

POSITIVITY IN T -EQUIVARIANT K -THEORY OF PARTIAL FLAG VARIETIES ASSOCIATED TO KAC-MOODY GROUPS

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Abstract: We prove sign-alternation of the product structure constants in the basis dual to the basis consisting of the structure sheaves of Schubert varieties in the torus-equivariant Grothendieck group of coherent sheaves on the partial flag varieties G/P associated to an arbitrary symmetrizable Kac-Moody group G , where P is any parabolic subgroup of finite type. This extends the previous work of Kumar from G/B to G/P . When G is of finite type, i.e., it is a semisimple group, then it was proved by Anderson-Griffeth-Miller.

1. INTRODUCTION

This is a continuation of Kumar [Kum17], the notation of which we freely use.

Let G be any symmetrizable Kac-Moody group over \mathbb{C} completed along the negative roots and $G^{\min} \subset G$ the ‘minimal’ Kac-Moody group as in [Kum02, §7.4]. Let B be the standard (positive) Borel subgroup, B^- the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus and W the Weyl group. Let $\bar{X} = G/B$ be the ‘thick’ flag variety (introduced by Kashiwara) which contains the standard flag variety $X = G^{\min}/B$. Let T be the adjoint torus, i.e., $T := H/Z(G^{\min})$, where $Z(G^{\min})$ denotes the center of G^{\min} and let $R(T)$ denote the representation ring of T . For any $w \in W$ (the Weyl group), we have the Schubert cell $C_w := BwB/B \subset X$, the Schubert variety $X_w := \overline{C_w} \subset X$, the opposite Schubert cell $C^w := B^-wB/B \subset \bar{X}$, and the opposite Schubert variety $X^w := \overline{C^w} \subset \bar{X}$. When G is a (finite-dimensional) semisimple group, it is referred to as the finite case.

Let P be any standard parabolic subgroup of G of finite type (i.e., the Levi component of P is finite dimensional) and set W^P to be the set of minimal length coset representatives of W/W_P , where W_P is the Weyl group of P . Then, for any $w \in W^P$, we have the Schubert cell $C_w^P := BwP/P \subset X_P$, Schubert variety $X_w^P := \overline{C_w^P} \subset X_P$, the opposite Schubert cell $C^w_P := B^-wP/P \subset \bar{X}_P$, and the opposite Schubert variety $X^w_P := \overline{C^w_P} \subset \bar{X}_P$ where $X_P = G^{\min}/P$ and $\bar{X}_P = G/P$.

Let $K_T^0(\bar{X}_P)$ be the Grothendieck group of T -equivariant coherent $\mathcal{O}_{\bar{X}_P}$ -modules of the nonquasi-compact scheme \bar{X}_P (cf. Definition 3.1). Let $\{[\xi_P^w]\}_{w \in W^P}$ be the ‘basis’ of $K_T^0(\bar{X}_P)$ given by $\xi_P^w := \mathcal{O}_{X_P^w}(-\partial X_P^w)$ (where $\partial X_P^w := X_P^w \setminus C_w^P$) and express the product in $K_T^0(\bar{X}_P)$ in this ‘basis’:

$$[\xi_P^u] \cdot [\xi_P^v] = \sum_{w \in W^P} d_{u,v}^w(P) [\xi_P^w], \quad \text{for some unique } d_{u,v}^w(P) \in R(T). \quad (1)$$

The above sum, in general, is infinite.

The following result is our main theorem (cf. Theorem 5.5). This was first conjectured by Graham-Kumar [GK08] in the finite case, proven in this case by Anderson-Griffeth-Miller [AGM11], and then proven in the general Kac-Moody case for G/B by Kumar [Kum17].

Theorem 1.1. *For any $u, v, w \in W^P$,*

$$(-1)^{l(u)+l(v)+l(w)} d_{u,v}^w(P) \in \mathbb{Z}_{\geq 0}[(e^{-\alpha_1} - 1), \dots, (e^{-\alpha_r} - 1)],$$

where $\{\alpha_1, \dots, \alpha_r\}$ are the simple roots, i.e., $(-1)^{l(u)+l(v)+l(w)}d_{u,v}^w(P)$ is a polynomial in the variables $x_1 = e^{-\alpha_1} - 1, \dots, x_r = e^{-\alpha_r} - 1$ with nonnegative integral coefficients.

The proof of the above theorem generally follows the proof in [Kum17] for the full flag variety \bar{X} . However, several of the geometric and cohomological results in the case of the full flag variety had to be generalized to the partial flag varieties \bar{X}_P . In particular, the definition of the basis in terms of the dualizing sheaves had to be modified in the parabolic case as the dualizing sheaf in this general case is not suitable.

In another related work, Baldwin-Kumar [BK17] proved an analogue of the above theorem for the basis consisting of the structure sheaves of the opposite Schubert varieties.

We have added an appendix to determine the dualizing sheaf $\omega_{X_P^w}$ of the Cohen-Macaulay scheme $X_P^w \subset \bar{X}_P$ for any $w \in W^P$. Even though this result is not used in the paper, we believe that it is interesting on its own.

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2. NOTATION

We work over the field \mathbb{C} of complex numbers. By a variety, we mean an algebraic variety over \mathbb{C} which is reduced, but not necessarily irreducible. For a scheme X and a closed subscheme Y , $\mathcal{O}_X(-Y)$ denotes the ideal sheaf of Y in X .

Let G be any symmetrizable Kac-Moody group completed along the negative roots (as opposed to along the positive roots as in [Kum02, Chapter 6]), and let $G^{\min} \subset G$ be the ‘minimal’ Kac-Moody group as in [Kum02, §7.4]. Let B be the standard (positive) Borel subgroup, B^- the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus and W the Weyl group [Kum02, Chapter 6]. For any standard parabolic subgroup P of G of finite type (i.e., the Levi component of P is finite dimensional), let

$$\bar{X}_P = G/P$$

be the ‘thick’ partial flag variety which contains the standard Kac-Moody partial flag ind-variety

$$X_P = G^{\min}/P.$$

If G is not of finite type, then \bar{X}_P is an infinite-dimensional nonquasi-compact scheme (cf. [Kas89, §4] for the case of $P = B$; the case of general P is similar) and X_P is an ind-projective variety [Kum02, §7.1]. The group G^{\min} (in particular, the maximal torus H) acts on X_P and \bar{X}_P . Let T be the quotient $H/Z(G^{\min})$, where $Z(G^{\min})$ is the center of G^{\min} . Then the action of H on \bar{X}_P (and X_P) descends to an action of T .

Let W^P denote the set of minimal length coset representatives in the quotient W/W_P , where $W_P \subset W$ is the Weyl group of P . Then, for any $w \in W^P$, we have the Schubert cell

$$C_w^P := BwP/P \subset X_P,$$

the Schubert variety

$$X_w^P := \overline{C_w^P} = \bigsqcup_{\substack{u \leq w \\ u \in W^P}} C_u^P \subset X_P,$$

the opposite Schubert cell

$$C_P^w := B^-wP/P \subset \bar{X}_P,$$

and the opposite Schubert variety

$$X_P^w := \overline{C_P^w} = \bigsqcup_{\substack{u \geq w \\ u \in W^P}} C_P^u \subset \bar{X}_P,$$

all endowed with the reduced subscheme structures. When $P = B$, we drop the qualification P , thus $\bar{X} = \bar{X}_B, X_w = X_w^B$ etc. Then, X_w^P is a (finite-dimensional) irreducible projective subvariety of X_P and X_P^w is a finite-codimensional irreducible subscheme of \bar{X}_P . We denote by Z_w the Bott-Samelson-Demazure-Hansen (BSDH) variety as in [Kum02, §7.1], which is a B -equivariant desingularization of X_w^P [Kum02, Prop. 7.1.15]. Further, X_w^P is normal and has rational singularities (in particular, Cohen-Macaulay for short CM) [Kum02, Theorem 8.2.2]. We also define the boundary of the Schubert variety by

$$\partial X_w^P := X_w^P \setminus C_w^P$$

with the reduced subscheme structure. For any $u, w \in W^P$ with $u \leq w$, we have the Richardson variety

$$X_w^u(P) := X_P^u \cap X_w^P \subset X_P$$

endowed with the reduced subvariety structure. By [KS14, Proposition 5.3] together with [BK05, Lemma 1.1.8], X_P is Frobenius split in any characteristic $p > 0$ compatibly splitting $\{X_w^P, X_P^w \cap X_P\}_{w \in W^P}$. In particular, any scheme theoretic intersection $X_{w_1}^P \cap \cdots \cap X_{w_m}^P \cap X_P^{v_1} \cap \cdots \cap X_P^{v_n}$ (for $m \geq 1$) is *reduced*.

We denote by Z_w^u the T -equivariant desingularization of X_w^u as in [Kum17, Theorem 6.8]. For $u, w \in W^P$, the canonical projection map $X_w^u \rightarrow X_w^u(P)$ is a proper birational morphism, where $C^u \cap C_w$ maps isomorphically onto a T -stable open subset of $C_P^u \cap C_w^P$ (cf. [Deo87, Lemma 4.3]). Hence, Z_w^u is a T -equivariant desingularization of $X_w^u(P)$.

We denote the representation ring of T by $R(T)$. Let $\{\alpha_1, \dots, \alpha_r\} \subset \mathfrak{h}^*$ be the set of simple roots, $\{\alpha_1^\vee, \dots, \alpha_r^\vee\} \subset \mathfrak{h}$ the set of simple coroots, and $\{s_1, \dots, s_r\} \subset W$ the corresponding set of simple reflections, where $\mathfrak{h} := \text{Lie}(H)$. Let $\rho \in \mathfrak{h}^*$ be any integral weight satisfying

$$\rho(\alpha_i^\vee) = 1, \text{ for all } 1 \leq i \leq r.$$

If G is a finite-dimensional semisimple group, ρ is unique, but for a general KM group G , it may not be unique. Denote by $\rho_Y := \sum_{i \in Y} \varpi_i$, where $\{\varpi_1, \dots, \varpi_r\}$ are fixed fundamental weights and $Y \subset \{1, \dots, r\}$ corresponds to the Levi component of P , i.e., $\{\alpha_i\}_{i \in Y}$ are the simple roots of the Levi component L of P containing H . (Observe that, in general, ϖ_i are not unique.) We then define the integral weight $\hat{\rho}_Y \in \mathfrak{h}^*$ by

$$\hat{\rho}_Y := \rho - \rho_Y.$$

For any integral weight λ , let \mathbb{C}_λ denote the one-dimensional representation of H on \mathbb{C} given by $h \cdot z = \lambda(h)z$ for $h \in H$ and $z \in \mathbb{C}$. It uniquely extends to a representation of B . We call λ a P -weight if $\dot{\lambda}(\alpha_i^\vee) = 0$ for all $i \in Y$, where $\dot{\lambda}$ is the derivative of λ . If λ is a P -weight, this action extends uniquely to an action of P and we define the G -equivariant line bundle $\mathcal{L}^P(\lambda)$ on \bar{X}_P by

$$\mathcal{L}^P(\lambda) := G \times^P \mathbb{C}_{-\lambda},$$

where for any representation V of P , $G \times^P V := (G \times V)/P$ where P acts on $G \times V$ by $(g, v) \cdot p = (gp, p^{-1}v)$ for $g \in G$, $v \in V$ and $p \in P$. We also define the line bundle

$$e^\lambda := \bar{X}_P \times \mathbb{C}_\lambda,$$

which while trivial as a non-equivariant line bundle, is equivariantly non-trivial with the diagonal action of H .

