INVARIANTS OF THE BOSONIC GHOST ALGEBRA UNDER FINITE GROUP ACTION

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Abstract. Vertex algebras are a class of non-associative and non-commutative algebraic structures. The Heisenberg vertex algebra, which serves as an algebraic model of a single free boson, is an example of such an algebra. In 2011, Andrew Linshaw studied the invariance of this algebra under finite group action and concluded that the ring of invariants of the Heisenberg algebra under the action of some reductive group is finitely generated. In 2016, Michael Penn, Hanbo Shao and the author classified the invariants of the free-fermion vertex algebra, an odd analogue of the Heisenberg algebra, under the action of the multiplicative group of integers modulo 2. Using similar techniques, this work aims to classify the invariants of another example of a vertex algebra, the bosonic ghost algebra, again under the action of $\mathbb{Z}/2\mathbb{Z}$.

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1. Introduction

Vertex algebras are an intriguing field, lying at the intersection of modern algebra and high-energy physics. Michael Penn, my thesis advisor, first introduced them to me as a potential area of study for a summer research project. His aim was to extend work done by a colleague, Andrew Linshaw, on classifying invariants of these algebras. My aim was, vaguely, to gain mathematical insight into physics and understand what it means to do research in mathematics. We were both more or less successful: Michael, Hanbo Shao and I classified the invariants of an interesting example of a vertex algebra under the action of $\mathbb{Z}/2\mathbb{Z}$; independently, I classified invariants of a different algebra under the same group action, leading to the work presented in this thesis; and, through a good deal of suffering, I gained some understanding of what research in mathematics entails.

You may currently be wondering whether you ought to read this paper. Let me convince you to continue: to begin with, the preliminary work necessary for this research covers a wide range of mathematics, including group representation theory, invariant theory from two different centuries, and various notions from the theory of commutative algebras, quantum algebras and operator algebras (these last three are conveniently all found in a paper titled "Commutative Quantum Operator Algebras"). I will review most of the necessary groundwork before presenting my own results. The breadth of material covered is interesting in its own right, particularly when you see how it is all tied together.

Even more interesting, perhaps, is the actual work done with vertex algebras. In working towards our results, I tangled with the difficulties of non-associative algebras for the first time. This was not very fun to do, but I am sure it will make for an interesting read. Due in good part to the inherent non-associativity of vertex algebras, the calculations necessary to prove our results are long and off-putting. They are not part of the reason why you should read this paper, unless you truly enjoy that sort of thing. In the end, despite the computational complexity, our results are quite simple and clean. If anything, you ought to continue reading to see this evolution from relative disorder to relative simplicity.

1.1. Historical Background.

To understand vertex operator algebras, we ought to understand their origin story. It begins with John McKay, who in 1978 embarked on an apparent goose chase when he noticed a connection between a coefficient of the elliptic modular j function:

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots = \sum c(n)q^n,$$
$$\text{where } q = e^{2\pi i\tau},$$

and the smallest complex representation of the monster group, a group with degree 196883. The monster group is the largest simple sporadic group, having order 8×10^{53}.

In fact, the difference between the degree of the monster and the first coefficient of the j function is so small that McKay felt it could, possibly, indicate the existence of some sort of relationship between the sporadic groups and modular functions, an idea which most of his colleagues promptly declared "moonshine." McKay's observation was dismissed until Thompson noticed that the next few coefficients in the modular j function could be written as linear combinations of dimensions of irreducible representations of the monster group [B].

The eventual result of McKay's goose chase was the moonshine module, a vertex algebra constructed by Frenkel, Lepowsky and Meurman in 1988. As it would turn out, the monster group is also the automorphism group of the moonshine module [B]. The pursuit of this new field was continued by Richard Borcherds, who first rigorously defined vertex algebras. Victor Kac, in his study of the theory of infinite-dimensional Lie algebras, extended and formalized Borcherds' definition. We rely mostly on Kac's definitions in this paper.

1.2. **Modern Work.** The technical framework established by Kac and his contemporaries has been used by many others, including Gregg Zuckerman, Bong Lian and Andrew Linshaw, three mathematicians whose work laid the foundation for what we will do in this paper.

In [LZ], Lian and Zuckerman bridged the gap between vertex algebras and quantum operator algebras, introducing much of the structure used in this paper. In 2011 Linshaw proved, for the invariant vertex algebra $\mathcal{H}(n)$, that $\mathcal{H}(n)^G$ is strongly finitely generated if G is a reductive group of automorphisms preserving the conformal structure of the Virasoro element [L]. An obvious question emerges from Linshaw's work: if we pick some reductive group G, can we find and classify all these generators?

1.3. **Overview.** This paper begins by covering necessary background material in invariant theory, drawing on two sources: *Invariant Theory*, published by Mara Neusel in 2007, and *The Classical Groups: Their Invariants and Representations*, published by Hermann Weyl in 1939. Neusel's clear presentation of invariant theory as seen through the lens of group representation theory is necessary to understand Weyl's somewhat opaque text, which will give us insight into the generators of the ring of polynomial invariants under the orthogonal group, $\mathcal{O}(n)$. Weyl also presents relations between these generators. We will find we are able to apply Weyl's work to our own via a series of isomorphisms.

At this point, we will be ready to truly enter the world of vertex algebras, beginning with several necessary but tedious definitions. These will come mostly from Victor Kac's 2003 lecture notes on the subject. We will also introduce two fundamental operations, the normal-ordered product and the circle product, and establish some identities. The reader is encouraged to hang on through this section, because it will all soon be applied!

Finally, we introduce our intended object of study: the bosonic ghost algebra, which we refer to as the $\beta-\gamma$ system. For the sake of timely submission of this paper we consider only a specific case of this algebra, but this will give us plenty to work with. Our aim will be to reduce the generating set of the ring of invariants of the $\beta-\gamma$ system acted on by $\mathbb{Z}/2\mathbb{Z}$. We first use a linear isomorphism to apply Weyl's results and obtain a rather broad generating set. Next, we are able to introduce a derivative structure that mimics the derivative structure of the $\beta-\gamma$ system. At this point, we are obliged to take a two page-long detour to prove some important results about the basis of a vector space of quadratic elements that we introduce solely for the sake of the proof. This detour is necessary because, once the proofs are complete, we are able to say something new (and better!) about the generating set of the ∂-algebra we constructed.

Now we are in the home stretch. In the final section of the paper, we make use of the non-associativity inherent to the $\beta-\gamma$ system to find "quantum" corrections that result from considering the relations between various pairings of elements. These thorny-looking relations allow us to further reduce our generating set. This final generating set is minimal so, at this point, we will be done.

2. Invariant Theory

We begin with introducing some fundamental concepts from group representation theory, a field of mathematics concerned with the study of groups via their action on various mathematical objects. We will focus on group actions on polynomials, as polynomial invariants will soon become relevant. The following definitions are due to M. Nuesel [N], while the examples are the author's.

2.1. **Group Representation Theory.**

Definition 2.1. Linear Representation of a Group
A linear representation of a group is a group homomorphism

$$\phi : G \to \mathrm{GL}(n, \mathbb{F})$$

where $\mathbb{F}$ is any field, the number n is the degree of the representation ϕ, and $\mathrm{GL}(n, \mathbb{F})$ is the general linear group, i.e., the group of $n \times n$ invertible matrices with entries from $\mathbb{F}$. For our purposes, we will always work over the complex numbers, $\mathbb{C}$.

Alternatively, and more generally, we may write $\mathrm{GL}(V)$, where V is the vector space over $\mathbb{F}$. In this case, we think of $\mathrm{GL}(V)$ as the group of invertible linear transforms of V. These representations may be of finite or infinite groups, but we are concerned only with representations of finite groups.

Example 1. The Cyclic Group $\mathbb{Z}_2$
Take $V = \mathbb{C}^n$. Let ρ be our linear representation, where

$$\rho : \mathbb{Z}_2 \to \mathrm{GL}_n(\mathbb{C})$$

such that $0 \mapsto I$ and $1 \mapsto -I$, where I is the identity matrix.

