2}} \beta_d^{-\frac{1}{2}} q_d^{\frac{1}{2}}$ and $\hat{q}_{d'}^d = e^{-\frac{\eta}{2}}\beta_d^{\frac{1}{2}} q_d^{\frac{1}{2}} \gamma_d^{-\frac{1}{2}} q_{d'}^{\frac{1}{2}}$. This implies that

$$a_d = e^{-\frac{\eta}{2}}\beta_d^{\frac{1}{2}}\gamma_d^{\frac{1}{2}}\theta_d q_{d'}^{\frac{1}{2}} q_d^{\frac{1}{2}}\left(1+\zeta\right) \tag{B129}$$

Note that the effort of each division manager depends on β_d or γ_d only through their geometric mean, $\left(\beta_d\gamma_d\right)^{\frac{1}{2}}$.

Consider the optimal contract that induces interior beliefs from both division managers.

$$\pi = \left(1-\beta_A-\gamma_B\right)\left(a_A+\zeta a_B\right)q_A + \left(1-\beta_B-\gamma_A\right)\left(a_B+\zeta a_A\right)q_B + \hat{U}_A + \hat{U}_B \tag{B130}$$

where $\hat{U}_d = \beta_d\left(a_d+\zeta a_{d'}\right)\hat{q}_d^d + \gamma_d\left(a_{d'}+\zeta a_d\right)\hat{q}_{d'}^d - \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) - \frac{a_d^2}{2\theta_d}$. Note that

$$\frac{d\pi}{d\beta_A} = \frac{\partial\pi}{\partial\beta_A} + \frac{\partial\pi}{\partial a_A}\frac{da_A}{d\beta_A} + \frac{\partial\pi}{\partial a_B}\frac{da_B}{d\beta_A}$$

where $\frac{\partial\pi}{\partial\beta_A} = -\left(a_A+\zeta a_B\right)q_A + \frac{\partial\hat{U}_A}{\partial\beta_A}$, and $\frac{\partial\hat{U}_A}{\partial\beta_A} = \left(a_A+\zeta a_B\right)\hat{q}_A^A - r\sigma^2\left(\beta_A+\rho\gamma_A\right)$. Further, $\frac{\partial\pi}{\partial a_A} = \left(1-\beta_A-\gamma_B\right)q_A + \zeta\left(1-\beta_B-\gamma_A\right)q_B + \frac{\partial\hat{U}_B}{\partial a_A}$, where $\frac{\partial\hat{U}_B}{\partial a_A} = \zeta\beta_B\hat{q}_B^B + \gamma_B\hat{q}_A^B$. Recall $\frac{\partial\hat{U}_A}{\partial a_A} = 0$ by the envelope theorem. Because a_A depends on the incentive contract only through $I_A \equiv \left(\beta_A\gamma_A\right)^{\frac{1}{2}}$, $\frac{da_A}{d\beta_A} = \frac{1}{2}\frac{\partial a_A}{\partial I_A}\left(\frac{\gamma_A}{\beta_A}\right)^{\frac{1}{2}}$. Thus,

$$\frac{d\pi}{d\beta_A} = -\left(a_A+\zeta a_B\right)\left(q_A-\hat{q}_A^A\right) - r\sigma^2\left(\beta_A+\rho\gamma_A\right) + \frac{\partial\pi}{\partial a_A}\frac{1}{2}\frac{\partial a_A}{\partial I_A}\left(\frac{\gamma_A}{\beta_A}\right)^{\frac{1}{2}}$$

Similarly,

$$\frac{d\pi}{d\gamma_A} = -\left(a_B+\zeta a_A\right)\left(q_B-\hat{q}_B^A\right) - r\sigma^2\left(\rho\beta_A+\gamma_A\right) + \frac{\partial\pi}{\partial a_A}\frac{1}{2}\frac{\partial a_A}{\partial I_A}\left(\frac{\beta_A}{\gamma_A}\right)^{\frac{1}{2}}$$

Therefore, $\frac{d\pi}{d\beta_A} = 0$ iff

$$\frac{\partial\pi}{\partial a_A}\frac{1}{2}\frac{\partial a_A}{\partial I_A}\left(\gamma_A\beta_A\right)^{\frac{1}{2}} = \beta_A\left(a_A+\zeta a_B\right)\left(q_A-\hat{q}_A^A\right) + r\sigma^2\left(\beta_A^2+\rho\gamma_A\beta_A\right)$$

and $\frac{d\pi}{d\gamma_A} = 0$ iff

$$\frac{\partial\pi}{\partial a_A}\frac{1}{2}\frac{\partial a_A}{\partial I_A}\left(\gamma_A\beta_A\right)^{\frac{1}{2}} = \gamma_A\left(a_B+\zeta a_A\right)\left(q_B-\hat{q}_B^A\right) + r\sigma^2\left(\rho\beta_A\gamma_A+\gamma_A^2\right)$$