3. IDENTIFICATION OF THE DUAL OF THE STRUCTURE SHEAF BASIS

Definition 3.1. For a quasi-compact scheme Y , an \mathcal{O}_Y -module \mathcal{S} is called *coherent* if it is finitely presented as an \mathcal{O}_Y -module and any \mathcal{O}_Y -submodule of finite type admits a finite presentation.

A subset $S \subset W^P$ is called an *ideal* if $x \in S$ and $v \leq x$ imply $v \in S$. An $\mathcal{O}_{\bar{X}_P}$ -module \mathcal{S} is called *coherent* if $\mathcal{S}|_{V^S}$ is a coherent \mathcal{O}_{V^S} -module for any finite ideal $S \subset W^P$, where V^S is the quasi-compact open subset of \bar{X}_P defined by

$$V^S = \bigcup_{w \in S} B^- wP/P = \bigcup_{w \in S} wU^-P/P,$$

where U^- is the unipotent part of B^- . Observe that $\{wU^-P/P\}_{w \in W^P}$ is an open cover of \bar{X}_P . Let $K_T^0(\bar{X}_P)$ denote the Grothendieck group of T -equivariant coherent $\mathcal{O}_{\bar{X}_P}$ -modules \mathcal{S} . Since the coherence condition on \mathcal{S} is imposed only for $\mathcal{S}|_{V^S}$ for finite ideals S , $K_T^0(\bar{X}_P)$ can be thought of as the inverse limit of $K_T^0(V^S)$ as S varies over all finite ideals of W^P [KS09, §2].

Similarly, define $K_0^T(X_P) := \text{Lim}_{n \rightarrow \infty} K_0^T(X_n^P)$, where $\{X_n^P\}_{n \geq 1}$ is the filtration of X_P giving X_P its ind-projective variety structure (i.e., $X_n^P = \bigcup_{l(w) \leq n, w \in W^P} BwP/P$) and $K_0^T(X_n^P)$ is the Grothendieck group of T -equivariant coherent sheaves on the projective variety X_n^P .

For any $w \in W^P$,

$$[\mathcal{O}_{X_w^P}] \in K_0^T(X_P),$$

where $[\mathcal{O}_{X_w^P}]$ denotes the class of $\mathcal{O}_{X_w^P}$ in $K_0^T(X_P)$.

Lemma 3.2. $\{[\mathcal{O}_{X_w^P}]\}_{w \in W^P}$ forms a basis of $K_0^T(X_P)$ as an $R(T)$ -module.

Proof. Apply [CG97, §5.2.14 and Theorem 5.4.17]. □

For any $u \in W^P$, $\mathcal{O}_{X_u^P}$ is a coherent $\mathcal{O}_{\bar{X}_P}$ -module. We record the following lemma due to Kashiwara-Shimozono [KS09, Proof of Lemma 8.1].

Lemma 3.3. Any T -equivariant coherent sheaf \mathcal{S} on V^u admits a free resolution in $\text{Coh}_T(\mathcal{O}_{V^u})$:

$$0 \rightarrow S_n \otimes \mathcal{O}_{V^u} \rightarrow \cdots \rightarrow S_1 \otimes \mathcal{O}_{V^u} \rightarrow S_0 \otimes \mathcal{O}_{V^u} \rightarrow \mathcal{S} \rightarrow 0,$$

where S_k are finite-dimensional T -modules, $V^u := uU^-P/P \subset \bar{X}_P$, and $\text{Coh}_T(\mathcal{O}_{V^u})$ is the abelian category of T -equivariant coherent \mathcal{O}_{V^u} -modules.

We define a pairing

$$\langle \cdot, \cdot \rangle : K_T^0(\bar{X}_P) \otimes K_0^T(X_P) \rightarrow R(T), \quad \langle [\mathcal{S}], [\mathcal{F}] \rangle = \sum_i (-1)^i \chi_T(X_n^P, \mathcal{F}or_i^{\mathcal{O}_{\bar{X}_P}}(\mathcal{S}, \mathcal{F})), \quad (2)$$

where \mathcal{S} is a T -equivariant coherent sheaf on \bar{X}_P and \mathcal{F} is a T -equivariant coherent sheaf on X_P supported on X_n^P for some n , and where χ_T is the T -equivariant Euler-Poincaré characteristic. This pairing is well defined by [Kum17, Proof of Lemma 3.5].

Definition 3.4. For any $u \in W^P$, set the T -equivariant sheaf on \bar{X}_P ,

$$\xi_P^u := \mathcal{O}_{X_P^u}(-\partial X_P^u), \text{ where } \partial X_P^u := X_P^u \setminus C_P^u \text{ with the reduced scheme structure.}$$

Proposition 3.5. $\{[\xi_P^u]\}_{u \in W^P}$ forms an infinite basis of $K_T^0(\bar{X}_P)$ as an $R(T)$ -module, where by infinite basis we mean that $K_T^0(\bar{X}_P) = \prod_{u \in W^P} R(T)[\xi_P^u]$.

Proof. By the same proof as in [KS09, §2] $\{[\mathcal{O}_{X_P^v}]\}_{v \in W^P}$ is an infinite basis of $K_T^0(\bar{X}_P)$. Moreover, $[\xi_P^u] = [\mathcal{O}_{X_P^u}] + \sum_{\substack{w' > u \\ w' \in W^P}} r_{w'}[\mathcal{O}_{X_P^{w'}}]$ for some $r_{w'} \in R(T)$, the lemma follows. \square

Proposition 3.6. For any finite union $Y = \bigcup_{i=1}^k X_P^{v_i}$ of opposite Schubert varieties and any w in W^P ,

- (a) $\mathcal{F}or_j^{\mathcal{O}_{\bar{X}_P}}(\mathcal{O}_Y, \mathcal{O}_{X_w^P}) = 0$ for all $j > 0$.
- (b) $H^j(X_n^P, \mathcal{O}_{Y \cap X_w^P}) = 0$ for all $j > 0$, where n is any positive integer such that $X_n^P \supset X_w^P$.

In particular, this applies to the case $Y = \partial X_P^u$.

Proof. This follows by the same argument as [Kum17, Proof of Corollary 5.7]. Observe that $X_w^u(P)$ is irreducible by Lemma 7.4. Moreover, the vanishing $H^j(X_n^P, \mathcal{O}_{X_w^v(P)}) = 0$, for all $j > 0$, follows from the corresponding result for \bar{X} (cf. [KS14, Corollary 3.2]) and applying the Leray spectral sequence for the bundle $\pi : \bar{X} \rightarrow \bar{X}_P$, since $\pi^{-1}(X_w^v(P)) = X_{ww_o^P}^v$, where w_o^P is the longest element of W_P . \square

Proposition 3.7. For any $v, w \in W^P$,

$$\mathcal{F}or_j^{\mathcal{O}_{\bar{X}_P}}(\xi_P^v, \mathcal{O}_{X_w^P}) = 0 \quad \text{for all } j > 0.$$

Proof. By Proposition 3.6(a), we have the vanishing

$$\mathcal{F}or_j^{\mathcal{O}_{\bar{X}_P}}(\mathcal{O}_{X_P^v}, \mathcal{O}_{X_w^P}) = 0 \quad \text{for all } j > 0.$$

The proposition follows from this together with Proposition 3.6 part (a) applied to $Y = \partial X_P^v$ and the long exact sequence for $\mathcal{F}or$ associated to the short exact sequence

$$0 \rightarrow \xi_P^v \rightarrow \mathcal{O}_{X_P^v} \rightarrow \mathcal{O}_{\partial X_P^v} \rightarrow 0. \quad \square$$

Proposition 3.8. For any $u, v \in W^P$,

$$\langle [\xi_P^u], [\mathcal{O}_{X_w^P}] \rangle = \delta_{u,v}.$$

Proof. For any $u, w \in W^P$, the pairing is by definition

$$\langle [\xi_P^u], [\mathcal{O}_{X_w^P}] \rangle = \sum_i (-1)^i \chi_T(X_n^P, \mathcal{F}or_i^{\mathcal{O}_{\bar{X}_P}}(\xi_P^u, \mathcal{O}_{X_w^P})),$$

with n taken such that $n \geq l(w)$. By Proposition 3.7, this becomes

$$\langle [\xi_P^u], [\mathcal{O}_{X_w^P}] \rangle = \chi_T(X_n^P, \xi_P^u \otimes_{\mathcal{O}_{\bar{X}_P}} \mathcal{O}_{X_w^P}). \quad (3)$$

From Proposition 3.6(a) and the definition $\xi_P^u := \mathcal{O}_{X_P^u}(-\partial X_P^u)$, we have the sheaf exact sequence

$$0 \rightarrow \xi_P^u \otimes_{\mathcal{O}_{\bar{X}_P}} \mathcal{O}_{X_w^P} \rightarrow \mathcal{O}_{X_P^u} \otimes_{\mathcal{O}_{\bar{X}_P}} \mathcal{O}_{X_w^P} \rightarrow \mathcal{O}_{\partial X_P^u} \otimes_{\mathcal{O}_{\bar{X}_P}} \mathcal{O}_{X_w^P} \rightarrow 0.$$

Hence,

$$\chi_T(X_n^P, \xi_P^u \otimes_{\mathcal{O}_{\bar{X}_P}} \mathcal{O}_{X_w^P}) = \chi_T(X_n^P, \mathcal{O}_{X_w^u(P)}) - \chi_T(X_n^P, \mathcal{O}_{(\partial X_P^u) \cap X_w^P}). \quad (4)$$

From Lemma 7.4, the Richardson variety $X_w^u(P)$ is irreducible (when nonempty, i.e., $u \leq w$) and hence $(\partial X_P^u) \cap X_w^P = \bigcup_{u < v \leq w} X_w^v(P)$ is connected (when nonempty) since $wP \in X_w^v(P)$ for all $u < v \leq w$. If $u \not\leq w$, $X_w^u(P)$ is empty, and hence Equations (3) and (4) imply $\langle [\xi_P^u], [\mathcal{O}_{X_w^P}] \rangle = 0$. Hence, we assume $u \leq w$ so that $X_w^u(P)$ is nonempty. Since $X_w^u(P)$ has rational singularities (Lemma 7.4)

$$H^i(X_n^P, \mathcal{O}_{X_w^u(P)}) = 0 \quad \text{for all } i > 0.$$

Further, by Proposition 3.6 (b) applied to ∂X_P^u ,

$$H^i(X_n^P, \mathcal{O}_{(\partial X_P^u) \cap X_w^P}) = 0 \quad \text{for all } i > 0.$$

Thus, for $u \leq w$,

$$\chi_T(X_n^P, \mathcal{O}_{X_w^u(P)}) = 1$$

and for $u < w$,

$$\chi_T(X_n^P, \mathcal{O}_{(\partial X_P^u) \cap X_w^P}) = 1.$$

Therefore, when $u < w$, $\langle [\xi_P^u], [\mathcal{O}_{X_w^P}] \rangle = 0$. Finally, if $u = w$, we have $\langle [\xi_P^w], [\mathcal{O}_{X_w^P}] \rangle = 1$ since $\partial X_P^w \cap X_w^P$ is empty in this case. \square

Let $\Delta : X_P \rightarrow X_P \times X_P$ be the diagonal map. Express the coproduct in $K_0^T(X_P)$:

$$\Delta_*[\mathcal{O}_{X_w^P}] = \sum_{u,v \in W^P} q_{u,v}^w(P) [\mathcal{O}_{X_u^P}] \otimes [\mathcal{O}_{X_v^P}], \quad \text{for } q_{u,v}^w \in R(T).$$