Example 2. The Dihedral Group D_4
Take $V = \mathbb{R}^2$. Let

$$\rho : D_4 \to \mathrm{GL}_2(\mathbb{R})$$

such that $r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Definition 2.2. Group Action
Let S be a set and G a group. A group action of G on S is a map

$$G \times S \to S, (g, s) \mapsto gs$$

such that $es = s$ and $(gh)s = g(hs)$ for all $g, h \in G$ and $s \in S$. We call S a G-set.

Example 3. $\mathbb{Z}_2$ as a group action and map
Take T to be $\mathbb{R}^2$, and G to be the multiplicative form of $\mathbb{Z}_2 = \{\pm 1\}$. Then, the map

$$\mathbb{Z}_2 \times \mathbb{R}^2 \to \mathbb{R}^2, (g, s) \mapsto gs$$

is a group action of $\mathbb{Z}_2$ on $\mathbb{R}^2$ for all $g \in \mathbb{Z}_2$ and $s \in \mathbb{R}^2$. We can check this quickly: take $x \in \mathbb{R}^2$. Then, $1(x) = x$ verifies the first condition and $(1 * -1)x = -x = 1(-1 * x)$ verifies the second. We say $\mathbb{R}^2$ is a $\mathbb{Z}_2$-set.

Moreover, if we take $S = \mathbb{R}$, another $\mathbb{Z}_2$-set, and let ϕ be multiplication by a vector in $\mathbb{R}^2$ (i.e., scalar multiplication, as all elements of $\mathbb{R}$ will be scalars) such that $\phi : \mathbb{R} \to \mathbb{R}^2$, then ϕ is a $\mathbb{Z}_2$-map. Again, this is easy to check. For $t_1, t_2 \in \mathbb{R}^2$ and $s \in \mathbb{R}$, we have $\phi(1 * s) = \binom{t_1}{t_2} 1 * s = 1 * \binom{t_1}{t_2} s = 1 * \phi(s)$. The calculation is essentially the same for the other member of $\mathbb{Z}_2$, -1.

It is worth nothing that a group action and a group representation, as defined above, are equivalent notions. We will use the terms interchangeably in this paper.

Definition 2.3. Dual Space
The **dual space** V^* is a vector space of dimension n consisting of all linear maps $V \to \mathbb{F}$. The **dual basis** $x_1, \cdots, x_n \in V^*$ is defined by

$$x_i(\mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [N], pg. 51. □

Remark 2.1. Linear Group Action
Take a finite group G and ρ, a linear representation of that group. Because ρ maps every group element to a matrix, $\rho(g)$ acts by matrix multiplication on the vectors $\mathbf{v} = (v_1, \cdots, v_n) \in \mathbb{F}^n$. So, for every element $g \in G$ we obtain a linear map

$$g : \mathbb{F}^n \to \mathbb{F}^n, \mathbf{v} \mapsto g\mathbf{v} := \rho(g)(v_1, \cdots, v_n)^t.$$

Using our previously defined terminology, we may say the map

$$G \times \mathbb{F}^n \to \mathbb{F}^n, (g, \mathbf{v}) \mapsto \rho(g)\mathbf{v}^t$$

defines a group action of G on the vector space $V = \mathbb{F}^n$. Similarly, we can define an action of G on elements of V^*:

$$gx_i(\mathbf{e}_j) := x_i(g^{-1}\mathbf{e}_j)$$

for all i, j. By extending this action to the entire vector space V^*, we induce a linear group action of G on V^*.

2.2. **Group Action on all Polynomial Functions.** We may further extend the group action defined above to all polynomials.

Denote by $\mathbb{F}[V] = \mathbb{C}[x_1, \cdots, x_n]$ the ring of polynomials in n polynomial functions on V, $x_1, \ldots x_n$, with coefficients taken from the field $\mathbb{C}$. A monomial in the ring of polynomials may be written as

$$\boldsymbol{x}^I = x_1^{i_1} \ldots x_n^{i_n}$$

where $i_1, \cdots i_n \in \mathbb{N}_0^n$ is an exponent sequence. Any polynomial $f \in \mathbb{F}[V]$ may be written as a finite sum of monomials:

$$f(x_1, \cdots, x_n) = \sum_I a_I \boldsymbol{x}^I$$

We extend our G-action multiplicatively, to define it on a monomial:

$$g(x_1^{i_1} \ldots x_n^{i_n}) = g(x_1)^{i_1} \ldots g(x_n)^{i_n}$$

and linearly, to define it on all polynomials:

$$gf = g(\sum_I a_I \boldsymbol{x}^I) = \sum_I a_I g(\boldsymbol{x}^I).$$

In summary, we have extended our G-action such that

$$gf(v) = f(\rho(g)^{-1}v)\ \forall g \in G, v \in V, \text{ and } f \in \mathbb{F}[V].$$

The degree of the polynomial in question remains the same no matter the group G.

Example 4. $\mathbb{Z}_2$ acting on an arbitrary polynomial
Take $\mathbb{O} : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n]$ by $\mathbb{O}(x_i) = -x_i$ and note that $\langle \mathbb{O} \rangle = \mathbb{Z}/2\mathbb{Z}$. Consider first the action of $\mathbb{O}$ on a monomial:

$$\mathbb{O}(x_1^{i_1} \dots x_n^{i_n}) = (-1)^n (x_1^{i_1} \dots x_n^{i_n}) = (-1)^n (\boldsymbol{x}^I).$$

Now, consider the same action on an arbitrary polynomial $p(x_1, \cdots, x_n)$ in $\mathbb{F}[V]$:

$$\mathbb{O}(p) = \sum_I a_I (-1)^n \boldsymbol{x}^I = (-1)^n \sum_I a_I \boldsymbol{x}^I,$$

where the action of the second group element on a polynomial uses the expression we wrote down for the same action on a monomial. It is an interesting property that the monomial will remain unchanged only for even powers of n.

Definition 2.4. We say a polynomial $f \in \mathbb{F}[V]$ is invariant under the group action of G if

$$g \cdot f = f \text{ for all } g \in G.$$

$\mathbb{F}[V]^G \subset \mathbb{F}[V]$ is the subset consisting of all polynomials invariant under G.

2.3. **Classical Invariant Theory.** Hermann Weyl's work in [W] produced what would become the first and second fundamental theorems of classical invariant theory. Specifically, Weyl obtained the generators for the ring of all polynomial invariants under the action of finite groups, including the orthogonal group $\mathcal{O}(n)$. Additionally, Weyl described the relations between elements in the generating set of such a ring as being of a common type, such that a $(n+1) \times (n+1)$ matrix of the elements has a determinant of 0.

Theorem 2.1. *The First Main Theorem for the Orthogonal Group*
Every even orthogonal invariant depending on m vectors $x^1, x^2, \cdots, x^m$ in the n-dimensional vector space is expressible in terms of the m^2 scalar products $(x^\alpha x^\beta)$. Every odd invariant is a sum of terms

$$[u^1 u^2 \dots u^n] \cdot f^*(x^1, \dots, x^m),$$

where $u^1, \dots, u^n$ are selected from the row $x^1, \dots, x^m$ and f^* is an even invariant.

If τ is the group $\mathcal{O}(n)$ of all proper and improper orthogonal transformations, then we have one basic type of invariant, the scalar product (xy). A typical relation among scalar products is the following, involving $n+1$ vectors x and $n+1$ vectors y:

$$J = \begin{vmatrix} (x_0y_0) & (x_0y_1) & \cdots & (x_0y_n) \\ (x_1y_0) & (x_1y_1) & \cdots & (x_1y_n) \\ \cdots & \cdots & \cdots & \cdots \\ (x_ny_0) & (x_ny_1) & \cdots & (x_ny_n) \end{vmatrix} = 0.$$

Proof. See H. Weyl, *The Classical Groups.* □

Theorem 2.2. *The Second Main Theorem for the Orthogonal Group*
Every relation among scalar products is an algebraic consequence of relations of type J.