Therefore, the optimal contract satisfies

$$\beta_A\left(a_A+\zeta a_B\right)\left(q_A-\hat{q}_A^A\right) + r\sigma^2\left(\beta_A^2+\rho\gamma_A\beta_A\right) = \gamma_A\left(a_B+\zeta a_A\right)\left(q_B-\hat{q}_B^A\right) + r\sigma^2\left(\rho\beta_A\gamma_A+\gamma_A^2\right)$$

Because we have interior beliefs, it must be that $\beta_A(a_A+\zeta a_B)\hat{q}_A^A = \gamma_A(a_B+\zeta a_A)\hat{q}_B^A$. Symmetric conditions hold for division B, with $q_A = q_B$, and we already showed $a_A = a_B$. Thus, the optimal contract must satisfy $f(\beta_A) = f(\gamma_A)$, where $f(x) = xa(1+\zeta)q + r\sigma^2 x^2$. Because f is monotonic, $\beta_A = \gamma_A$. Therefore, the optimal contract when $H_d \in [e^{-\eta}, e^{\eta}]$ and $\gamma_d > 0$ is equity, with $H_d = 1$. Substituting back into the FOCs, it can be shown that the optimal incentive level is $\beta = \frac{e^{-\frac{\eta}{2}}\theta(1+\zeta)^2q^2}{4e^{-\frac{\eta}{2}}\theta(1+\zeta)^2q^2 - 3e^{-\eta}\theta q^2(1+\zeta)^2 + 2r\sigma^2(1+\rho)}$.

Next consider the optimal contract with $H_d \in [e^{-\eta}, e^{\eta}]$ but $\gamma_d < 0$. From Lemma 2, $\hat{q}_d^d = (e^{-\eta}H_d)^{\frac{1}{2}} q_d$ but $\hat{q}_{d'}^d = \left[2 - \left(e^{-\eta}\frac{1}{H_d}\right)^{\frac{1}{2}}\right] q_{d'}$. Substituting into the FOC for effort, $a_d = \theta_d\left[\beta_d \hat{q}_d^d + \gamma_d\zeta\hat{q}_{d'}^d\right]$ and applying symmetry,

$$a_d = \theta_d q\left\{(1-\zeta)e^{-\frac{\eta}{2}}\beta_d^{\frac{1}{2}}|\gamma_d|^{\frac{1}{2}} + 2\gamma_d\zeta\right\} \tag{B131}$$

Note that a is increasing in β on this region, and we are on this region only if $\beta < e^{\eta}|\gamma|$, which implies that

$$a_d \leq \theta q|\gamma|(1-3\zeta) \tag{B132}$$

Therefore, for $\zeta \geq \frac{1}{3}$, $a_d \leq 0$ on this region. That is, for $\zeta \geq \frac{1}{3}$, there is no contract that induces effort with $\beta > 0$ and $\gamma < -e^{-\eta}\beta$. Any such contract would be dominated by setting $\beta = \gamma = 0$.

Finally, let us consider the optimal contract with corner beliefs, $H_d \leq e^{-\eta}$, so that $\hat{q}_d^d = e^{-\eta}q_d$ and $\hat{q}_{d'}^d = q_{d'}$. In this case,

$$a_d = \theta_d\left[\beta_d e^{-\eta}q + \gamma_d\zeta q\right] \tag{B133}$$

Note the HQ has payoff

$$\pi = (1-\beta_A-\gamma_B)(a_A+\zeta a_B)q_A + (1-\beta_B-\gamma_A)(a_B+\zeta a_A)q_B + \hat{U}_A + \hat{U}_B \tag{B134}$$

Thus,

$$\frac{d\pi}{d\beta_A} = \frac{\partial\pi}{\partial\beta_A} + \frac{\partial\pi}{\partial a_A}\frac{da_A}{d\beta_A} \tag{B135}$$