Also, express the product in $K_T^0(\bar{X}_P)$

$$[\xi_P^u] \cdot [\xi_P^v] = \sum_{w \in W^P} d_{u,v}^w(P) [\xi_P^w], \quad \text{for } d_{u,v}^w(P) \in R(T). \quad (5)$$

Proposition 3.9. *For any $u, v, w \in W^P$,*

$$q_{u,v}^w(P) = d_{u,v}^w(P).$$

Proof. Let $\bar{\Delta} : \bar{X}_P \rightarrow \bar{X}_P \times \bar{X}_P$ be the diagonal map. Then, for any $w \in W^P$,

$$\begin{aligned} \langle \bar{\Delta}^*[\xi_P^u \boxtimes \xi_P^v], [\mathcal{O}_{X_w^P}] \rangle &= \langle [\xi_P^u \boxtimes \xi_P^v], \Delta_*[\mathcal{O}_{X_w^P}] \rangle \\ &= \langle [\xi_P^u \boxtimes \xi_P^v], \sum_{u', v' \in W^P} q_{u', v'}^w(P) [\mathcal{O}_{X_{u'}^P}] \otimes [\mathcal{O}_{X_{v'}^P}] \rangle \\ &= q_{u, v}^w(P), \quad \text{by Proposition 3.8.} \end{aligned}$$

On the other hand, since $[\xi_P^u] \cdot [\xi_P^v] = \bar{\Delta}^*[\xi_P^u \boxtimes \xi_P^v]$, we also have

$$\begin{aligned} \langle \bar{\Delta}^*[\xi_P^u \boxtimes \xi_P^v], [\mathcal{O}_{X_w^P}] \rangle &= \langle [\xi_P^u] \cdot [\xi_P^v], [\mathcal{O}_{X_w^P}] \rangle \\ &= \left\langle \sum_{w' \in W^P} d_{u, v}^{w'}(P) [\xi_{P'}^{w'}], [\mathcal{O}_{X_w^P}] \right\rangle \\ &= d_{u, v}^w(P), \quad \text{by Proposition 3.8} \quad \square \end{aligned}$$

4. THE MIXING SPACE AND MIXING GROUP

In this section, we introduce the mixing space $(X_P)_\mathbb{P}$, which is a bundle over a product of projective spaces with fiber X_P . This allows the reduction from the T -equivariant K -theory to the non-equivariant K -theory. We then introduce the mixing group Γ whose action is sufficient to allow for a transversality result used to prove part of our main technical result.

Fix a large N and let $\mathbb{P} := (\mathbb{P}^N)^r$, where $r = \dim T$. For any $\mathbf{j} = (j_1, \dots, j_r) \in [N]^r$, where $[N] := \{0, 1, \dots, N\}$, set

$$\mathbb{P}^{\mathbf{j}} = \mathbb{P}^{N-j_1} \times \dots \times \mathbb{P}^{N-j_r} \quad \text{and} \quad \mathbb{P}_{\mathbf{j}} = \mathbb{P}^{j_1} \times \dots \times \mathbb{P}^{j_r}.$$

We also define the boundary of $\mathbb{P}_{\mathbf{j}}$ by

$$\partial \mathbb{P}_{\mathbf{j}} := (\mathbb{P}^{j_1-1} \times \mathbb{P}^{j_2} \times \dots \times \mathbb{P}^{j_r}) \cup \dots \cup (\mathbb{P}^{j_1} \times \mathbb{P}^{j_2} \times \dots \times \mathbb{P}^{j_r-1}),$$

where we interpret $\mathbb{P}^{-1} := \emptyset$ as the empty set. Throughout the paper, we fix an identification $T \cong (\mathbb{C}^*)^r$ under $t \mapsto (e^{\alpha_1}(t), \dots, e^{\alpha_r}(t))$.

Definition 4.1. Let $E(T)_\mathbb{P} := (\mathbb{C}^{N+1} \setminus \{0\})^r$ be the total space of the standard principal T -bundle $E(T)_\mathbb{P} \rightarrow \mathbb{P}$. Let $\pi_{X_P} : (X_P)_\mathbb{P} := E(T)_\mathbb{P} \times^T X_P \rightarrow \mathbb{P}$ be the fibration with fiber X_P associated to the principal T -bundle $E(T)_\mathbb{P} \rightarrow \mathbb{P}$, where we twist the action of T on X_P via

$$t \odot x = t^{-1}x. \quad (6)$$

For any T -subscheme $Y \subset X_P$, denote $Y_\mathbb{P} := E(T)_\mathbb{P} \times^T Y \subset (X_P)_\mathbb{P}$.

The following proposition follows easily by using [CG97, §5.2.14 and Theorem 5.4.17] applied to the vector bundle $(BwP/P)_\mathbb{P} \rightarrow \mathbb{P}$.

Proposition 4.2. $K_0((X_P)_\mathbb{P}) := \text{Lim}_{n \rightarrow \infty} K_0((X_n^P)_\mathbb{P})$ is a free module over the ring $K_0(\mathbb{P}) = K^0(\mathbb{P})$ with basis $\{[\mathcal{O}_{(X_w^P)_\mathbb{P}}]\}_{w \in W^P}$. Therefore, $K_0((X_P)_\mathbb{P})$ has a \mathbb{Z} -basis

$$\{\pi_{X_P}^*([\mathcal{O}_{\mathbb{P}_{\mathbf{j}}}] \cdot [\mathcal{O}_{(X_w^P)_\mathbb{P}}])\}_{\mathbf{j} \in [N]^r, w \in W^P},$$

where we view $[\mathcal{O}_{\mathbb{P}_{\mathbf{j}}}]$ as an element of $K_0(\mathbb{P}) = K^0(\mathbb{P})$.

Let $Y_P := X_P \times X_P$. The diagonal map $\Delta : X_P \rightarrow Y_P$ gives rise to the embedding

$$\tilde{\Delta} : (X_P)_{\mathbb{P}} \rightarrow (Y_P)_{\mathbb{P}} = E(T)_{\mathbb{P}} \times^T Y_P \cong (X_P)_{\mathbb{P}} \times_{\mathbb{P}} (X_P)_{\mathbb{P}}.$$

Therefore, we have (denoting the projection $(Y_P)_{\mathbb{P}} \rightarrow \mathbb{P}$ by π_{Y_P})

$$\tilde{\Delta}_*[\mathcal{O}_{(X_P)_{\mathbb{P}}}] = \sum_{\substack{u,v \in W^P \\ \mathbf{j} \in [N]^r}} c_{u,v}^w(\mathbf{j}) \pi_{Y_P}^*([\mathcal{O}_{\mathbb{P}^{\mathbf{j}}}] \cdot [\mathcal{O}_{(X_u^P \times X_v^P)_{\mathbb{P}}}] \in K_0((Y_P)_{\mathbb{P}}) \quad (7)$$

for some $c_{u,v}^w(\mathbf{j}) \in \mathbb{Z}$.

Let $\bar{Y}_P = \bar{X}_P \times \bar{X}_P$ and let $K^0((\bar{Y}_P)_{\mathbb{P}})$ denote the Grothendieck group associated to the semi-group of coherent $\mathcal{O}_{(\bar{Y}_P)_{\mathbb{P}}}$ -modules. Define, for $u, v \in W^P$,

$$\widetilde{\xi_P^u \boxtimes \xi_P^v} := \mathcal{O}_{(X_P^u \times X_P^v)_{\mathbb{P}}}(-\partial((X_P^u \times X_P^v)_{\mathbb{P}})) \in K^0((\bar{Y}_P)_{\mathbb{P}}),$$

where $\partial((X_P^u \times X_P^v)_{\mathbb{P}}) := ((\partial X_P^u \times X_P^v) \cup (X_P^u \times \partial X_P^v))_{\mathbb{P}}$.

Lemma 4.3. *With the notation as above,*

$$c_{u,v}^w(\mathbf{j}) = \langle \pi_{\bar{Y}_P}^*[\mathcal{O}_{\mathbb{P}^{\mathbf{j}}}(-\partial\mathbb{P}^{\mathbf{j}})] \cdot [\widetilde{\xi_P^u \boxtimes \xi_P^v}], \tilde{\Delta}_*[\mathcal{O}_{(X_P^w)_{\mathbb{P}}}] \rangle, \quad (8)$$

where $\pi_{\bar{Y}_P} : (\bar{Y}_P)_{\mathbb{P}} \rightarrow \mathbb{P}$ is the projection and the pairing $\langle \cdot, \cdot \rangle : K^0((\bar{Y}_P)_{\mathbb{P}}) \otimes K_0((Y_P)_{\mathbb{P}}) \rightarrow \mathbb{Z}$ is defined similar to (2) above. Explicitly,

$$\langle [\mathcal{S}], [\mathcal{F}] \rangle = \sum_i (-1)^i \chi((\bar{Y}_P)_{\mathbb{P}}, \mathcal{T}or_i^{\mathcal{O}_{(\bar{Y}_P)_{\mathbb{P}}}}(\mathcal{S}, \mathcal{F})), \quad (9)$$

where χ denotes the (non-equivariant) Euler-Poincaré characteristic.

Proof. We compute

$$\begin{aligned} & \langle \pi_{\bar{Y}_P}^*[\mathcal{O}_{\mathbb{P}^{\mathbf{j}}}(-\partial\mathbb{P}^{\mathbf{j}})] \cdot [\widetilde{\xi_P^u \boxtimes \xi_P^v}], \tilde{\Delta}_*[\mathcal{O}_{(X_P^w)_{\mathbb{P}}}] \rangle \\ &= \langle \pi_{\bar{Y}_P}^*[\mathcal{O}_{\mathbb{P}^{\mathbf{j}}}(-\partial\mathbb{P}^{\mathbf{j}})] \cdot [\widetilde{\xi_P^u \boxtimes \xi_P^v}], \sum_{\substack{u',v' \in W^P \\ \mathbf{j}' \in [N]^r}} c_{u',v'}^w(\mathbf{j}') \pi_{\bar{Y}_P}^*([\mathcal{O}_{\mathbb{P}^{\mathbf{j}'}}] \cdot [\mathcal{O}_{(X_{u'}^P \times X_{v'}^P)_{\mathbb{P}}}] \rangle \\ &= c_{u,v}^w(\mathbf{j}) \quad \text{by Proposition 3.8 and [Kum17, identity (20)].} \quad \square \end{aligned}$$

Definition 4.4. Let T act on B via

$$t \cdot b = t^{-1}bt$$

where $t \in T$ and $b \in B$. Then there is a natural action of ΔT on $B \times B$. Let $(B^2)_{\mathbb{P}}$ be the ind-scheme over \mathbb{P} :

$$(B^2)_{\mathbb{P}} := E(T)_{\mathbb{P}} \times^T (B \times B) \rightarrow \mathbb{P}$$

and let Γ_0 denote the group of global sections of $(B^2)_{\mathbb{P}}$ under pointwise multiplication. Thus, Γ_0 can be identified with the set of regular maps $f : E(T)_{\mathbb{P}} \rightarrow B \times B$ such that $f(e \cdot t) = t^{-1} \cdot f(e)$ for all $e \in E(T)_{\mathbb{P}}$ and $t \in T$. Now, $GL(N+1)^r$ acts canonically on $(B^2)_{\mathbb{P}}$ in a way compatible with its action on \mathbb{P} and acts on Γ_0 via its pull-back. We define the mixing group

$$\Gamma = \Gamma_{B^2} := \Gamma_0 \rtimes GL(N+1)^r:$$

$$1 \rightarrow \Gamma_0 \rightarrow \Gamma \rightarrow GL(N+1)^r \rightarrow 1.$$

By the comments following [Kum17, Lemmas 4.7 and 4.8], we have the following two lemmas.