Proof. See H. Weyl, *The Classical Groups*, [W]. □

2.4. **Interpreting Classical Invariant Theory.** We rely on the known isomorphism $\mathcal{O}(1) \cong \mathbb{Z}/2\mathbb{Z}$ to draw Weyl's results into our work.

Definition 2.5. The Orthogonal Group of Dimension n

The orthogonal group $\mathcal{O}(n)$ is defined as as the group of $n \times n$ orthogonal matrices, i.e., matrices such that $A^T = A^{-1}$, where the group action is matrix multiplication. It is well known that $\det(A) = \det(A^T)$ for a matrix of any degree. Moreover, if that matrix is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$. Every matrix in the orthogonal group is, by definition, invertible, so this condition applies. We need one last (practically tautological) observation: if two matrices are equal, then their determinants are also equal. So now, for $A \in \mathcal{O}(n)$, we have:

$$(\det(A))^2 = \det(A)\det(A^T) = \det(A)\det(A^{-1}) = \frac{\det(A)}{\det(A)} = 1.$$

This implies that $\det(A) = \pm 1$. Now, we refine this understanding of $\mathcal{O}(n)$ to $\mathcal{O}(1)$.

Definition 2.6. The Orthogonal Group of Dimension 1

We define $\mathcal{O}(1)$ as the group of 1×1 matrices such that $A^T = A^{-1}$. Right away, we can write down that $\mathcal{O}(1) = \{x\}$ with $\det(x) = \pm 1$. This implies that $\mathcal{O}(1) = \{1, -1\}$, which is the multiplicative form of $\mathbb{Z}/2\mathbb{Z}$. We conclude that $\mathcal{O}(1) = \mathbb{Z}/2\mathbb{Z}$, meaning we are now able to apply Weyl's theorems to polynomials in one variable. However, it will be useful to us later in this paper to be able to apply them to a polynomial in two variables. In order to extend his theorems to our setting, we demonstrate the following linear isomorphism on the level of commutative algebras:

$$\begin{gathered} \mathbb{C}[x(0), x(1), \dots] \cong \mathbb{C}[x_0(0), x_1(0), x_0(1), x_1(1), \dots] \\ x(m) \mapsto x_r(q) \end{gathered} \tag{2.1}$$

where $m = 2q + r$ for $m, q \geq 0$ and $0 \leq r < 2$.

We will soon demonstrate the isomorphism $\mathbb{C}[x_0(m), x_1(n)] \cong A(1)$, allowing us to lift Weyl's theorems into the realm of vertex algebras. In order to do so, however, we need to present some basic results and definitions concerning these algebras.

3. Vertex Algebras

3.1. **Definitions.** The following definitions are due to V. Kac [K].

Definition 3.1. Formal Distribution
Given a vector space V over the complex numbers, for $a_n \in V$, a V-valued formal distribution is an expression

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^n \tag{3.1}$$

where z is a formal variable and by distribution we mean a linear map $T : D(R) \mapsto R$. We define the formal residue $\operatorname{Res}_z a(z) := a_{-1}$, from which it follows that we may define $a_{(n)} = \text{Res}_z\, z^n a(z)$ and rewrite 3.1 as

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \tag{3.2}$$

We may extend this definition to multiple formal variables. For example, in two variables, we have:

$$a(z,w) = \sum_{n,m\in\mathbb{Z}} a_{n,m} z^n w^m \tag{3.3}$$

for $a_{m,n} \in V$.

Definition 3.2. Graded Vector Space
A graded vector space is a vector space with a decomposition into the direct sum of vector subspaces.

Definition 3.3. Super Space
A super space is a vector space with a $\mathbb{Z}/2\mathbb{Z}$ grading, i.e., a decomposition $V = V_0 \oplus V_1$.

Remark 3.1. Parity of Vertex Algebras
Vertex algebras may be of even or odd parity. An odd vertex algebra (c.f. [PSC]) is constructed from a super space as defined above. An even vertex algebra, such as the bosonic ghost algebra we will introduce shortly, is constructed from a vector space without a grading.

3.2. **Construction.** Let V be a vector space over $\mathbb{C}$. We define $QO(V)$ to be the space of all linear maps

$$V \mapsto V[[z,z^{-1}]] := \left\{ \sum_{n\in\mathbb{Z}} v(n)z^{-n-1} | v(n) \in \text{End}(V), v(n) = 0 \text{ for } n >> 0 \right\}$$

where z is a formal variable. Each element $a \in QO(v)$ can be represented as a formal power series $a(z) \in QO(V)$, written as:

$$a = a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \in \text{End}(V)[[z,z^{-1}]].$$

We also impose the following truncation condition: for all $a \in QO(V)$ and $v \in V$, there is an $N \in \mathbb{N}$ such that $a(n)v = 0$ for all $n \geq N$. In terms of formal power series, this gives us:

$$a(z)v \in V((z)) = V[[z]][z^{-1}].$$

We endow QO(V) with infinitely many bilinear products known as circle products. For homogeneous $a, b \in \text{QO}(V)$, we define the n^{th} circle product for $n \in \mathbb{Z}$:

$$a(w) \circ_n b(w) = Res_z a(z)b(w)(z-w)^n - Res_z b(w)a(z)(-w+z)^n. \tag{3.4}$$

The graded subspace $\mathcal{A}$ of $\text{QO}(V)$, closed under all circle products $\circ_n$ and containing the identity operator, is called a quantum operator algebra. Such an algebra is closed under formal differentiation, which will allow us to derive an identity for negative and non-negative circle products shortly. First, note that ∂ is a formal differentiation operator with respect to z and ${}^\circ_\circ - {}^\circ_\circ$ denotes the normal ordered product:

$${}^\circ_\circ a(z)b(w) {}^\circ_\circ := a(z)_+ b(w) + p(a,b)b(w)a(z)_-$$

where

$$a(z)_+ = \sum_{n\leq -1} a_{(n)} z^{-1-n}$$

$$a(z)_- = \sum_{n\geq 0} a_{(n)} z^{-1-n}$$

The normal ordered product is iterated from left to right, so that $:a_1(z)\dots a_k(z):$ is defined by

$$:a_1(z)\dots a_k(z): = :a_1(z)b(z):,$$

where $b(z) = :a_2(z)\dots a_k(z):$.

Lemma 3.1. *Circle Product*

$$a(z)\circ_n b(z) = \begin{cases} \frac{1}{(-n-1)!}:\partial^{-n-1}a(z)b(z): & \text{if } n<0 \\ [(\sum_{m=0}^{n} a(m)(-z)^{n-m}), b(z)] & \text{if } n\geq 0 \end{cases} \tag{3.5}$$

where

$$\partial a(z) = \frac{\partial}{\partial z} a(z) \tag{3.6}$$

and, naturally,

$$\partial(a(z)b(z)) = \frac{\partial}{\partial z} a(z)b(z) + a(z)\frac{\partial}{\partial z} b(z)$$

Proof. We consider two cases: $n \geq 0$ and $n < 0$. First, for $n \geq 0$, we have

$$(w-z)^n = \sum_{k=0}^{n} \binom{n}{k} (-z)^{n-k} w^k = (-z+w)^n$$

So we may write

$$\begin{aligned} a(z)\circ_n b(z) &= \mathrm{Res}_w (w-z)^n [a(w), b(z)] \\ &= \mathrm{Res}_w \sum_{k=0}^{n} \binom{n}{k} (-z)^{n-k} w^k [\sum_{m\in\mathbb{Z}} a(m) w^{-m-1}, b(z)] \\ &= \mathrm{Res}_w \sum_{k=0}^{n} \binom{n}{k} (-z)^{n-k} [\sum_{m\in\mathbb{Z}} a(m) w^{k-m-1}, b(z)] \\ &= \sum_{k=0}^{n} \binom{n}{k} (-z)^{n-k} [\mathrm{Res}_w \sum_{m\in\mathbb{Z}} a(m) w^{k-m-1}, b(z)] \end{aligned}$$

To take the formal residue, we observe that $k-m-1=-1$ only when $k=m$, so:

$$\begin{aligned} &= \sum_{k=0}^{n} \binom{n}{k} (-z)^{n-k} [a(k), b(z)] \\ &= [\sum_{k=0}^{n} \binom{n}{k} a(k)(-z)^{n-k}, b(z)] \end{aligned}$$

This proves the first part of equation (3.4).