Note that $\frac{\partial\pi}{\partial\beta_A} = -(a_A+\zeta a_B)q_A + \frac{\partial\hat{U}_A}{\partial\beta_A}$, where $\frac{\partial\hat{U}_A}{\partial\beta_A} = (a_A+\zeta a_B)e^{-\eta}q_A - r\sigma^2(\beta_A+\rho\gamma_A)$. Thus, $\frac{\partial\pi}{\partial\beta_A} = -(a_A+\zeta a_B)q_A(1-e^{-\eta}) - r\sigma^2(\beta_A+\rho\gamma_A)$. Further, $\frac{\partial\pi}{\partial a_A} = (1-\beta_A-\gamma_B)q_A + (1-\beta_B-\gamma_A)\zeta q_B + \frac{\partial\hat{U}_B}{\partial a_A}$, where $\frac{\partial\hat{U}_B}{\partial a_A} = \gamma_B q_A + \zeta\beta_B e^{-\eta}q_B$: by the envelope theorem, $\frac{d\hat{U}_A}{da_A} = 0$. Because $\frac{da_A}{d\beta_A} = \theta e^{-\eta}q$, $\frac{\partial\pi}{\partial\beta_A} = 0$ iff

$$\frac{\partial\pi}{\partial a_A}\theta_A e^{-\eta}q_A = (a_A+\zeta a_B)q_A(1-e^{-\eta}) + r\sigma^2(\beta_A+\rho\gamma_A) \tag{B136}$$

Similarly,

$$\frac{d\pi}{d\gamma_A} = \frac{\partial\pi}{\partial\gamma_A} + \frac{\partial\pi}{\partial a_A}\frac{da_A}{d\gamma_A} \tag{B137}$$

because $\frac{\partial\pi}{\partial\gamma_A} = -(a_B+\zeta a_A)q_B + \frac{\partial\hat{U}_A}{\partial\gamma_A}$ and $\frac{\partial\hat{U}_A}{\partial\gamma_A} = (a_B+\zeta a_A)q_B - r\sigma^2(\rho\beta+\gamma)$, $\frac{\partial\pi}{\partial\gamma_A} = -r\sigma^2(\rho\beta+\gamma)$. Because, $\frac{da_A}{d\gamma_A} = \theta_A\zeta q_B$, $\frac{d\pi}{d\gamma_A} = 0$ iff

$$\frac{d\pi}{da_A}\theta_A\zeta q_B = r\sigma^2(\rho\beta_A+\gamma_A)$$

Thus, $\frac{\partial\pi}{\partial\beta_A} = 0$ and $\frac{d\pi}{d\gamma_A} = 0$ implies that

$$\frac{d\pi}{da_A}\theta q = a(1+\zeta)e^{\eta}q(1-e^{-\eta}) + e^{\eta}r\sigma^2(\beta+\rho\gamma) = \frac{1}{\zeta}r\sigma^2(\rho\beta+\gamma) \tag{B138}$$

Substituting in for optimal effort, $a = \theta\left[\beta e^{-\eta}q + \gamma\zeta q\right]$, and rearranging,

$$\theta\beta e^{-\eta}\zeta(1+\zeta)e^{\eta}q^2(1-e^{-\eta}) + r\sigma^2(\zeta e^{\eta}-\rho)\beta = r\sigma^2(1-\zeta e^{\eta}\rho)\gamma - \theta\gamma\zeta^2(1+\zeta)e^{\eta}q^2(1-e^{-\eta}) \tag{B139}$$

We will guess and verify that $r\sigma^2(1-\zeta e^{\eta}\rho) > \theta\zeta^2(1+\zeta)e^{\eta}q^2(1-e^{-\eta})$, so that we are not dividing by zero. This implies that $\gamma = m\beta$, where

$$m = \frac{\theta\zeta(1+\zeta)q^2(1-e^{-\eta}) + r\sigma^2(\zeta e^{\eta}-\rho)}{r\sigma^2(1-\zeta e^{\eta}\rho) - \theta\zeta^2(1+\zeta)e^{\eta}q^2(1-e^{-\eta})}.$$

If $\rho > 0$, note that the numerator is strictly increasing in ζ, while the denominator is strictly decreasing in ζ.

When $\zeta = 0$, $m = -\rho$, so the numerator is negative for small values of ζ and positive for large values of ζ. Note that $m = 0$ iff $\rho = \frac{1}{r\sigma^2}\theta\zeta(1+\zeta)q^2\left(1-e^{-\eta}\right)+\zeta e^{\eta}$. In this case, the denominator is $D = r\sigma^2\left(1-\zeta^2 e^{2\eta}\right)+\zeta^2\theta q^2\left(1-e^{-\eta}\right)(1+\zeta)e^{\eta}(1-\zeta)$ which is strictly positive (note that, for $\zeta \geq e^{-\eta}$, the numerator is bigger than $\theta e^{-\eta}q^2\left(1-e^{-2\eta}\right)+r\sigma^2(1-\rho)$, which is strictly positive for all $\rho \in [-1,1]$, so it must be that $\zeta < e^{-\eta}$ when $m = 0$). Finally, note that we are on this region iff $\gamma < e^{-\eta}\beta$, or equivalently, iff $m < e^{-\eta}$. Because the numerator is strictly positive as the denominator approaches 0, m explodes, so it must be for ζ smaller than that $m > e^{-\eta}$. When $\rho < 0$, the numerator is strictly positive, and the denominator is increasing then decreasing, so m is still well-defined by the same argument. Thus, m solves $f(m) = 0$,