Lemma 4.5. Γ is connected.

Lemma 4.6. Given any $\bar{e} \in \mathbb{P}$ and any (b_1, b_2) in the fiber of $(B^2)_{\mathbb{P}}$ over \bar{e} , there exists a section $\gamma \in \Gamma_0$ such that $\gamma(\bar{e}) = (b_1, b_2)$.

Define the action of Γ on $(Y_P)_{\mathbb{P}}$ by

$$(\gamma, g) \cdot [e, (y, y')] = [ge, \gamma(ge) \cdot (y, y')]$$

for $\gamma \in \Gamma_0$, $g \in GL(N+1)^r$, $e \in E(T)_{\mathbb{P}}$, and $(y, y') \in Y_P$, where the action of Γ_0 is via the standard action of B^2 on Y_P . From Lemma 4.6, it follows that the orbits of the Γ -action on $(Y_P)_{\mathbb{P}}$ are precisely $\{(C_u^P \times C_v^P)_{\mathbb{P}}\}_{u, v \in W^P}$.

Proposition 4.7. For any coherent sheaf \mathcal{S} on \mathbb{P} , and any $u, v \in W^P$,

$$\pi^*[\mathcal{S}] \cdot [\widetilde{\xi_P^u \boxtimes \xi_P^v}] = [\pi^*(\mathcal{S}) \otimes_{\mathcal{O}_{(\bar{Y}_P)_{\mathbb{P}}}} (\widetilde{\xi_P^u \boxtimes \xi_P^v})] \in K^0((\bar{Y}_P)_{\mathbb{P}}),$$

where we abbreviate $\pi_{\bar{Y}_P}$ by π and $\pi^*(\mathcal{S}) = \mathcal{O}_{(\bar{Y}_P)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{S}$. In particular,

$$\pi^*[\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)] \cdot [\widetilde{\xi_P^u \boxtimes \xi_P^v}] = [\pi^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes_{\mathcal{O}_{(\bar{Y}_P)_{\mathbb{P}}}} (\widetilde{\xi_P^u \boxtimes \xi_P^v})].$$

Proof. This follows from the same argument as [Kum17, Proposition 4.9]. \square

5. STATEMENT OF MAIN RESULTS

The following is our main technical result. The proof of its two parts are given in Sections 6 and 10 respectively.

Theorem 5.1. For general $\gamma \in \Gamma$, $\mathbf{j} \in [N]^r$, and any $u, v, w \in W^P$,

(A) $\mathcal{F}or_i^{\mathcal{O}_{(\bar{Y}_P)_{\mathbb{P}}}}(\pi^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes (\widetilde{\xi_P^u \boxtimes \xi_P^v}), \gamma_*\tilde{\Delta}_*(\mathcal{O}_{(X_w^P)_{\mathbb{P}}})) = 0$ for all $i > 0$, where $\gamma \in \Gamma$ is viewed as an automorphism of the scheme $(Y_P)_{\mathbb{P}}$.

(B) For $c_{u,v}^w(\mathbf{j}) \neq 0$, where $c_{u,v}^w(\mathbf{j})$ is defined by (7) (see also (8)),

$$H^p((\bar{Y}_P)_{\mathbb{P}}, \pi^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes (\widetilde{\xi_P^u \boxtimes \xi_P^v}) \otimes \gamma_*\tilde{\Delta}_*(\mathcal{O}_{(X_w^P)_{\mathbb{P}}})) = 0$$

for all $p \neq |\mathbf{j}| + l(w) - (l(u) + l(v))$, where $|\mathbf{j}| = \sum_{i=1}^r j_i$.

Since Γ is connected (cf. Lemma 4.5), we have the following result as an immediate corollary of Lemma 4.3, Proposition 4.7, and Theorem 5.1.

Corollary 5.2. $(-1)^{|\mathbf{j}|+l(w)-(l(u)+l(v))} c_{u,v}^w(\mathbf{j}) \in \mathbb{Z}_{\geq 0}$.

Recall the definition of the structure constants $d_{u,v}^w(P) \in R(T)$ (cf. (5)) for the product in $K_T^0(\bar{X}_P)$.

Lemma 5.3. For any $u, v, w \in W^P$, $d_{u,v}^w(P) \in \mathbb{Z}[(e^{-\alpha_1} - 1), \dots, (e^{-\alpha_r} - 1)]$.

Proof. By [GK08, Proposition 3.5], which is valid in the symmetrizable Kac-Moody case by the same proof, for any $u, v, w \in W^P$,

$$d_{u,v}^w(P) = \sum_{u' \in uW_P, v' \in vW_P} d_{u',v'}^w(B). \quad (10)$$

Now, $d_{u',v'}^w(B) \in \mathbb{Z}[(e^{-\alpha_1} - 1), \dots, (e^{-\alpha_r} - 1)]$ by [Kum17, Lemma 4.12]. Hence, the lemma follows from the identity (10). \square

The following lemma follows easily from the identity (7), Proposition 3.9, [GK08, Lemma 6.2] (which is valid in the Kac-Moody case as well) (see also [AGM11, §3]).

Lemma 5.4. *For any $u, v, w \in W^P$, we can choose a large enough N (depending on u, v, w) and write (cf. Lemma 5.3)*

$$d_{u,v}^w(P) = \sum_{\mathbf{j}} d_{u,v}^w(\mathbf{j})(e^{-\alpha_1} - 1)^{j_1} \dots (e^{-\alpha_r} - 1)^{j_r}$$

for some unique $d_{u,v}^w(\mathbf{j}) \in \mathbb{Z}$, where $\mathbf{j} = (j_1, \dots, j_r)$. Then

$$d_{u,v}^w(\mathbf{j}) = (-1)^{|\mathbf{j}|} c_{u,v}^w(\mathbf{j}). \quad (11)$$

The following main theorem of this paper is an immediate consequence of Corollary 5.2 and Lemma 5.4, which was proved in the G/B case by Kumar [Kum17].

Theorem 5.5. *For any $u, v, w \in W^P$, and any symmetrizable Kac-Moody group G and parabolic subgroup P of finite type, the structure constants in $K_T^0((\bar{X}_P)_{\mathbb{P}})$ satisfy*

$$(-1)^{l(u)+l(v)+l(w)} d_{u,v}^w(P) \in \mathbb{Z}_{\geq 0}[(e^{-\alpha_1} - 1), \dots, (e^{-\alpha_r} - 1)]. \quad (12)$$

6. PROOF OF PART (A) OF THEOREM 5.1

Proof of Theorem 5.1(A). Note that since the assertion is local in \mathbb{P} , we assume $(\bar{Y}_P)_{\mathbb{P}} \simeq \mathbb{P} \times \bar{Y}_P$. Then,

$$\pi^* \mathcal{O}_{\mathbb{P}_j}(-\partial \mathbb{P}_j) \simeq \mathcal{O}_{\mathbb{P}_j}(-\partial \mathbb{P}_j) \boxtimes \mathcal{O}_{\bar{Y}_P}, \quad (13)$$

$$\widetilde{\xi_P^u \boxtimes \xi_P^v} \simeq \mathcal{O}_{\mathbb{P}} \boxtimes (\xi_P^u \boxtimes \xi_P^v), \quad (14)$$

$$\mathcal{O}_{(X_w^P \times X_w^P)_{\mathbb{P}}} \simeq \mathcal{O}_{\mathbb{P}} \boxtimes (\mathcal{O}_{X_w^P} \boxtimes \mathcal{O}_{X_w^P}). \quad (15)$$

We first show that for any $\mathcal{O}_{(Y_w^P)_{\mathbb{P}}}$ -module \mathcal{S} (where $(Y_w^P)_{\mathbb{P}} := (X_w^P \times X_w^P)_{\mathbb{P}}$),

$$\begin{aligned} \mathcal{F}or_i^{\mathcal{O}_{(\bar{Y}_P)_{\mathbb{P}}}}(\pi^*(\mathcal{O}_{\mathbb{P}_j}(-\partial \mathbb{P}_j)) \otimes (\widetilde{\xi_P^u \boxtimes \xi_P^v}), \mathcal{S}) \\ \simeq \mathcal{F}or_i^{\mathcal{O}_{(Y_w^P)_{\mathbb{P}}}}(\mathcal{O}_{(Y_w^P)_{\mathbb{P}}} \otimes_{\mathcal{O}_{(\bar{Y}_P)_{\mathbb{P}}}}(\pi^*(\mathcal{O}_{\mathbb{P}_j}(-\partial \mathbb{P}_j)) \otimes (\widetilde{\xi_P^u \boxtimes \xi_P^v})), \mathcal{S}). \end{aligned} \quad (16)$$

To prove (16), we observe the following: Let R, S be commutative rings with a ring homomorphism $R \rightarrow S$, M an R -module and N an S -module. Then $N \otimes_S (S \otimes_R M) \simeq N \otimes_R M$. This gives the following isomorphism, provided $\text{Tor}_j^R(S, M) = 0$ for all $j > 0$:

$$\text{Tor}_i^R(M, N) \simeq \text{Tor}_i^S(S \otimes_R M, N). \quad (17)$$

Then (16) follows with $R = \mathcal{O}_{(\bar{Y}_P)_\mathbb{P}}$, $S = \mathcal{O}_{(Y_w^P)_\mathbb{P}}$, $M = \pi^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes (\xi_P^u \boxtimes \xi_P^v)$, and $N = S$, provided that $\mathcal{T}or_j^R(S, M) = 0$ for all $j > 0$. But this follows from the Kunnetth formula, and the isomorphisms (13) - (15) together with Proposition 3.7.

By Proposition 3.7 and isomorphism (17) applied to $R = \mathcal{O}_{\bar{X}_P}$, $S = \mathcal{O}_{X_w^P}$, $M = \xi_P^u$, and $N = \mathcal{O}_{X_x^P}$ (for $x \leq w, x \in W^P$), we have

$$\mathcal{T}or_j^{\mathcal{O}_{X_w^P}}(\mathcal{O}_{X_w^P} \otimes_{\mathcal{O}_{\bar{X}_P}} \xi_P^u, \mathcal{O}_{X_x^P}) = 0 \quad \text{for all } x \leq w, j > 0. \quad (18)$$

By Lemma 4.6, the closures of the Γ -orbits in $(Y_w^P)_\mathbb{P}$ are precisely $(X_x^P \times X_y^P)_\mathbb{P}$ for $x, y \leq w$ and $x, y \in W^P$. Setting $\mathcal{F} := \mathcal{O}_{(Y_w^P)_\mathbb{P}} \otimes_{\mathcal{O}_{(\bar{Y}_P)_\mathbb{P}}} (\pi^* \mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j) \otimes (\xi_P^u \boxtimes \xi_P^v))$, the identities (13)–(15) and (18) imply that \mathcal{F} is homologically transverse to the closures of the Γ -orbits in $(Y_w^P)_\mathbb{P}$. Then applying [AGM11, Theorem 2.3] (with their $G = \Gamma$, $X = (Y_w^P)_\mathbb{P}$, $\mathcal{E} = \tilde{\Delta}_* \mathcal{O}_{(X_w^P)_\mathbb{P}}$ and their $\mathcal{F} = \mathcal{F}$), we have the following:

$$\mathcal{T}or_i^{\mathcal{O}_{(Y_w^P)_\mathbb{P}}}(\mathcal{O}_{(Y_w^P)_\mathbb{P}} \otimes_{\mathcal{O}_{(\bar{Y}_P)_\mathbb{P}}} (\pi^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes (\xi_P^u \boxtimes \xi_P^v)), \gamma_* \tilde{\Delta}_* \mathcal{O}_{(X_w^P)_\mathbb{P}}) = 0 \quad \text{for all } i > 0. \quad (19)$$

(We note here that although Γ is infinite-dimensional, its action on $(Y_w^P)_\mathbb{P}$ factors through a finite-dimensional quotient group $\bar{\Gamma}$ of Γ .) Finally, observe that $\gamma(\tilde{\Delta}(X_w^P)_\mathbb{P}) \subset (Y_w^P)_\mathbb{P}$, and thus (16) and (19) imply

$$\mathcal{T}or_i^{\mathcal{O}_{(\bar{Y}_P)_\mathbb{P}}}(\pi^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes (\xi_P^u \boxtimes \xi_P^v), \gamma_* \tilde{\Delta}_*(\mathcal{O}_{(X_w^P)_\mathbb{P}})) = 0 \quad \text{for all } i > 0. \quad \square$$

This proves Theorem 5.1(A).