We begin the proof of the identity when $n < 0$ with some mild re-indexing, letting $n \to -n-1$. Now we are trying to show, for $n < 0$,

$$a(z) \circ_{-n-1} b(z) \frac{1}{(n)!} {}^{\circ}_{\circ} \partial^n a(z) b(z) {}^{\circ}_{\circ}.$$

Again, we get an expression for the circle product in terms of the residue, this time with respect to z, and we write:

$$\begin{aligned} a(z) \circ_{-n-1} b(z) &= \mathrm{Res}_z((z-w)^{-n-1})[a(z), b(w)] - \mathrm{Res}_z((-z+w)^{-n-1})[b(w), a(z)] \\ &= \mathrm{Res}_z[a(z)b(w)(\sum_{i \geq 0} \binom{-n-1}{i} (-w)^i z^{-n-1-i})] \\ &- \mathrm{Res}_z[b(w)a(z)(\sum_{i \geq 0} \binom{-n-1}{i} z^i (-w)^{-n-1-i})] \end{aligned}$$

Using the fact that $a(z) = \sum_{m \in \mathbb{Z}} a_{(m)} z^{-m-1}$ where $a_{(m)} = \mathrm{Res}_z\, z^m a(z)$, we may write:

$$\begin{aligned} &= \mathrm{Res}_z[(\sum_{m \in \mathbb{Z}} a_{(m)} z^{-m-1})(b(w))(\sum_{i \geq 0} \binom{-n-1}{i} (-w)^i z^{-n-1-i})] \\ &- \mathrm{Res}_z[b(w)(\sum_{m \in \mathbb{Z}} a_{(m)} z^{-m-1})(\sum_{i \geq 0} \binom{-n-1}{i} z^i (-w)^{-n-1-i})] \end{aligned}$$

We consider the two terms separately. For the first term, in order to take the formal residue, we need $-m-1-n-1-i = -1$ or, more concisely, $m = -n-1-i$. We are restricted by the fact that $n \geq 0$ and $i \geq 0$. These three restrictions reduce the first term to:

$$\left(\binom{-n-1}{0} (-w)^0 a_{-n-1} + \binom{-n-1}{1} (-w)^1 a_{-n-2} + \dots \right) b(w).$$

Using the fact that, for l and $k \in \mathbb{Z}$, we have:

$$\frac{1}{k!} \partial^k x^l = \binom{l}{k} \frac{l!}{(l-k)!},$$

we may write:

$$\begin{aligned} &= \left(\frac{1}{n!} \partial^n w^{-n-1} a_0 + \frac{1}{n!} \partial^n w^{-n-1} a_{-1} + \dots \right) b(w) \\ &= \left(\sum_{n \leq -1} \frac{1}{n!} \partial^n a_n w^{-n-1} \right) b(w) \\ &= a(z)_- b(w). \end{aligned}$$

Now we consider the second term. In order to take the formal residue, we need $m = -i$. With the same restrictions as the first term, we have:

$$b(w) \left(\binom{-n-1}{0} a_0 (-w)^{-n-1} + \binom{-n-1}{1} a_1 (-w)^{-n-2} + \dots \right)$$

Again rewriting the binomial coefficients in terms of derivatives, as well as factoring out some negatives, we obtain:

$$
\begin{aligned}
&= b(w)(-1)\left(\frac{1}{n!}\partial^n a_0 w^{-n-1} + \frac{1}{n!}\partial^n a_1 w^{-n-1} + \dots\right)\\
&= -b(w)\left(\sum_{n\geq 0}\frac{1}{n!}\partial^n a_n w^{-n-1}\right)\\
&= -b(w)a(z)_+.
\end{aligned}
$$

Subtracting the second term from the first term gives us the desired result. □

The set $QO(V)$ contains the identity map $\mathbb{1} : V \to V$, given by

$$\mathbb{1}(z) = \sum_{n\in\mathbb{Z}} \mathbb{1}(n)z^{-n-1} = \mathbb{1}.$$

If a linear subspace $A \subset QO(V)$ contains $\mathbb{1}$ and is closed under taking circle products, it is known as a quantum operator algebra. Furthermore, a quantum operator algebra whose elements are pairwise mutually local is called a commutative quantum operator algebra. We say two quantum operators $a(z), b(z)$ are mutually local if $a(z) \circ_n b(z) = 0$ for all n save for a finite set of positive n. The notion of a commutative quantum operator algebra is equivalent to that of a vertex algebra, and from now on we will use the latter term.
For $a(z), b(w) \in A$, where A is a vertex algebra, we define the operator product expansion as follows.

Definition 3.4. Operator Product Expansion
The operator product expansion (OPE) for a local pair of g-valued formal distributions is defined as:

$$a(z)b(w) = \sum_{n\geq 0} a(w) \circ_n b(w)(z-w)^{-n-1} + {}^{\circ}_{\circ}a(z)b(w)^{\circ}_{\circ} \tag{3.7}$$

For the remainder of the paper, we will abbreviate $a(z) \in A$ to $a \in A$. Using the normal ordered product and circle product as defined above, we may write down several identities describing the interaction between these products. These identities have been used by several authors; c.f. [L] and others. For $a, b, c \in A$, $m \geq 0$, we have:

$$({}^{\circ}_{\circ}ab^{\circ}_{\circ}) \circ_m c = \sum_{k\geq 0} \frac{1}{k!} {}^{\circ}_{\circ}(\partial^k a)(b \circ_{m+k} c)^{\circ}_{\circ} + \sum_{k\geq 0} b \circ_{m-k-1} (a \circ_k c) \tag{3.8}$$

$$a \circ_m ({}^{\circ}_{\circ}bc^{\circ}_{\circ}) = {}^{\circ}_{\circ}(a \circ_m b)c^{\circ}_{\circ} + {}^{\circ}_{\circ}b(a \circ_m c)^{\circ}_{\circ} + \sum_{k=1}^{m} \binom{m}{k} (a \circ_{m-k} b) \circ_{k-1} c \tag{3.9}$$

(3.10)

$${}^{\circ}_{\circ}{}^{\circ}_{\circ}ab^{\circ}_{\circ}c^{\circ}_{\circ} - {}^{\circ}_{\circ}abc^{\circ}_{\circ} = \sum_{k\geq 0} \frac{1}{(k+1)!} \left({}^{\circ}_{\circ}\left(\partial^{k+1}a\right)(b \circ_k c)^{\circ}_{\circ} + (-1)^{|a||b|} {}^{\circ}_{\circ}\left(\partial^{k+1}b\right)(a \circ_k c)^{\circ}_{\circ} \right)$$

$${}^{\circ}_{\circ}ab^{\circ}_{\circ} - (-1)^{|a||b|} {}^{\circ}_{\circ}ba^{\circ}_{\circ} = \sum_{k\geq 0} \frac{(-1)^k}{(k+1)!} \partial^{k+1}(a \circ_k b) \tag{3.11}$$

Our goal in this paper is to find a minimal generating set for an example of a vertex algebra, so we must clarify what it means for a vertex algebra to be generated by something. We say a vertex algebra A is generated by a subset $B \subset A$ and write

$$A = \langle B \rangle$$

if any element of A can be written as a finite linear combination of terms of the form

$$b_1 \circ_{i_1} b_2 \circ_{i_2} b_3 \circ_{i_3} \cdots \circ_{i_{n-1}} b_n \circ_{i_n} \mathbf{1}$$

for $b_j \in B$ and $i_j \in \mathbb{Z}$.

