Invariants of the Free-Fermion Vertex Algebra under the Action of $\mathbb{Z}/2\mathbb{Z}$

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Abstract

Many authors, most famously H. Weyl in the early 20th century, have studied rings of polynomial invariants. More recently, A. Linshaw and co-authors adapted classical invariant theory to study the invariance of vertex algebras. Drawing on Linshaw's methods, our work describes a linear isomorphism from classically invariant polynomial rings to quantum operator algebras that allows us to apply the first fundamental theorem of invariant theory. Specifically, we study the invariance of the rank n free-fermion vertex algebra under the action of the $\mathbb{Z}/2$ group and obtain its minimal generating set.

Vertex Algebras

A vertex algebra is defined as a set of formal power series closed under formal derivatives

$$a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \in \mathrm{End}(V)[[z, z^{-1}]]$$

whose coefficients are linear maps on a vector space together with non-associative bilinear operations:

$$\circ_m : V \times V \to V$$

defined by

$$a(w)\circ_n b(w) = \begin{cases} \frac{1}{(-n-1)!} {:}\partial^{-n-1}u(z)v(z){:} & \text{if } n < 0 \\ [(\sum_{m=0}^{n} u(m)(-z)^{n-m}), v(z)] & \text{if } n \geq 0 \end{cases}$$

where

$${:}a(z)b(w){:} = a(z)_{-}b(w) + (-1)^{|a||b|}b(w)a(z)_{+}$$

where

$$a(z)_{+/-} = \sum_{n\geq 0/n<0} a(n)z^{-n-1}$$

A subset $S = \{a_i | i \in I\}$ is said to generate A if every element $a \in A$ can be written as a non-associative linear combination of elements a_i and $\circ_n$, for $n \in \mathbb{Z}$.

S strongly generates A if every every element $a \in A$ can be written as a non-associative linear combination of elements a_i and $\circ_n$, for $n < 0$.

The Free-Fermion Algebra

The free-fermion vertex algebra, denoted $\mathcal{F}(n)$, is an algebra generated by n odd fields:

$$\langle \varphi_1(z), \dots, \varphi_n(z) \rangle$$

where $\mathcal{F}(n)$ is endowed with a differential structure such that:

$$\partial a = \frac{\partial}{\partial z} a(z)$$

Additionally, the non-negative circle products are greatly simplified:

$$\partial^a \varphi_i \circ_m \partial^b \varphi_j = (-1)^b (a+b)!$$

if $m = a + b$ and $i = j$, and 0 otherwise. The resulting non-associativity relation is described by:

$$\begin{aligned} &{:}{:}\partial^a\varphi_i\partial^b\varphi_j{:}\partial^c\varphi_k{:} - {:}\partial^a\varphi_i{:}\partial^b\varphi_j\partial^c\varphi_k{:}{:} \\ &= \frac{(-1)^c\delta_{j,k}}{b+c+1}\partial^{a+b+c+1}\varphi_i \\ &+ \frac{(-1)^c\delta_{i,k}}{a+b+1}\partial^{a+b+c+1}\varphi_j \end{aligned}$$

and the non-commutativity relation is described by:

$${:}\partial^a\varphi_i\partial^b\varphi_j{:} = -{:}\partial^b\varphi_j\partial^a\varphi_i{:}$$

These relations can be used to show that $\mathcal{F}(n)$ is strongly generated:

$$\mathcal{F}(n) = \langle \varphi_1(z), \dots, \varphi_n(z) \rangle_S$$

Classical Invariants

Using a corollary to H. Weyl's results describing the ring of invariants of $\mathcal{O}(n)$, we describe the ring of invariants of $\Lambda = \Lambda(y_i(m) | 1 \leq i \leq n, m \geq 0)$ under the action of $\mathbb{Z}/2$, given by

$$y_i(m) \mapsto -y_i(m).$$

in terms of quadratic generators:

$$p_{i,j}(a,b) = y_i(a)y_j(b)$$

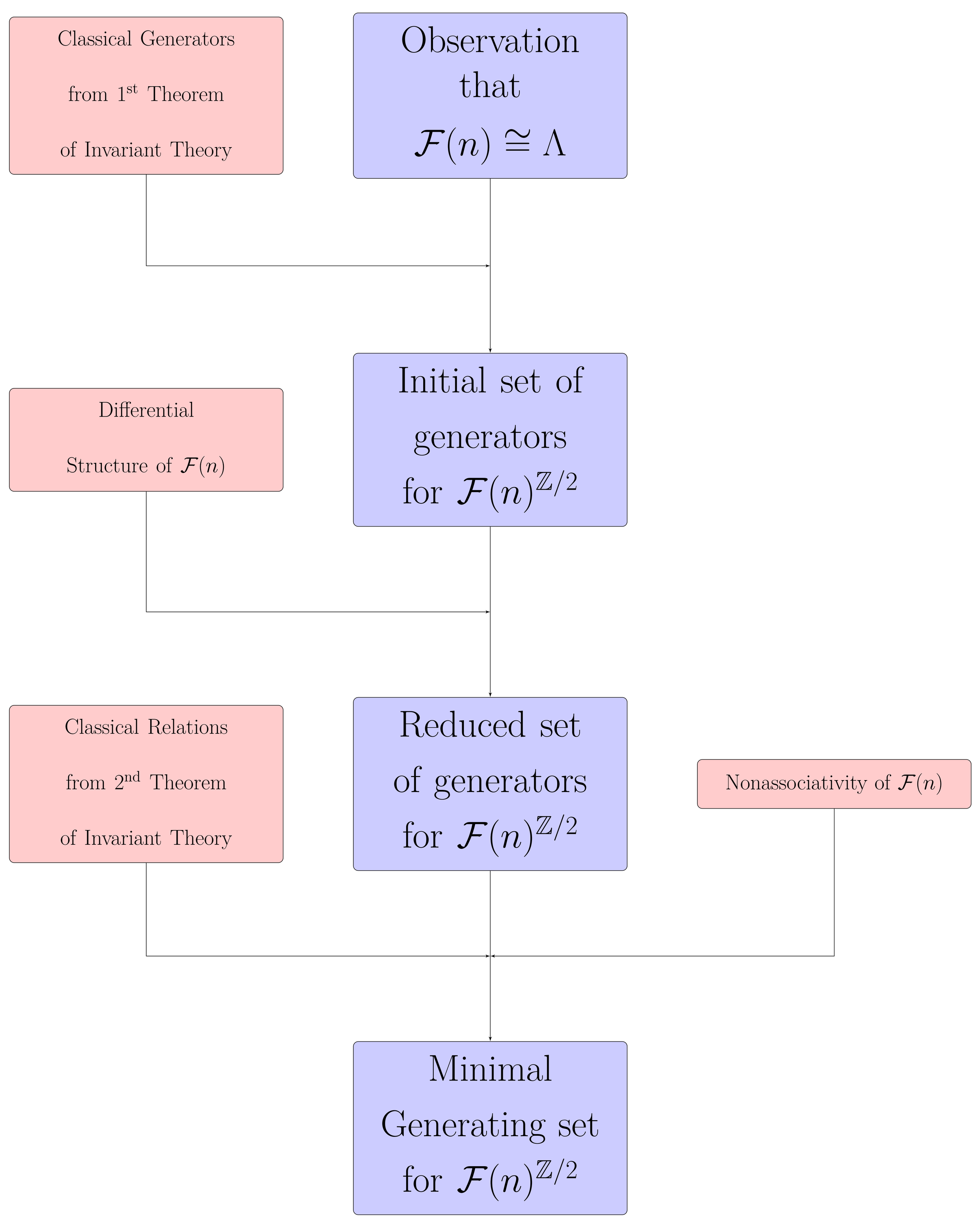
with relations given by:

$$p_{i,j}(a,b) = -p_{j,i}(b,a)$$

and

$$p_{i_0,j_0}(a_0,b_0)p_{i_1,j_1}(a_1,b_1) + p_{i_0,j_1}(a_0,b_1)p_{i_1,j_0}(a_1,b_0) = 0$$

where $(0,1) \leq (a_0,i_0) \leq (a_1,i_1)$ and $(0,1) \leq (b_0,j_0) \leq (b_1,j_1)$, ordered lexicographically.



Quantum Corrections

Finally, we describe a linear isomorphism between the invariant subspaces $\mathcal{F}(n)^{\mathbb{Z}/2}$ and $\wedge_\partial$:

$$y_{i_1}(m_1)\cdots y_{i_k}(m_k) \mapsto {:}\partial^{m_1}\varphi_{i_1}\cdots\partial^{m_k}\varphi_{i_k}{:}$$

and infer that $\mathcal{F}(n)^{\mathbb{Z}/2}$ is spanned by monomials of the form ${:}\partial^{m_1}\varphi_{i_1}\cdots\partial^{m_{2k}}\varphi_{i_{2k}}{:}$.

Under this isomorphism, $\mathcal{F}(n)$ inherits a natural $\mathbb{Z}/2$ action The above flowchart serves as an outline for our argument to find a minimal generating set for $\mathcal{F}(n)^{\mathbb{Z}/2}$

Letting $\psi_{i,j}(a,b) = {:}\partial^a\varphi_i\partial^b\varphi_j{:}$, we observe that we use this differential structure to reduce the generating set of $\mathcal{F}(n)^{\mathbb{Z}/2}$ to

$$\psi_{i,i}(0, 2a+1) \text{ for } 1 \leq i \leq n \text{ and } a \geq 0$$
$$\psi_{i,j}(0, b) \text{ for } 1 \leq i < j \leq n \text{ and } b \geq 0.$$

Results

We use the non-associative and non-commutative structure of $\mathcal{F}(n)$ to further reduce this set and obtain our final result, which we divide into two cases: For the case when $0 \leq n \leq 2$:

$$\mathcal{F}(1)^{\mathbb{Z}/2\mathbb{Z}} = \langle \psi_{1,1}(0,1) \rangle_S$$

and

$$\mathcal{F}(2)^{\mathbb{Z}/2\mathbb{Z}} = \langle \psi_{1,1}(0,1), \psi_{2,2}(0,1), \psi_{1,2}(0,0), \psi_{1,2}(0,1) \rangle_S.$$

For the case when $n > 2$:

$$\mathcal{F}(n)^{\mathbb{Z}/2} = \langle \psi_{i,i}(0,1), \psi_{j,k}(0,0) \rangle_S$$

for $1 \leq i \leq n, 1 \leq j < k \leq n$.

where all of the above generating sets are minimal.

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