7. STUDY OF THE RICHARDSON VARIETIES

For any $u, v \leq w$ in W^P set $X_w^{u,v} := X_w^u \times X_w^v$ where $X_w^u := X^u \cap X_w$. Similarly, we set $X_w^{u,v}(P) := X_w^u(P) \times X_w^v(P)$ where $X_w^u(P) := X_P^u \cap X_w^P$. We also write $X_w^2(P) := X_w^P \times X_w^P$ and from now on we denote by $Y_{\mathbf{j}}$ (for any T -stable subscheme $Y \subset \bar{X}_P$) the inverse image of $\mathbb{P}_{\mathbf{j}}$ under the standard quotient map $E(T)_\mathbb{P} \times^T Y \rightarrow \mathbb{P}$.

The action of B on X_w^P factors through the action of a finite dimensional quotient group $\bar{B} = B_w$ containing the maximal torus H . Similarly, the action of Γ on $(X_w^2(P))_\mathbb{P}$ descends to an action of a finite dimensional quotient group $\bar{\Gamma} = \Gamma_w$:

$$\Gamma \rightarrow \bar{\Gamma} = \Gamma_w \rightarrow GL(N+1)^r.$$

Further, we can (and do) take $\bar{\Gamma} = \bar{\Gamma}_0 \times GL(N+1)^r$, where $\bar{\Gamma}_0$ is the group of global sections of the bundle $(\bar{B}^2)_\mathbb{P} := E(T)_\mathbb{P} \times^T \bar{B}^2 \rightarrow \mathbb{P}$.

Lemma 7.1. *For any $\mathbf{j} = (j_1, \dots, j_r) \in [N]^r$ and $u, v \leq w \in W^P$, the map*

$$m : \bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbf{j}} \rightarrow (X_w^2(P))_{\mathbb{P}}$$

is flat, where $m(\gamma, x) = \gamma \cdot \pi_2(x)$ and $\pi_2 : (X_w^{u,v}(P))_{\mathbf{j}} \rightarrow (X_w^2(P))_{\mathbb{P}}$ is the map induced from the inclusion $p : X_w^u(P) \times X_w^v(P) \rightarrow X_w^2(P)$. Similarly, its restriction $\hat{m} : \bar{\Gamma} \times \partial((X_w^{u,v}(P))_{\mathbf{j}}) \rightarrow (X_w^2(P))_{\mathbb{P}}$ is also flat, where

$$\partial((X_w^{u,v}(P))_{\mathbf{j}}) := ((\partial X_P^{u,v}) \cap (X_w^2(P))_{\mathbf{j}}) \cup (X_w^{u,v}(P))_{\partial\mathbb{P}_{\mathbf{j}}},$$

$(X_w^{u,v}(P))_{\partial\mathbb{P}_{\mathbf{j}}}$ denotes the inverse image of $\partial\mathbb{P}_{\mathbf{j}}$ under the quotient map $E(T)_{\mathbb{P}} \times^T (X_w^{u,v}(P)) \rightarrow \mathbb{P}$ and $\partial X_P^{u,v} := (\partial X_P^u \times X_P^v) \cup (X_P^u \times \partial X_P^v)$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} \bar{\Gamma}_0 \times X_w^{u,v}(P) & \longrightarrow & \bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbf{j}} & \longrightarrow & \mathrm{GL}(N+1)^r \times \mathbb{P}_{\mathbf{j}} \\ \downarrow m' & & \downarrow m & & \downarrow m'' \\ X_w^2(P) & \longrightarrow & (X_w^2(P))_{\mathbb{P}} & \longrightarrow & \mathbb{P} \end{array}$$

Here the two right horizontal maps are locally trivial fibrations with fibers in the leftmost spaces. From this and the fact that m'' is smooth (see the proof of [Kum17, Lemma 6.11]), to show that m is flat it is enough to show that m' is a flat morphism. From Lemma 4.6, it suffices to show that $(\bar{B}^2) \times X_w^{u,v}(P) \rightarrow X_w^2(P)$ is flat. This follows from the proof of [Kum17, Lemma 6.10] for \bar{X}_P .

Observe also that, by the same proof as that of [Kum17, Lemma 6.10], the map $(\bar{B}^2) \times ((\partial X_P^{u,v}) \cap X_w^2(P)) \rightarrow X_w^2(P)$ is flat. Hence to show that \hat{m} is flat, we first observe that the restrictions to the components $\Gamma_1 := \bar{\Gamma} \times ((\partial X_P^{u,v}) \cap X_w^2(P))_{\mathbf{j}}$, $\Gamma_2 := \bar{\Gamma} \times (X_w^{u,v}(P))_{\partial\mathbb{P}_{\mathbf{j}}}$, and $\Gamma_1 \cap \Gamma_2$ are all flat maps (following the same argument as the first part of this proof). Therefore \hat{m} is flat on $\Gamma_1 \cup \Gamma_2$, since for any affine scheme $Y = Y_1 \cup Y_2$ with closed subschemes Y_1, Y_2 and a morphism $f : Y \rightarrow X$ of schemes, there is an exact sequence of $k[X]$ -modules:

$$0 \rightarrow k[Y] \rightarrow k[Y_1] \oplus k[Y_2] \rightarrow k[Y_1 \cap Y_2] \rightarrow 0. \quad \square$$

As in Section 2, for $v \leq w \in W^P$, let

$$\pi_w^v : Z_w^v \rightarrow X_w^v(P)$$

be the T -equivariant desingularization of $X_w^v(P)$. For $u, v, w \in W^P$, let $Z_w^{u,v} := Z_w^u \times Z_w^v$ under the diagonal action of T and $(Z_w^{u,v})_{\mathbf{j}}$ is the inverse image of $\mathbb{P}_{\mathbf{j}}$ under the map $E(T)_{\mathbb{P}} \times^T Z_w^{u,v} \rightarrow \mathbb{P}$.

We record the following from [Kum17, Lemma 6.11]:

Lemma 7.2. *For any $\mathbf{j} \in [N]^r$ and $u, v \leq w \in W^P$, the map*

$$\tilde{m} : \bar{\Gamma} \times (Z_w^{u,v})_{\mathbf{j}} \rightarrow (Z_w^2)_{\mathbb{P}}$$

is a smooth morphism, where \tilde{m} is defined similarly to the map m in Lemma 7.1.

Lemma 7.3. *For any $u \in W^P$, X_P^u is normal and CM.*

Proof. The standard projection map $p : G/B \rightarrow G/P$ is a locally trivial fibration with fibers isomorphic to the finite dimensional flag variety P/B (since, by assumption, P is of finite type). By [KS09, Propositions 3.2 and 3.4] the lemma follows for X^u . Hence, the lemma follows for X_P^u since $X^u \rightarrow X_P^u$ is a locally trivial fibration with fiber P/B . Thus, locally X^u is isomorphic with $X_P^u \times \mathbb{A}^n$, for $n = \dim P/B$. Alternatively, see [sta, Lemma 10.164.3] for normality and [sta, Lemma 75.26.5] for CM property. \square

Lemma 7.4. *For any symmetrizable Kac-Moody group G and any $u \leq w$ in W^P , the Richardson variety $X_w^u(P) \subset \bar{X}_P$ is irreducible, normal, and CM with rational singularities. Moreover, $C_w^P \cap C_P^u$ is an open dense subset of $X_w^u(P)$.*

Proof. Observe that under the locally trivial fibration $\pi : \bar{X} \rightarrow \bar{X}_P$ with fiber P/B , $\pi^{-1}(X_w^u(P)) = X_{\hat{w}}^u$, where \hat{w} is the longest element in the coset wW_P . Hence, the irreducibility, normality and

CM property of $X_w^u(P)$ follows, as in the above lemma 7.3, from the corresponding properties of X_w^u (cf. [Kum17, Proposition 6.6]). Similarly, the rational singularities of $X_w^u(P)$ follows from the corresponding result for X_w^u (cf. [KS14, Theorem 3.1]).

Finally, $C_w^P \cap C_P^u$ is clearly an open subset of $X_w^u(P)$ and therefore dense since it is nonempty by [Kum02, Lemma 7.3.10]. \square

Lemma 7.5. $\mathcal{O}_{X_P^v}(-\partial X_P^v)$ is a CM $\mathcal{O}_{X_P^v}$ -module and $\mathcal{O}_{\partial X_P^v}$ is a CM ring.

Proof. From the locally trivial fibration $G/B \rightarrow G/P$ with fiber P/B , we see that locally

$$\bar{X} \simeq \bar{X}_P \times \mathbb{A}^n, \quad X^v \simeq X_P^v \times \mathbb{A}^n, \quad \text{and} \quad \partial X^v \simeq \partial X_P^v \times \mathbb{A}^n. \quad (20)$$

Thus, locally as $\mathcal{O}_{X^v} \simeq \mathcal{O}_{X_P^v} \otimes \mathcal{O}_{\mathbb{A}^n}$ -modules, we get

$$\mathcal{O}_{X^v}(-\partial X^v) \simeq (\mathcal{O}_{X_P^v}(-\partial X_P^v)) \otimes \mathcal{O}_{\mathbb{A}^n}. \quad (21)$$

Since $\mathcal{O}_{X^v}(-\partial X^v)$ is a CM \mathcal{O}_{X^v} -module (cf. [Kum17, Corollary 10.5]), from the above decomposition (21), we get that $\mathcal{O}_{X_P^v}(-\partial X_P^v)$ is a CM $\mathcal{O}_{X_P^v}$ -module.

Now, $\mathcal{O}_{\partial X^v}$ being a CM ring (cf. [Kum17, Corollary 10.5]), so is $\mathcal{O}_{\partial X_P^v}$ by using the decompositions (20). \square

Remark Following the same proof as that of Lemma 7.4, replacing X_P^u by ∂X_P^u and observing that $\partial X^u \cap X_w^u$ is CM (cf. [Kum17, Remark 6.7]), we also have that $(\partial X_P^u) \cap X_w^P$ is CM.