Furthermore, we say A is strongly generated by a subset B and write

$$A = \langle B \rangle_S$$

if in (3.4) we may take $i_j < 0$. By the definition of our circle products, this implies that A is spanned by monomials of the form

$$:\partial^{i_1} a_1 \cdots \partial^{i_n} a_n:.$$

4. The Bosonic Ghost Algebra

In this section, we turn to our principal object of study: the bosonic ghost algebra, which we will henceforth refer to as the $\beta - \gamma$ system. Specifically, we consider the rank-1 algebra, denoted $A(1)$. $A(1)$ is generated by two fields, $\beta(z)$ and $\gamma(z)$, subject to operator product expansion (OPE):

$$\beta(z)\beta(w) = :\beta(z)\beta(w): + 0 \tag{4.1}$$

$$\gamma(z)\gamma(w) = :\gamma(z)\gamma(w): + 0 \tag{4.2}$$

$$\beta(z)\gamma(w) = :\beta(z)\gamma(w): + \frac{1}{(z-w)^2} \tag{4.3}$$

A general element $a \in A(1)$ is a monomial of the form $\beta \circ_{m_1} \cdots \circ_{m_k} \beta \circ_{n_1} \gamma \circ_{n_2} \cdots \circ_{n_l} \gamma$. It is well known that this reduces to an expansion in only the negative circle products, as the non-negative circle products become constant terms, leaving us with a monomial of the form $:\partial^{m_1}\beta ... \partial^{m_k}\beta\partial^{n_1}\gamma ... \partial^{n_l}\gamma:$. This gives us the following linear isomorphism between a classical polynomial ring and $A(1)$:

$$\begin{gathered} \mathbb{C}[x_0(m), x_1(n) | m, n \geq 0] \cong A(1), \\ x_0(m_1) \cdots x_0(m_k) x_1(n_1) \cdots x_1(n_l) \mapsto :\partial^{m_1}\beta \cdots \partial^{m_k}\beta d^{n_1}\gamma \cdots \partial^{n_l}\gamma:. \end{gathered} \tag{4.4}$$

There is a natural action of $\mathbb{Z}/2\mathbb{Z}$ on $A(1)$ given by

$$\beta(z) \mapsto -\beta(z) \text{ and } \gamma(z) \mapsto -\gamma(z). \tag{4.5}$$

Our first isomorphism, (4.4), induces a linear isomorphism of the invariant subspaces

$$\mathbb{C}[x_0(m), x_1(n) | m, n \geq 0]^{\mathbb{Z}/2\mathbb{Z}} \cong \mathcal{A}(1)^{\mathbb{Z}/2\mathbb{Z}}, \tag{4.6}$$

allowing us to study the invariants of $\mathbb{C}[x_0(m), x_1(n) | m, n \geq 0]$ and apply our results to $A(1)$. Throughout this paper, we will use the following shorthand:

$$\begin{aligned} \omega_1(a,b) &= :\partial^a\beta\partial^b\beta: \\ \omega_2(a,b) &= :\partial^a\gamma\partial^b\gamma: \\ \omega_3(a,b) &= :\partial^a\beta\partial^b\gamma:. \end{aligned} \tag{4.7}$$

We will also refer to the ω_1 and ω_2 terms as "homogeneous" terms and ω_3 terms as "heterogeneous" terms.

4.1. **Applying Invariant Theory.** The results from classical invariant theory we built up earlier, specifically (2.1), allow us to conclude that the invariants of $\mathbb{C}[x_0(m), x_1(n) \mid 0 \le m \le n]^{\mathbb{Z}/2\mathbb{Z}}$ are generated by quadratics

$$\begin{aligned} q_{0,0}(a,b) &= x_0(a)x_0(b) \\ q_{1,1}(a,b) &= x_1(a)x_1(b) \\ q_{0,1}(a,b) &= x_0(a)x_1(b) \end{aligned} \tag{4.8}$$

In analogy to the elements of $A(1)$, will refer to the first two terms as homogeneous terms of types 1 and 2 and the last term as a heterogeneous term. Right away, this representation and (4.4) give us a new generating set for $A(1)^{\mathbb{Z}/2\mathbb{Z}}$:

$$A(1)^{\mathbb{Z}/2\mathbb{Z}} = \langle \omega_1(a_1,b_1), \omega_2(a_2,b_2), \omega_3(a_3,b_3) | a_i, b_j \ge 0 \rangle_S . \tag{4.9}$$

The second theorem from [W], (2.2), gives us the following relation between elements of $\mathbb{C}[x_0(m), x_1(n)|m,n \ge 0]$, described by:

$$x_0(a_0)x_1(a_0)x_0(a_1)x_1(a_1) - x_0(a_0)x_1(a_1)x_0(a_0)x_1(a_1) = 0 \tag{4.10}$$

for $0 \le a_0 \le a_1$ and $0 \le b_0 \le b_1$. We will put off using this relation until section 4.3, where will we use it to motivate further reduction of the generating set.

4.2. **Introducing the Derivative Structure.** Recall that for an algebra A over $\mathbb{C}$ a derivation $\partial : A \to A$ is a $\mathbb{C}$-linear map that satisfies

$$\partial(ab) = a\partial(b) + \partial(a)b \tag{4.11}$$

for any $a, b \in A$. The pair (A, ∂) is known as a ∂-algebra and will be denoted by A_∂.

We now endow $\mathbb{C}[x_0(m), x_1(n)|m,n \ge 0]$ (and thus $\mathbb{C}[x_0(m), x_1(n)|m,n \ge 0]^{\mathbb{Z}/2\mathbb{Z}}$) with the structure of a ∂-algebra via the derivation described by

$$\begin{aligned} \partial : \mathbb{C}[x_0(m), x_1(n)|m,n \ge 0] &\to \mathbb{C}[x_0(m), x_1(n)|m,n \ge 0] \\ x_i(m) &\mapsto x_i(m+1). \end{aligned} \tag{4.12}$$

We write $\mathbb{C}_\partial[x_0(m), x_1(n)|m,n \ge 0]$ for the associated ∂-algebra. This extra structure will allow us to describe $\mathbb{C}_\partial[x_0(m), x_1(n)|m,n \ge 0]^{\mathbb{Z}/2\mathbb{Z}}$ using a smaller generating set. Of use for the proof will be the following vector space of quadratic elements

$$A_{i,j}(m) = \{q_{i,j}(a,b)|a+b=m\}. \tag{4.13}$$

where the $q_{i,j}$ terms are as defined in (4.8).

Lemma 4.1. *The set*

$$\{\partial^{2m-2k} q_{i,i}(0,2k)|0 \le k \le m\} \tag{4.14}$$

is a basis of $A_{i,i}(2m)$ for $1 \le i \le n$. Moreover, the set

$$\{\partial^{2m-2k+1} q_{i,i}(0,2k)|0 \le k \le m\} \tag{4.15}$$

is a basis of $A_{i,i}(2m+1)$ for $1 \le i \le n$.

Proof. We begin with several observations concerning the bases of $A_{i,i}(2m)$ and $A_{i,i}(2m+1)$ as defined above.

Consider all basis vectors of $A_{i,i}(2m)$ under ∂. An arbitrary basis vector is $q_{i,i}(a, 2m-a), 0 \leq a \leq m$. Applying ∂, we see that

$$\partial q_{i,i}(a, 2m-a) = q_{i,i}(a+1, 2m-a) + q_{i,i}(a, 2m-a+1).$$

The right hand side of this equation is in the basis of $A_{i,i}(2m+1)$; it follows that $\partial A_{i,i}(2m)$ is a subset of $A_{i,i}(2m+1)$. Furthermore, $\dim A_{i,i}(2m) = \dim A_{i,i}(2m+1) = m+1$. As $\partial A_{i,i}(2m)$ is a subset of $A_{i,i}(2m+1)$ and both have the same dimension, we can say they are equal as vector spaces.