$$f(m) = m\left[r\sigma^2(1-\zeta e^{\eta}\rho)-\theta\zeta^2(1+\zeta)e^{\eta}q^2\left(1-e^{-\eta}\right)\right]-\theta\zeta(1+\zeta)q^2\left(1-e^{-\eta}\right)-r\sigma^2(\zeta e^{\eta}-\rho)$$

Note $f' = r\sigma^2(1-\zeta e^{\eta}\rho)-\theta\zeta^2(1+\zeta)e^{\eta}q^2\left(1-e^{-\eta}\right)$. We already proved above that $f' > 0$ for all $m < e^{-\eta}$.

$$\begin{aligned}\frac{\partial f}{\partial \zeta} &= -m\left[r\sigma^2 e^{\eta}\rho+\theta e^{\eta}q^2\left(1-e^{-\eta}\right)\left(2\zeta+3\zeta^2\right)\right]\\ &\quad -\theta e^{-\eta}e^{\eta}q^2\left(1-e^{-\eta}\right)(1+2\zeta)-r\sigma^2 e^{\eta}\end{aligned}$$

Thus, $\frac{\partial f}{\partial \zeta} < 0$. By the implicit function theorem, note that $\frac{df}{d\zeta} = 0$, because f is uniformly zero. Because $\frac{df}{d\zeta} = \frac{\partial f}{\partial \zeta}+f'\frac{dm}{d\zeta}$ is uniformly zero, $\frac{dm}{d\zeta} = -\frac{\frac{\partial f}{\partial \zeta}}{f'} > 0$. Therefore, an increase in the synergy increases the exposure of the contract to cross-pay under corner beliefs. Similarly,

$$\frac{\partial f}{\partial \eta} = m\left[-r\sigma^2\zeta e^{\eta}\rho-\theta\zeta^2(1+\zeta)e^{\eta}q^2\right]-\theta\zeta(1+\zeta)q^2 e^{-\eta}-r\sigma^2\zeta e^{\eta}$$

which is likewise negative. Therefore, $\frac{dm}{d\eta} > 0$: and an increase in uncertainty increases cross-pay with synergies. Finally, note

$$\begin{aligned}f(m)|_{\zeta=e^{-\eta}} &= m\left[r\sigma^2(1-\rho)-\theta e^{-2\eta}\left(1+e^{-\eta}\right)e^{\eta}q^2\left(1-e^{-\eta}\right)\right]-\theta e^{-\eta}\left(1+e^{-\eta}\right)q^2\left(1-e^{-\eta}\right)-r\sigma^2(1-\rho)\\ &= (m-1)\left[r\sigma^2(1-\rho)-\theta e^{-\eta}q^2\left(1-e^{-2\eta}\right)\right]\end{aligned}$$

If $r\sigma^2(1-\rho) > \theta e^{-\eta}q^2\left(1-e^{-2\eta}\right)$, this implies that $m = 1$ when $\zeta = e^{-\eta}$. If $r\sigma^2(1-\rho) \leq \theta e^{-\eta}q^2\left(1-e^{-2\eta}\right)$, then there exists a $\zeta \leq e^{-\eta}$ such that $r\sigma^2(1-\zeta e^{\eta}\rho)-\theta\zeta^2(1+\zeta)e^{\eta}q^2\left(1-e^{-\eta}\right) = 0$, and $\lim_{\zeta\to\tilde{\zeta}} m = +\infty$. Because $\frac{dm}{d\zeta} > 0$, there exists a unique $\hat{\zeta} < e^{-\eta}$ such that $m < e^{-\eta}$ iff $\zeta < \hat{\zeta}$. Therefore, if $\zeta > \hat{\zeta}$, it is not locally optimal to select $\gamma < e^{-\eta}\beta$, and the HQ will shift to the first region, $H_d \in \left[e^{-\eta}, e^{\eta}\right]$.

Therefore, define $\bar{\zeta} = \max\left\{\hat{\zeta}, \frac{1}{3}\right\}$. For all $\zeta > \bar{\zeta}$, the optimal contract will be equity. ■