8. THE SCHEMES \mathcal{Z}_P AND $\tilde{\mathcal{Z}}$

Let $u, v \leq w \in W^P$. As in Section 7, denote by $Z_w^{u,v} := Z_w^u \times Z_w^v$, where Z_w^u is the T -equivariant desingularization of $X_w^u(P)$ as in Section 2. We also let $Z_w^2 := Z_w \times Z_w$, where Z_w is a BSDH variety as in [Kum02, §7.1.3]. For any $\mathbf{j} \in [N]^r$, let $(X_w^{u,v}(P))_{\mathbf{j}}$ and $(Z_w^{u,v})_{\mathbf{j}}$ denote the inverse image of $\mathbb{P}_{\mathbf{j}}$ through the maps $E(T)_{\mathbb{P}} \times^T X_w^{u,v}(P) \rightarrow \mathbb{P}$ and $E(T)_{\mathbb{P}} \times^T Z_w^{u,v} \rightarrow \mathbb{P}$ respectively.

We define the scheme $\tilde{\mathcal{Z}}$ to be the fiber product $(\bar{\Gamma} \times (Z_w^{u,v})_{\mathbf{j}}) \times_{(Z_w^2)_{\mathbb{P}}} \tilde{\Delta}((Z_w)_{\mathbb{P}})$ and \mathcal{Z}_P to be the fiber product $(\bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbf{j}}) \times_{(X_w^2(P))_{\mathbb{P}}} \tilde{\Delta}((X_w^P)_{\mathbb{P}})$ as in the commutative diagram:

$$\begin{array}{ccc}
\tilde{\mathcal{Z}} & \xrightarrow[\text{(smooth)}]{\tilde{\mu}} & \tilde{\Delta}((Z_w)_{\mathbb{P}}) \\
\downarrow \tilde{i} & \square & \downarrow \\
\bar{\Gamma} \times (Z_w^{u,v})_{\mathbf{j}} & \xrightarrow[\text{(smooth)}]{\tilde{m}} & (Z_w^2)_{\mathbb{P}} \\
\downarrow \theta & & \downarrow \beta \\
\bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbf{j}} & \xrightarrow[\text{(flat)}]{m} & (X_w^2(P))_{\mathbb{P}} \\
\uparrow i & \square & \uparrow \\
\mathcal{Z}_P & \xrightarrow[\text{(flat)}]{\mu} & \tilde{\Delta}((X_w^P)_{\mathbb{P}})
\end{array}$$

$f: \tilde{\mathcal{Z}} \rightarrow \mathcal{Z}_P$ (via \tilde{i})
 $\tilde{\pi}: \tilde{\mathcal{Z}} \rightarrow \bar{\Gamma}$
 $\pi: \mathcal{Z}_P \rightarrow \bar{\Gamma}$

The map θ is induced by the product of the desingularizations $\pi_w^u : Z_w^u \rightarrow X_w^u(P)$ and $\pi_w^v : Z_w^v \rightarrow X_w^v(P)$. In particular, θ is a birational map for $u, v \leq w \in W^P$. The map β is induced from the BSDH desingularization $Z_w \rightarrow X_w^P$. The maps π and $\tilde{\pi}$ are obtained from the projections to the $\bar{\Gamma}$ -factor via the maps i and \tilde{i} respectively. The map f is the restriction of θ to $\tilde{\mathcal{Z}}$ (via \tilde{i}) with image inside \mathcal{Z}_P .

Proposition 8.1. *The schemes \mathcal{Z}_P and $\tilde{\mathcal{Z}}$ are irreducible and the map $f : \tilde{\mathcal{Z}} \rightarrow \mathcal{Z}_P$ is a proper birational map. Thus $\tilde{\mathcal{Z}}$ is a desingularization of \mathcal{Z}_P . Moreover, \mathcal{Z}_P is CM with*

$$\dim(\mathcal{Z}_P) = |\mathbf{j}| + l(w) - (l(u) + l(v)) + \dim(\bar{\Gamma}), \quad (22)$$

where $|\mathbf{j}| = \sum_{i=1}^r j_i$ for $\mathbf{j} = (j_1, \dots, j_r)$.

Proof. We first note that the top Cartesian square is precisely the same as that of [Kum17, §7], and hence (as in [Kum17, Equation 59])

$$\dim(\tilde{\mathcal{Z}}) = \dim(\bar{\Gamma}) + |\mathbf{j}| + l(w) - l(u) - l(v) \quad (23)$$

and $\tilde{\mathcal{Z}}$ is irreducible from the proof of [Kum17, Proposition 7.4].

We next show that \mathcal{Z}_P is of pure dimension. Since m is a flat morphism, $\text{Im}(m)$ is an open subset of $(X_w^2(P))_{\mathbb{P}}$ ([Har77, Chap. III, Exercise 9.1]). Moreover, clearly $\text{Im}(m) \supset (C_w^2(P))_{\mathbb{P}}$, thus $\text{Im}(m)$ intersects $\tilde{\Delta}((X_w^2(P))_{\mathbb{P}})$. Applying [Har77, Chap. III, Corollary 9.6] first to the morphism $m : \bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbf{j}} \rightarrow \text{Im}(m)$ and then its restriction μ to \mathcal{Z}_P , we see that \mathcal{Z}_P is pure dimensional.

We next show that \mathcal{Z}_P is irreducible. Since $\tilde{\mathcal{Z}}$ is irreducible, so is its open subset $\tilde{\mathcal{Z}} \cap \tilde{m}^{-1}((C_w^2(P))_{\mathbb{P}})$. Now $\tilde{\mathcal{Z}} \cap \tilde{m}^{-1}((C_w^2(P))_{\mathbb{P}})$ maps surjectively onto the open subset $\mathcal{Z}_P \cap m^{-1}((C_w^2(P))_{\mathbb{P}})$ of \mathcal{Z}_P under f and hence $\mathcal{Z}_1 := \overline{\mathcal{Z}_P \cap m^{-1}((C_w^2(P))_{\mathbb{P}})}$ is an irreducible component of \mathcal{Z}_P . Let \mathcal{Z}_2 be another irreducible component of \mathcal{Z}_P so that $\mu(\mathcal{Z}_2) \subset \tilde{\Delta}((X_w^P \setminus C_w^P)_{\mathbb{P}})$. But $\dim(\tilde{\Delta}((X_w^P \setminus C_w^P)_{\mathbb{P}})) < \dim(\tilde{\Delta}((X_w^P)_{\mathbb{P}}))$ and all of the fibers of $\mu|_{\mathcal{Z}_2}$ have dimension no more

than any fiber of μ , so $\dim(\mathcal{Z}_2) < \dim(\mathcal{Z}_1)$, which is a contradiction since \mathcal{Z}_P is of pure dimension. Thus $\mathcal{Z}_P = \mathcal{Z}_1$ is irreducible.

The map f is clearly proper and an isomorphism when restricted to the open subset

$$\tilde{\mathcal{Z}} \cap (\bar{\Gamma} \times ((C_P^u \cap C_w^P) \times (C_P^v \cap C_w^P))_{\mathbf{j}}),$$

where we identify the inverse image $(\pi_w^u)^{-1}(C_P^u \cap C_w^P)$ inside Z_w^u with $C_P^u \cap C_w^P$. Hence $\dim(\mathcal{Z}_P) = \dim(\tilde{\mathcal{Z}})$ and (22) holds by the equation (23). Therefore,

$$\text{codim}_{\mathcal{Z}_P}(\bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbf{j}}) = \text{codim}_{\tilde{\Delta}((X_w^P)_{\mathbb{P}})}((X_w^2(P))_{\mathbb{P}}) = l(w).$$

The fact that \mathcal{Z}_P is CM follows from the fact that m is flat as well as [Kum17, Lemma 7.2] and Lemma 7.4. \square

Lemma 8.2. *The scheme \mathcal{Z}_P is normal, irreducible, and CM.*

Proof. The only thing that remains to be shown is that \mathcal{Z}_P is normal. Consider the map

$$\mu_o : G \times^{U^-} X_P^u(S'_u) \rightarrow \bar{X}_P, \quad [g, x] \mapsto g \cdot x,$$

where $X_P^u(S'_u) := X_P^u \cap V^{S'_u}$, $V^{S'_u} := \bigcup_{w \in S'_u} B^- w x_o$, x_o is the base point $1.P$ of \bar{X}_P , and $S'_u := \{v \in W^P \mid l(v) \leq l(u) + 1\}$. Since μ_o is G -equivariant, it is a locally trivial fibration with fibers $F^u := \bigcup_{u \leq v; l(v) \leq l(u)+1} P v^{-1} U^- / U^-$. Since X_P^u is normal by Lemma 7.3 and any B^- orbit in $X_P^u(S'_u)$ has codimension ≤ 1 in X_P^u , $X_P^u(S'_u)$ is smooth and so is F^u . (Here the smoothness of F^u means that there exists a closed normal subgroup B_1 of B of finite codimension such that B_1 acts freely and properly on F^u and the quotient $B_1 \backslash F^u$ is a smooth scheme of finite type over \mathbb{C} .) Therefore μ_o is a smooth morphism. Hence so is the restriction of μ_o to the open subset $B \times X_P^u(S'_u)$, and further so is the restriction to the inverse image of X_w^P , $\mu_o(w) : B \times (X_P^u(S'_u) \cap X_w^P) \rightarrow X_w^P$. Clearly $\mu_o(w)$ factors through a smooth morphism $\bar{\mu}_o(w) : \bar{B} \times (X_P^u(S'_u) \cap X_w^P) \rightarrow X_w^P$, where \bar{B} is a finite-dimensional quotient of B . Hence, $\bar{\mu}_o(w)^{-1}(\hat{X}_w^P) = \bar{B} \times (X_P^u(S'_u) \cap (\hat{X}_w^P))$ is a smooth variety, where $\hat{X}_w^P := X_w^P \setminus \Sigma_w^P$ and Σ_w^P is the singular locus of X_w^P .

Following the same argument as in the proof of Lemma 7.1 and restricting the middle vertical map to the open subset $\bar{m} : \bar{\Gamma} \times ((X_P^u(S'_u) \times X_P^v(S'_v)) \cap X_w^2(P))_{\mathbf{j}} \rightarrow (X_w^2(P))_{\mathbb{P}}$ and the left vertical map accordingly, we have that \bar{m} is a smooth morphism (with open image Y). Hence, the restriction of \bar{m} to $\hat{m} : \bar{m}^{-1}(\tilde{\Delta}((X_w^P)_{\mathbb{P}})) \rightarrow \tilde{\Delta}((X_w^P)_{\mathbb{P}})$ is also smooth. (Observe that Y does intersect $\tilde{\Delta}((X_w^P)_{\mathbb{P}})$ since Y being a $\bar{\Gamma}$ -stable open subset of $(X_w^2(P))_{\mathbb{P}}$, $(C_w^2(P))_{\mathbb{P}} \subset Y$.) Therefore the open set $\hat{m}^{-1}(\tilde{\Delta}((\hat{X}_w^P)_{\mathbb{P}}))$ in $\mathcal{Z}_P = m^{-1}(\tilde{\Delta}((X_w^P)_{\mathbb{P}}))$ is a smooth variety. Denote the complement of $\bar{\Gamma} \times ((X_P^u(S'_u) \times X_P^v(S'_v)) \cap X_w^2(P))_{\mathbf{j}}$ in $\bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbf{j}}$ by F and denote $\hat{m}^{-1}(\tilde{\Delta}((\Sigma_w^P)_{\mathbb{P}}))$ by F' . Then F' has codimension ≥ 2 in $\bar{m}^{-1}(\tilde{\Delta}((X_w^P)_{\mathbb{P}}))$, and hence in \mathcal{Z}_P . Also, F is of codimension ≥ 2 in $\bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbf{j}}$. If F is nonempty, then the restriction of m to F is flat (following the proof of Lemma 7.1) with image an open subset of $(X_w^2(P))_{\mathbb{P}}$ intersecting $\tilde{\Delta}((X_w^P)_{\mathbb{P}})$. Therefore the codimension of $F \cap \mathcal{Z}_P$ in \mathcal{Z}_P is ≥ 2 and hence so is the complement of the smooth locus of \mathcal{Z}_P in \mathcal{Z}_P . Finally, since \mathcal{Z}_P is CM, it is normal by Serre's criterion [Har77, Ch. II, Theorem 8.22(A)]. \square

Proposition 8.3. *The scheme \mathcal{Z}_P has rational singularities.*

Proof. Since μ is flat and $\tilde{\Delta}((X_w^P)_{\mathbb{P}})$ has rational singularities [Kum02, Theorem 8.2.2(c)], it is enough to show (by [Elk78, Théorème 5]) that the fibers of μ are disjoint unions of irreducible varieties with rational singularities.