Similarly, $\partial A_{i,i}(2m-1)$ is a subset of $A_{i,i}(2m)$. It follows from (1) that the codimension of $A_{i,i}(2m)$ and $\partial A_{i,i}(2m-1)$ is 1. It follows from the above observations that

$$\begin{aligned} &A_{i,i}(2m)/\partial A_{i,i}(2m-1) \\ &= \partial A_{i,i}(2m-1) + \langle q(0,2m)\rangle \\ &= \mathbb{C}q_{i,i}(0,2m) + \partial A_{i,i}(2m-1) \end{aligned}$$

This implies that $A_{i,i}(2m) = \partial^2 A_{i,i}(2m-2) \bigoplus \mathbb{C}q_{i,i}(0,2m)$. Similarly, $A_{i,i}(2m+1) = \partial^3 A_{i,i}(2m-2) \bigoplus \mathbb{C}\partial q(0,2m)$.

We proceed with proof by induction on m, splitting our proof into two cases.

For the even case, we begin by observing that $A_{i,i}(0)$ has a basis spanned by $\{q_{i,i}(0,0)\}$. Now, suppose the the basis of $A_{i,i}(2k)$ is $\{\partial^{2i} q_{i,i}(0, 2k-2i) | 0 \leq i \leq k\}$. Then, as $A_{i,i}(2k+2) = \partial^2 A_{i,i}(2k) + \mathbb{C}q_{i,i}(0, 2k+2)$, we can say that $\partial^2 A_{i,i}(2k) = \partial^2 \ A_{i,i}(2k)$. Using the fact that the basis of a direct sum is the union of the corresponding bases, we can say $A_{i,i}(2k+2) = \{\partial^{2i+2} q_{i,i}(0, 2k-2i) | 0 \leq i \leq k\} \cup \{q_{i,i}(0, 2k+2)\}$.

Now we re-index, letting $i \to i-1$, and we obtain

$$\begin{aligned} &= \{\partial^{2i} q_{i,i}(0, 2k-2i+2) | 1 \leq i \leq k+1\} \cup \{q_{i,i}(0, 2k+2)\} \\ &= \{\partial^{2i} q_{i,i}(0, 2k-2i+2) | 0 \leq i \leq k+1\}, \end{aligned}$$

which is what we wanted to show.

For $A_{i,i}(2m+1)$, the odd case, we begin by noting that it follows from the previous proof that the basis of $A_{i,i}(2k) = \{\partial^{2i} q(0, 2k-2i) | 0 \leq i \leq k\}$.

Using the fact that the basis of $A_{i,i}(2k+1)$ is the same as the basis of $\partial A_{i,i}(2k)$, we can write the former as:

$$\{\partial^{2i+1} q(0, 2k-2i) | 0 \leq i \leq k\}.$$

□

Lemma 4.2. *The set*

$$\{\partial^{m-k} q_{i,j}(0,k) | 0 \leq k \leq m\} \tag{4.16}$$

is a basis of $A_{i,j}(m)$ *for* $1 \leq i < j \leq n$.

Proof. We begin by obtaining a new vector space decomposition for $A_{i,j}(m)$. We would like to show that we can write an arbitrary basis element of $A_{i,j}(m+1)$ as an element in $\partial A_{i,j}(m) \bigoplus \mathbb{C}q_{i,j}(0, m+1)$. We proceed via proof by induction.

First, consider the element $q_{i,j}(1,m)$ in the basis of $A_{i,j}(m+1)$.
Note that we can write $q_{i,j}(1,m) = (q_{i,j}(0,m+1) + q_{i,j}(1,m)) - (q_{i,j}(0,m+1))$, which is a linear combination of elements in $\partial A_{i,j}(m)$ and $\mathbb{C}q_{i,j}(0,m+1)$.

Now suppose, for some $1 \leq d \leq m+1$, that $q_{i,j}(d, m+1-d) \in \partial A_{i,j}(m+) + \mathbb{C}q_{i,j}(0,m+1)$. It follows, then, that $q_{i,j}(a+1, m-a) = q_i, j(a+1, m-a) + q_{i,j}(a, m+1-a) - q_{i,j}(a, m+1-a)$. By our assumption, this is a linear combination of elements in $\partial A_{i,j}(m)$ and $\mathbb{C}q_{i,j}(0, m+1-a)$.

We conclude that $A_{i,j}(m+1) = \partial A_{i,j}(m) \bigoplus \mathbb{C}q_{i,j}(0, m_1)$.

Now that we have obtained the decomposition we needed, we may once again proceed by induction on m. We begin by noting that that the basis of $A_{i,j}(0) = \{q_{i,j}(0,0)\}$. Now, suppose the basis of $A_{i,j}(k) = \{\partial^i q_{i,j}(0,0) | 0 \leq i \leq k\}$. Then, using the fact that basis $A_{i,j}(k+1) = \partial A_{i,j}(k) \bigoplus \mathbb{C}q_{i,j}(0,k+1)$, we can say $A_{i,j}(k+1) = \{\partial^i + 1 q_{i,j}(0, k-i) | 0 \leq i \leq k\} \cup \{q_{i,j}(0, k+1))\}$.

If we re-index and let $i = i-1$, we obtain the following basis for $A_{i,j}(k+1)$

$$\{\partial^i q_{i,j}(0, k+i)) | 0 \leq i \leq k+1\}.$$

□

The following proposition follows immediately from lemmas (4.1) and (4.2).

Proposition 4.1. *The invariant ∂-algebra $\mathbb{C}_\partial[x_0(m), x_1(n) | m, n \geq 0]^{\mathbb{Z}/2\mathbb{Z}}$ is generated by monomials*

$$\begin{aligned} &q_{i,i}(0,2a) \text{ for } 1 \leq i \leq n \text{ and } a \geq 0 \\ &q_{i,j}(0,b) \text{ for } 1 \leq i < j \leq n \text{ and } b \geq 0. \end{aligned} \tag{4.17}$$

We apply (4.17) directly to obtain a new generating set for $A(1)^{\mathbb{Z}/2\mathbb{Z}}$:

$$A(1)^{\mathbb{Z}/2\mathbb{Z}} = \langle \omega_1(0,2a), \omega_2(0,2b), \omega_3(0,c) \rangle_S \,. \tag{4.18}$$

4.3. **Non-Associativity and Quantum Corrections.** Now, finally, we will use the non-associative structure of $A(1)$ to further reduce our generating set. We will do this in two steps: first, calculating non-associativity corrections using (3.10) and, secondly, constructing relations between sets of terms, which we will call quantum corrections, and using these relations to solve for terms we suspect may not be in our final generating set.

Note that, by way of the OPE (4.1, 4.2, and 4.3) we have:

$$\begin{aligned} \beta\beta &= {}^\circ_\circ\beta\beta^\circ_\circ + 0 \\ \gamma\gamma &= {}^\circ_\circ\gamma\gamma^\circ_\circ + 0 \\ \beta\gamma &= {}^\circ_\circ\beta\gamma^\circ_\circ + \frac{1}{(z-w)^2} \end{aligned} \tag{4.19}$$