Let $x \in \tilde{\Delta}((C_{w'}^P)_{\mathbb{P}})$, where $w' \leq w \in W^P$. Then from [Kum17, Thoerem 7.6] and Lemma 4.6,

$$\text{Stab}(x) \setminus \mu^{-1}(x) \simeq ((X_P^u \cap C_{w'}^P) \times (X_P^v \cap C_{w'}^P))_{\mathbf{j}}, \quad (24)$$

where the stabilizer $\text{Stab}(x)$ is taken with respect to the $\bar{\Gamma}$ action on $(X_w^2(P))_{\mathbb{P}}$. From [Ser58, Proposition 3, §2.5], the map $\bar{\Gamma} \rightarrow \text{Stab}(x) \setminus \bar{\Gamma}$ is locally trivial in the étale topology.

We have the pull-back diagram

$$\begin{array}{ccc} \mu^{-1}(x) & \subseteq & \bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbf{j}} \\ \downarrow & & \downarrow \\ \text{Stab}(x) \setminus \mu^{-1}(x) & \subseteq & (\text{Stab}(x) \setminus \bar{\Gamma}) \times (X_w^{u,v}(P))_{\mathbf{j}}, \end{array}$$

where the left vertical map is a locally trivial fibration in the étale topology since so is the right side map. Therefore, since $\text{Stab}(x)$ is smooth and $\text{Stab}(x) \setminus \mu^{-1}(x)$ has rational singularities (by the equation (24) and Lemma 7.4), $\mu^{-1}(x)$ is a disjoint union of irreducible varieties with rational singularities by [KM08, Corollary 5.11]. \square

Proposition 8.4. *The scheme $\partial\mathcal{Z}_P$ is of pure codimension 1 in \mathcal{Z}_P and is CM, where we define the closed subscheme $\partial\mathcal{Z}_P$ of \mathcal{Z}_P as*

$$\partial\mathcal{Z}_P := (\bar{\Gamma} \times \partial((X_w^{u,v}(P))_{\mathbf{j}})) \times_{(X_w^2(P))_{\mathbb{P}}} \tilde{\Delta}((X_w^P)_{\mathbb{P}}),$$

where $\partial((X_w^{u,v}(P))_{\mathbf{j}})$ is as in Lemma 7.1.

Proof. First recall that $\bar{\Gamma} \times \partial((X_w^{u,v}(P))_{\mathbf{j}}) \xrightarrow{\hat{m}} (X_w^2(P))_{\mathbb{P}}$ is a flat morphism (cf. Lemma 7.1) and observe that $\partial((X_w^{u,v}(P))_{\mathbf{j}})$ is pure of codimension 1 in $(X_w^{u,v}(P))_{\mathbf{j}}$. We consider two cases: $\partial\mathbb{P}_{\mathbf{j}} \neq \emptyset$ or $\partial\mathbb{P}_{\mathbf{j}} = \emptyset$. In the first case, $\text{Im } \hat{m} = \text{Im } m$. In the latter case, we have

$$\text{Im } \hat{m} \supset \left(\left(\left(\bigcup_{u \rightarrow u' \leq \theta \leq w} C_{\theta}^P \right) \times \left(\bigcup_{v \leq \theta' \leq w} C_{\theta'}^P \right) \right) \cup \left(\left(\bigcup_{u \leq \theta \leq w} C_{\theta}^P \right) \times \left(\bigcup_{v \rightarrow v' \leq \theta' \leq w} C_{\theta'}^P \right) \right) \right)_{\mathbb{P}}.$$

In either case we have that, if nonempty, $\text{Im } \hat{m}$ intersects $\tilde{\Delta}((X_w^P)_{\mathbb{P}})$. Moreover, since \hat{m} is flat, $\text{Im } \hat{m}$ is open in $(X_w^2(P))_{\mathbb{P}}$. Thus, by [Har77, Ch. III, Corollary 9.6], each fiber of \hat{m} (if nonempty) is pure of dimension

$$\dim(\bar{\Gamma}) + \dim((X_w^{u,v}(P))_{\mathbf{j}}) - \dim((X_w^2(P))_{\mathbb{P}}) - 1.$$

Applying [Har77, Ch. III, Corollary 9.6] again to the restriction of \hat{m} to $\partial\mathcal{Z}_P$ via μ , we have that $\partial\mathcal{Z}_P$ is pure of dimension

$$\dim(\bar{\Gamma}) + \dim((X_w^{u,v}(P))_{\mathbf{j}}) - \dim((X_w^2(P))_{\mathbb{P}}) - 1 + \dim(\tilde{\Delta}((X_w^P)_{\mathbb{P}})).$$

Hence, from (22) it is clear that $\partial\mathcal{Z}_P$ is pure of codimension 1 in \mathcal{Z}_P . From Lemma 7.4 and the remark following Lemma 7.5, both of $((\partial X_P^u) \cap X_w^P) \times X_w^v(P)$ and $X_w^u(P) \times ((\partial X_P^v) \cap X_w^P)$ are CM. Hence so is their intersection, which is of pure codimension 1 in each of them. Thus their union is CM (cf. [Kum02, Theorem A.36]) and hence so is $((\partial X_P^{u,v}) \cap X_w^2(P))_{\mathbf{j}}$. We also have that $(X_w^{u,v}(P))_{\partial\mathbb{P}_{\mathbf{j}}}$ and the intersection

$$((\partial X_P^{u,v}) \cap X_w^2(P))_{\mathbf{j}} \cap (X_w^{u,v}(P))_{\partial\mathbb{P}_{\mathbf{j}}} = ((\partial X_P^{u,v}) \cap X_w^2(P))_{\partial\mathbb{P}_{\mathbf{j}}}$$

are CM since $\partial\mathbb{P}_{\mathbf{j}}$ is. Since $((\partial X_P^{u,v}) \cap X_w^2(P))_{\partial\mathbb{P}_{\mathbf{j}}}$ is CM of pure codimension 1 in each of $(X_w^{u,v}(P))_{\partial\mathbb{P}_{\mathbf{j}}}$ and $((\partial X_P^{u,v}) \cap X_w^2(P))_{\mathbf{j}}$, their union $\partial((X_w^{u,v}(P))_{\mathbf{j}})$ is CM. Finally, applying [Kum17, Lemma 7.2] to \hat{m} shows $\partial\mathcal{Z}_P$ is CM. \square

Corollary 8.5. *Assume that $c_{u,v}^w(\mathbf{j}) \neq 0$, where $c_{u,v}^w(\mathbf{j})$ is defined by the equation (7) (also see Lemma 4.3). Then for general $\gamma \in \bar{\Gamma}$, the fiber $N_\gamma := \pi^{-1}(\gamma) \subset \mathcal{Z}_P$ is CM of pure dimension, where $\pi : \mathcal{Z}_P \rightarrow \bar{\Gamma}$ is defined at the beginning of this section. In fact, for any $\gamma \in \bar{\Gamma}$ such that N_γ is pure of dimension*

$$\dim(N_\gamma) = \dim(\mathcal{Z}_P) - \dim(\bar{\Gamma}) = |\mathbf{j}| + l(w) - l(u) - l(v), \quad (25)$$

N_γ is CM (and this condition holds for general γ).

Similarly, if $|\mathbf{j}| + l(w) - l(u) - l(v) > 0$, then for general $\gamma \in \bar{\Gamma}$, the fiber $M_\gamma := \pi_1^{-1}(\gamma) \subset \partial\mathcal{Z}_P$ is CM of pure codimension 1 in N_γ , where π_1 is the restriction of π to $\partial\mathcal{Z}_P$. If $|\mathbf{j}| + l(w) - l(u) - l(v) = 0$, then for general $\gamma \in \bar{\Gamma}$, M_γ is empty.

In particular, for general $\gamma \in \bar{\Gamma}$,

$$\mathcal{E}xt_{\mathcal{O}_{N_\gamma}}^i(\mathcal{O}_{N_\gamma}(-M_\gamma), \omega_{N_\gamma}) = 0 \quad \text{for all } i > 0. \quad (26)$$

Proof. This follows the same argument as the proof of [Kum17, Corollary 7.9]. \square

9. STUDY OF $R^p f_*(\omega_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}}))$

Throughout this section we assume $c_{u,v}^w(\mathbf{j}) \neq 0$, where $c_{u,v}^w(\mathbf{j})$ is defined by the identity (7). We also follow the notation of the commutative diagram at the beginning of Section 8.

Lemma 9.1. *For $u \in W^P$, the line bundle $\mathcal{L}^P(\hat{\rho}_Y)|_{X_P^u}$ has a section with zero set precisely equal to ∂X_P^u . In particular,*

$$\mathcal{L}^P(\hat{\rho}_Y)|_{X_w^u(P)} \sim \sum_i b_i X_i \quad \text{for some } b_i > 0,$$

where the X_i are the irreducible components of $(\partial X_P^u) \cap X_w^P$.

Proof. Let $L(\hat{\rho}_Y)$ denote the integrable highest weight G^{\min} -module with highest weight $\hat{\rho}_Y$ and let $L(\hat{\rho}_Y)^\vee$ be its restricted dual, i.e., the direct sum of the dual of its weight spaces. Also, let \mathcal{L} be the tautological line bundle over $\mathbb{P}(L(\hat{\rho}_Y))$. Consider the linear map $\beta : L(\hat{\rho}_Y)^\vee \rightarrow H^0(\mathbb{P}(L(\hat{\rho}_Y)), \mathcal{L}^*)$, where $\beta(f)(x) = (x, f|_x)$ for $f \in L(\hat{\rho}_Y)^\vee$ and $x \in \mathbb{P}(L(\hat{\rho}_Y))$. Further, let $i^* : H^0(\mathbb{P}(L(\hat{\rho}_Y)), \mathcal{L}^*) \rightarrow H^0(\bar{X}_P, \mathcal{L}^P(\hat{\rho}_Y))$ be induced from $i : \bar{X}_P \rightarrow \mathbb{P}(L(\hat{\rho}_Y))$ taking $gP \mapsto [ge_{\hat{\rho}_Y}]$, where $e_{\hat{\rho}_Y}$ is a highest weight vector in $L(\hat{\rho}_Y)$. Set $\chi := i^* \circ \beta : L(\hat{\rho}_Y)^\vee \xrightarrow{\sim} H^0(\bar{X}_P, \mathcal{L}^P(\hat{\rho}_Y))$, which is the Borel-Weil isomorphism (see [Kum02, §8.1.21]) given by $\chi(f)(gP) = [g, f(ge_{\hat{\rho}_Y})]$.