In other words, homogeneous terms are associative and do not result in any corrections when we re-associate within our normal ordered products. Heterogeneous terms, on the other hand, are not associative. When we re-associate products, we must introduce a correction term. We calculate the following non-associativity corrections, using (3.10):

$$
\begin{aligned}
{}^{\circ}_{\circ}\omega_1(a,b)\omega_2(c,d){}^{\circ}_{\circ} &= {}^{\circ\circ}_{\circ\circ}\partial^a\beta\partial^b\beta{}^{\circ\circ}_{\circ\circ}\partial^c\gamma\partial^d\gamma{}^{\circ\circ}_{\circ\circ} \\
&= {}^{\circ}_{\circ}\partial^a\beta\partial^b\beta\partial^c\gamma\partial^d\gamma{}^{\circ}_{\circ} \\
&+ (-1)^b\left(\frac{{}^{\circ}_{\circ}\partial^{a+b+c+1}\beta\partial^d\gamma{}^{\circ}_{\circ}}{b+c+2} + \frac{{}^{\circ}_{\circ}\partial^{a+b+d+1}\beta\partial^c\gamma{}^{\circ}_{\circ}}{b+d+2}\right) \\
&- (-1)^a\left(\frac{{}^{\circ}_{\circ}\partial^{a+b+c+1}\beta\partial^d\gamma{}^{\circ}_{\circ}}{a+c+2} + \frac{{}^{\circ}_{\circ}\partial^{a+b+d+1}\beta\partial^c\gamma{}^{\circ}_{\circ}}{a+d+2}\right)
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
{}^{\circ}_{\circ}\omega_3(a,b)\omega_3(c,d){}^{\circ}_{\circ} &= {}^{\circ\circ}_{\circ\circ}\partial^a\beta\partial^b\gamma{}^{\circ\circ}_{\circ\circ}\partial^c\beta\partial^d\gamma{}^{\circ\circ}_{\circ\circ} \\
&= {}^{\circ}_{\circ}\partial^a\beta\partial^b\gamma\partial^c\beta\partial^d\gamma{}^{\circ}_{\circ} \\
&+ (-1)^b\left(\frac{{}^{\circ}_{\circ}\partial^{a+b+c+1}\beta\partial^d\gamma{}^{\circ}_{\circ}}{b+c+2} + \frac{{}^{\circ}_{\circ}\partial^{a+b+d+1}\beta\partial^c\beta{}^{\circ}_{\circ}}{b+d+2}\right) \\
&- (-1)^a\left(\frac{{}^{\circ}_{\circ}\partial^{a+b+c+1}\gamma\partial^d\gamma{}^{\circ}_{\circ}}{a+c+2} + \frac{\partial^{a+b+d+1}\gamma\partial^c\beta}{a+d+2}\right)
\end{aligned}
\tag{4.21}
$$

Now, we lift the second theorem from [W], which was applicable to our classical polynomial ring, into the realm of vertex algebras. Specifically, we consider relations of the type described in (4.10). In the classical setting, our terms associate. Clearly, in $A(1)$, they do not. This non-associativity allow us to find some interesting quantum corrections. These corrections result from evaluating terms of the form

$$
{}^{\circ}_{\circ}\omega_i(a,b)\omega_j(c,d){}^{\circ}_{\circ} - {}^{\circ}_{\circ}\omega_k(a,c)\omega_l(b,d){}^{\circ}_{\circ},
$$

in analogy to (4.10). The preliminary work here was done for specific terms using a Mathematica package written by Kris Thielemans of London's Imperial College Theoretical Physics Group in 1992. Eventually, once we had an idea of the types of relations we ought to be considering, we calculated the following general corrections.

$$
\begin{aligned}
{}^{\circ}_{\circ}\omega_3(a,b)\omega_3(c,d){}^{\circ}_{\circ} - {}^{\circ}_{\circ}\omega_2(a,c)\omega_1(b,d){}^{\circ}_{\circ} &= \\
&+ (-1)^b\left(\frac{{}^{\circ}_{\circ}\partial^{a+b+c+2}\beta\partial^d\gamma{}^{\circ}_{\circ}}{b+c+2} + \frac{{}^{\circ}_{\circ}\partial^{a+b+d+1}\beta\partial^c\beta{}^{\circ}_{\circ}}{b+d+2}\right) \\
&- (-1)^a\left(\frac{{}^{\circ}_{\circ}d^{a+b+c+1}\gamma\partial^d\beta{}^{\circ}_{\circ}}{a+c+2} + \frac{{}^{\circ}_{\circ}\partial^{a+b+d+1}\gamma\partial^c\beta{}^{\circ}_{\circ}}{a+d+2}\right) \\
&- (-1)^c\left(\frac{{}^{\circ}_{\circ}\partial^{a+b+c+1}\beta\partial^d\gamma{}^{\circ}_{\circ}}{b+c+2} + \frac{{}^{\circ}_{\circ}\partial^{a+c+d+1}\beta\partial^b\gamma{}^{\circ}_{\circ}}{c+d+2}\right) \\
&+ (-1)^a\left(\frac{{}^{\circ}_{\circ}\partial^{a+b+c+1}\beta\partial^d\gamma{}^{\circ}_{\circ}}{a+b+2} + \frac{{}^{\circ}_{\circ}\partial^{a+c+d+1}\beta\partial^b\beta{}^{\circ}_{\circ}}{a+d+2}\right)
\end{aligned}
\tag{4.22}
$$

$$\begin{aligned}
&{}^\circ_\circ\omega_1(a,b)\omega_2(c,d)^\circ_\circ - {}^\circ_\circ\omega_3(a,c)\omega_3(b,d)^\circ_\circ = \\
&\quad + (-1)^b\left(\frac{{}^\circ_\circ\partial^{a+b+c+1}\beta\partial^d\gamma^\circ_\circ}{b+c+2} + \frac{{}^\circ_\circ\partial^{a+b+d+1}\beta\partial^c\gamma^\circ_\circ}{b+d+2}\right) \\
&\quad - (-1)^a\left(\frac{{}^\circ_\circ\partial^{a+b+c+1}\beta\partial^d\gamma^\circ_\circ}{a+c+2} + \frac{{}^\circ_\circ\partial^{a+b+d+1}\beta\partial^c\gamma^\circ_\circ}{a+d+2}\right) \\
&\quad - (-1)^a\left(\frac{{}^\circ_\circ\partial^{a+b+c+1}\gamma\partial^d\gamma^\circ_\circ}{a+b+2} - \frac{{}^\circ_\circ\partial^{a+c+d+1}\gamma\partial^d\beta^\circ_\circ}{a+d+2}\right) \\
&\quad - (-1)^c\left(\frac{{}^\circ_\circ\partial^{a+b+c+1}\beta\partial^d\gamma^\circ_\circ}{b+c+2} + \frac{{}^\circ_\circ\partial^{a+c+d+1}\beta\partial^d\beta^\circ_\circ}{c+d+2}\right)
\end{aligned} \tag{4.23}$$

Armed with (4.22) and (4.23), we are ready to further reduce our generator set. The proof is in two steps: first for homogeneous terms (those of the form $\omega_1(a,b)$ and $\omega_2(a,b)$) and next for heterogeneous terms (those of the form $\omega_3(a,b)$).

Theorem 4.1. *The invariant sub-algebra $A(1)^{\mathbb{Z}/2\mathbb{Z}}$ is minimally strongly generated by the fields $\omega_1(0,0), \omega_1(0,2), \omega_2(0,0), \omega_2(0,2), \omega_3(0,0),$ and $\omega_3(0,1)$. More concisely,*

$$A(1)^{\mathbb{Z}/2\mathbb{Z}} = \langle \omega_1(0,0), \omega_1(0,2), \omega_2(0,0), \omega_2(0,2), \omega_3(0,0), \omega_3(0,1)\rangle_S . \tag{4.24}$$

Proof. By (4.18), we know that $\omega_1(a,b)$ may be written in terms of $\omega_1(0,2m)$ and various ω_3 terms. The following proof is by induction. First, consider the case where $m=2$. We have the following relationship:

$$\begin{aligned}
&{}^\circ_\circ\omega_1(0,1)\omega_3(0,1)^\circ_\circ - {}^\circ_\circ\omega_1(0,0)\omega_3(1,1)^\circ_\circ \\
&\qquad = \frac{1}{2}\omega_1(4,0) - \frac{2}{3}\omega_1(3,1)
\end{aligned}$$

where, in this case, $\omega_1(4,0) = \omega_1(0,4)$, so we may write:

$$\omega_1(0,4) = 2({}^\circ_\circ\omega_1(0,1)\omega_3(0,1)^\circ_\circ - {}^\circ_\circ\omega_1(0,0)\omega)3(1,1)^\circ_\circ) + \frac{4}{3}\omega_1(1,3)$$

where $\omega_1(1,3) = \partial\omega_1(0,3) - \omega_1(0,4)$. So, finally, we have:

$$\begin{aligned}
\omega_1(0,4) &= \frac{6}{7}({}^\circ_\circ\omega_1(0,1)\omega_3(0,1)^\circ_\circ - {}^\circ_\circ\omega_1(0,0)\omega_3(1,1)^\circ_\circ) + \frac{4}{7}\partial\omega_1(0,3) \\
&\in \langle \omega_1(0,0), \omega_1(0,2), \omega_3(0,b)\rangle_S .
\end{aligned}$$

Now suppose, for all $a \le b$, $\omega_1(0,2a) \in \langle w_1(0,0), w_1(0,2)\rangle_S$. Note that

$$\begin{aligned}
&{}^\circ_\circ\omega_1(0,1)\omega_3(0,2b-1)^\circ_\circ - {}^\circ_\circ\omega_1(0,0)\omega_3(1,2b-1)^\circ_\circ \\
&\qquad = \frac{-5}{6}\omega_3(3,2b-1) - \frac{1}{3}\omega_3(2b+2,0) - \frac{1}{2b+1}\omega_1(0,2b+2)
\end{aligned}$$

and this implies:

$$\begin{aligned}
\omega_1(0,2b+2) &= (2b+1)(-{}^\circ_\circ\omega_1(0,1)\omega_3(0,2b-1)^\circ_\circ + {}^\circ_\circ\omega_1(0,0)\omega_3(1,2b-1)^\circ_\circ) \\
&\quad - \frac{10b+5}{6}\omega_3(3,2b-1) - \frac{2b+1}{3}\omega_3(2b+2,0) \\
&\in \langle \omega_1(0,0), \omega_1(0,2), \omega_3(0,b)\rangle_S .
\end{aligned}$$

The proof for homogeneous terms of type $\omega_2(a,b)$ follows in the same manner. Now, we consider heterogeneous terms. By a previous argument we know that $\omega_3(a,b)$

may be written in terms of $\omega_3(0,m)$ and homogeneous terms. Again, the proof is by induction. First, note the case when $m=2$:

$$\begin{aligned} &{}^\circ_\circ\omega_1(0,0)\omega_2(0,0)^\circ_\circ - {}^\circ_\circ\omega_3(0,0)\omega_3(0,0)^\circ_\circ \\ &\qquad = \frac{3}{2}\omega_3(0,0) - \frac{1}{2}\omega_3(0,2) \end{aligned}$$

where

$$\partial^2\omega_3(0,0) = \omega_3(0,2) + 2\omega_3(0,0) + \omega_3(2,0)$$

and this implies:

$$\begin{aligned} \omega_3(2,0) &= \partial^2\omega_3(0,0) - \omega_3(0,0) - 2(\omega_3(0,1) - \omega_3(0,2)) \\ &\in \langle \omega_3(0,0), \omega_3(0,1)\rangle_S . \end{aligned}$$

Now suppose, for all $m \geq 2$, we have $\omega_3(0,m) \in \langle \omega_3(0,0), \omega_3(0,1)\rangle_S$. We consider the following relation:

$$\begin{aligned} &{}^\circ_\circ\omega_1(0,0)\omega_3(1,m-3)^\circ_\circ - {}^\circ_\circ\omega_1(1,0)\omega_3(0,m-3)^\circ_\circ \\ &\qquad = \frac{-5}{6}\omega_3(3,m-3) + \frac{1}{m-1}\omega_3(m-1,1) \\ &\qquad - \frac{1}{m-1}\omega_3(m,0) - \frac{1}{m-1}\omega_1(m-1,1) - \frac{1}{m-1}\omega_1(m,0) \end{aligned}$$

where

$$\partial^m\omega_3(0,0) = \omega_3(0,n) + \binom{n}{1}\omega_3(1,n) + \ldots + \binom{1}{n-1}\omega_3(n-1,1) + \omega_3(n,0)$$

and this implies

$$\omega_3(n,0) = -\omega_3(0,n) + \partial^m\omega_3(0,0) - \left(\binom{n}{1}\omega_3(1,n) + \ldots + \binom{1}{n-1}\omega_3(n-1,1)\right).$$

We would like to show that the right hand side of this equation is always in our generating set. To do so will acquire an additional proof by induction. The $m=2$ case follows from the above calculation for a term of the form $\omega_3(2,0)$. Now suppose, for the case when $m=n-1$:

$$\partial^{n-1}\omega_3(0,0) = \omega_3(0,n-1) + \binom{n}{1}\omega_3(1,n-2) + \cdots + \binom{1}{n}\omega_3(n-2,1) + \omega_3(n-1,0)$$

which implies

$$\begin{aligned} \omega_3(n-1,0) &= \omega_3(0,n-1) + \partial^{n-1}\omega_3(0,0) - \left(\binom{n-1}{1}\omega_3(1,n-2) + \cdots + \binom{1}{n-1}\omega_3(n-2,1)\right) \\ &\in \langle \omega_3(0,0), \omega_3(0,1)\rangle_S . \end{aligned}$$

Then, for the case when $m=n$, we may write

$$\partial^n\omega_3(0,0) = \partial(\partial^{n-1}\omega_3(0,0)) = \omega_3(0,n) + \binom{n}{1}\omega_3(1,n-1) + \cdots + \binom{1}{n}\omega_3(n-1,1) + \omega_3(n,0)$$

implying

$$\omega_3(n,0) = -\omega_3(0,n) + d(\partial^{n-1}\omega_3(0,0)) - (\binom{n}{1}\omega_3(1,n-1) + \cdots + \binom{1}{n}\omega_3(n-1,1))$$

where

$$\partial\omega_3(0,n-1) = \omega_3(1,n-1) + \omega_3(0,n)$$

so it follows that

$$\omega_3(1, n-1) = \partial\omega_3(0, n-1) - \omega_3(0, n)$$

which in our generating set, by definition. Also,

$$\partial\omega_3(n-1, 0) = \omega_3(n-1, 1) + \omega_3(n, 0)$$

implies

$$\omega_3(n-1, 1) = \partial\omega_3(n-1, 0) - \omega_3(n, 0)$$

which, according to our hypothesis, is also in our generating set. So we may conclude, for the entire right hand expression:

$$-\omega_3(0,n)+\partial^m\omega_3(0,0)-(\binom{n}{1}\omega_3(1,n-1)+\cdots+\binom{1}{n}\omega_3(n-1,1)) \in \langle\omega_3(0,0),\omega_3(0,1)\rangle_S$$

In other words, $\omega_3(n, 0) \in \langle\omega_1(0, 0), \omega_2(0, 0), \omega_3(0, 0), \omega_3(0, 1)\rangle_S$.

With this result we may return to our original proof, where we left off with this expression:

$$\begin{aligned}\omega_3(m, 0) = &-(m-1){}^\circ_\circ\omega_1(0, 0)\omega_3(1, m-3){}^\circ_\circ - {}^\circ_\circ\omega_1(1, 0)\omega_3(0, m-3){}^\circ_\circ \\ &- \frac{-5}{6}(m-1)\omega_3(3, m-3) - \omega_3(m-1, 1) - \omega_1(m-1, 1) - \omega_1(m, 0)\end{aligned}$$

Simplifying and solving for the $\omega_3(0, m)$ term, we obtain

$$\begin{aligned}\omega_3(0, m) &= \partial^m\omega_3(0, 0) - (\binom{m}{1}o_3(1, m-1) + \cdots + \binom{1}{m}\omega_3(m-1, 1)) \\ &= +(m-1){}^\circ_\circ\omega_1(0, 0)\omega_3(1, m-3){}^\circ_\circ - {}^\circ_\circ\omega_1(1, 0)\omega_3(0, m-3){}^\circ_\circ \\ &+ \frac{5}{6}(m-1)\omega_3(3, m-3) + \omega_3(m-1, 1) + \omega_1(m-1, 1) - \omega_1(m, 0) \\ &\in \langle\omega_3(0, 0), \omega_3(0, 1)\rangle_S\,.\end{aligned}$$

□

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