Let $e_{u\hat{\rho}_Y}$ be the extremal weight vector of $L(\hat{\rho}_Y)$ with weight $u\hat{\rho}_Y$ and $e_{u\hat{\rho}_Y}^*$ the linear form which takes value 1 on $e_{u\hat{\rho}_Y}$ and 0 on any weight vector of weight different from $u\hat{\rho}_Y$. It is easy to see (from [Kum02, Lemma 8.3.3 and Proposition 1.4.2(a)]) that the section $\chi(e_{u\hat{\rho}_Y}^*)|_{X_P^u}$ has zero set exactly ∂X_P^u . This proves the lemma. \square

Recall that a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on an irreducible projective variety X is said to be *nef* (resp. *big*) if D has nonnegative intersection with every irreducible curve in X (resp. $\dim(H^0(X, \mathcal{O}_X(mD))) > cm^{\dim(X)}$ for some $c > 0$ and $m \gg 1$). Note that if D is ample, then it is *nef* and *big* (cf. [KM08, Proposition 2.61]).

For a proper morphism $\pi : X \rightarrow Y$ between schemes and a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on X , D is said to be π -*nef* (resp. π -*big*) if D has nonnegative intersection with every irreducible curve

in X contracted by π (resp. $\pi_*\mathcal{O}_X(mD) > cm^n$ for some $c > 0$ and $m \gg 1$, where n is the dimension of a general fiber of π).

Proposition 9.2. *There exists a nef and big line bundle \mathcal{M} on $(Z_w^{u,v})_{\mathbf{j}}$ with a section with support $\partial((Z_w^{u,v})_{\mathbf{j}})$, where $\partial((Z_w^{u,v})_{\mathbf{j}})$ is defined to be the inverse image of $\partial((X_w^{u,v}(P))_{\mathbf{j}})$ under the canonical map $(Z_w^{u,v})_{\mathbf{j}} \rightarrow (X_w^{u,v}(P))_{\mathbf{j}}$ induced by the T -equivariant map $\pi_w^{u,v} : Z_w^{u,v} \rightarrow X_w^{u,v}(P)$. Moreover, \mathcal{M} can be chosen to be the pull-back of an ample line bundle \mathcal{M}' on $(X_w^{u,v}(P))_{\mathbf{j}}$.*

Proof. Let \mathcal{H} be an ample line bundle on $\mathbb{P}_{\mathbf{j}}$ with a section with support precisely $\partial\mathbb{P}_{\mathbf{j}}$. Let $\mathcal{L}_{Z_w^{u,v}}^P(\hat{\rho}_Y \boxtimes \hat{\rho}_Y)$ be the pull-back of the line bundle $\mathcal{L}^P(\hat{\rho}_Y) \boxtimes \mathcal{L}^P(\hat{\rho}_Y)$ on $\bar{X}_P \times \bar{X}_P$ via the morphism $Z_w^{u,v} \rightarrow \bar{X}_P \times \bar{X}_P$. The line bundle $e^{u\hat{\rho}_Y + v\hat{\rho}_Y} \mathcal{L}_{Z_w^{u,v}}^P(\hat{\rho}_Y \boxtimes \hat{\rho}_Y)$ is T -equivariant, hence we have the line bundle

$$\tilde{\mathcal{L}}_{Z_w^{u,v}}^P(-\hat{\rho}_Y \boxtimes -\hat{\rho}_Y) := E(T)_{\mathbf{j}} \times^T e^{u\hat{\rho}_Y + v\hat{\rho}_Y} \mathcal{L}_{Z_w^{u,v}}^P(\hat{\rho}_Y \boxtimes \hat{\rho}_Y) \rightarrow (Z_w^{u,v})_{\mathbf{j}}.$$

Set $\mathcal{M} := \tilde{\mathcal{L}}_{Z_w^{u,v}}^P(-\hat{\rho}_Y \boxtimes -\hat{\rho}_Y) \otimes \pi^*(\mathcal{H}^N)$, where π is the standard quotient map $E(T)_{\mathbf{j}} \times^T Z_w^{u,v} \rightarrow \mathbb{P}_{\mathbf{j}}$ and $N \gg 0$. Take $\theta : (Z_w^{u,v})_{\mathbf{j}} \rightarrow \tilde{\mathcal{L}}_{Z_w^{u,v}}^P(-\hat{\rho}_Y \boxtimes -\hat{\rho}_Y)$ to be the section given by

$$[e, z] \mapsto [e, 1_{u\hat{\rho}_Y + v\hat{\rho}_Y} \otimes (\bar{\chi}(e_{u\hat{\rho}_Y}^*) \boxtimes \bar{\chi}(e_{v\hat{\rho}_Y}^*))(z)],$$

where $e \in E(T)_{\mathbf{j}}$, $z \in Z_w^{u,v}$, $1_{u\hat{\rho}_Y + v\hat{\rho}_Y}$ is the constant section of the trivial line bundle over $Z_w^{u,v}$ with H action given by $u\hat{\rho}_Y + v\hat{\rho}_Y$, and $\bar{\chi} \boxtimes \bar{\chi}$ is the pull-back of the Borel-Weil isomorphism as in the proof of Lemma 9.1. Let σ be any section of \mathcal{H}^N with support precisely $\partial\mathbb{P}_{\mathbf{j}}$ and $\hat{\sigma}$ its pull-back to $(Z_w^{u,v})_{\mathbf{j}}$. It is easy to see that the support of $\theta \otimes \hat{\sigma}$ is precisely $\partial((Z_w^{u,v})_{\mathbf{j}})$.

Moreover, the line bundle \mathcal{M} is the pull-back of the line bundle $\mathcal{M}' := \tilde{\mathcal{L}}'(-\hat{\rho}_Y \boxtimes -\hat{\rho}_Y) \otimes \pi_1^*(\mathcal{H}^N)$ on $E(T)_{\mathbf{j}} \times^T X_w^{u,v}(P)$ via the morphism

$$E(T)_{\mathbf{j}} \times^T Z_w^{u,v} \rightarrow E(T)_{\mathbf{j}} \times^T X_w^{u,v}(P),$$

where π_1 is the standard quotient map $E(T)_{\mathbf{j}} \times^T X_w^{u,v}(P) \rightarrow \mathbb{P}_{\mathbf{j}}$ and

$$\tilde{\mathcal{L}}'(-\hat{\rho}_Y \boxtimes -\hat{\rho}_Y) := E(T)_{\mathbf{j}} \times^T \left(e^{u\hat{\rho}_Y + v\hat{\rho}_Y} (\mathcal{L}^P(\hat{\rho}_Y) \boxtimes \mathcal{L}^P(\hat{\rho}_Y))|_{X_w^{u,v}(P)} \right).$$

From [KM08, Proposition 1.45 and Theorems 1.37 and 1.42], \mathcal{M}' is ample on $(X_w^{u,v}(P))_{\mathbf{j}}$ for large enough $N > 0$. Since π is a birational morphism and \mathcal{M}' is ample, \mathcal{M} is nef and big by [Deb01, §1.29]. \square

Proposition 9.3. *For $\tilde{\pi} : \tilde{\mathcal{Z}} \rightarrow \bar{\Gamma}$,*

$$R^p \tilde{\pi}_*(\omega_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}})) = 0 \quad \text{for all } p > 0,$$

where $\partial\tilde{\mathcal{Z}} := f^{-1}(\partial\mathcal{Z}_P)$ and $\omega_{\tilde{\mathcal{Z}}}(\partial\tilde{\mathcal{Z}}) := \mathcal{H}om_{\mathcal{O}_{\tilde{\mathcal{Z}}}}(\mathcal{O}_{\tilde{\mathcal{Z}}}(-\partial\tilde{\mathcal{Z}}), \omega_{\tilde{\mathcal{Z}}})$. Here we take $\partial\mathcal{Z}_P$ with the reduced scheme structure.

(Observe that f being a desingularization of a normal scheme \mathcal{Z}_P and $\partial\mathcal{Z}_P$ being reduced, $\partial\tilde{\mathcal{Z}}$ is also a reduced scheme.)

Proof. As guaranteed by Proposition 9.2, let \mathcal{M} be a nef and big line bundle on $(Z_w^{u,v})_{\mathbf{j}}$ with divisor $\sum_{i=1}^d b_i Z_i$ (where $b_i > 0$ for all i) supported precisely in $\partial((Z_w^{u,v})_{\mathbf{j}})$. Further, \mathcal{M} can be taken to be the pull-back of an ample line bundle \mathcal{M}' on $(X_w^{u,v}(P))_{\mathbf{j}}$. Let N be an integer so that $N > b_i$ for all i .

By the proof of Proposition 9.2, the line bundle $\epsilon \boxtimes \mathcal{M}'$ over $\bar{\Gamma} \times (X_w^{u,v}(P))_{\mathbb{J}}$ restricted to \mathcal{Z}_P via i has a section with support $\partial \mathcal{Z}_P$, where $\partial \mathcal{Z}_P$ is defined in Proposition 8.4 and ϵ is the trivial line bundle over $\bar{\Gamma}$. Hence, the pull-back of the line bundle $f^*(i^*(\epsilon \boxtimes \mathcal{M}'))$ over $\tilde{\mathcal{Z}}$ has a section with support $f^{-1}(\partial \mathcal{Z}_P) = \partial \tilde{\mathcal{Z}}$. Since $\partial \tilde{\mathcal{Z}}$ is the zero set of a line bundle on $\tilde{\mathcal{Z}}$, $\partial \tilde{\mathcal{Z}}$ is a pure scheme of codimension 1 in $\tilde{\mathcal{Z}}$. Let \mathcal{L} be the line bundle on the smooth scheme $\tilde{\mathcal{Z}}$ associated to the reduced divisor $\partial \tilde{\mathcal{Z}}$ and let D be the divisor $\sum_i (N - b_i) \tilde{Z}_i$ on $\tilde{\mathcal{Z}}$, where

$$\tilde{Z}_i := (\bar{\Gamma} \times Z_i) \times_{(Z_w^2)_{\mathbb{P}}} \tilde{\Delta}((Z_w)_{\mathbb{P}}).$$

Then, as proved in [Kum17, Proof of Proposition 8.4], each \tilde{Z}_i is a smooth irreducible divisor of $\tilde{\mathcal{Z}}$, and for any collection $\tilde{Z}_{i_1}, \dots, \tilde{Z}_{i_q}$, $1 \leq i_1 < \dots < i_q \leq d$, the intersection $\bigcap_{j=1}^q \tilde{Z}_{i_j}$ (if non-empty) is smooth and of pure codimension q in $\tilde{\mathcal{Z}}$. (Observe that $\tilde{\mathcal{Z}}$ and \tilde{Z}_i here coincide with the same in [Kum17, Proposition 8.4].) Hence, the \tilde{Z}_i 's are all distinct and $\partial \tilde{\mathcal{Z}} = \sum \tilde{Z}_i$ is a simple normal crossing divisor. Clearly,

$$\mathcal{L}^N(-D) = \mathcal{O}_{\tilde{\mathcal{Z}}} \left(\sum_i b_i \tilde{Z}_i \right) \simeq \tilde{i}^* \left(\mathcal{O}_{\bar{\Gamma} \times (Z_w^{u,v})_{\mathbb{J}}} \left(\sum_i b_i (\bar{\Gamma} \times Z_i) \right) \right).$$

By [Deb01, §1.6], since $\sum_i b_i Z_i$ is a nef divisor (by assumption), and since \tilde{i} is injective, $\mathcal{L}^N(-D)$ is $\tilde{\pi}$ -nef.

We next show that $\mathcal{L}^N(-D)$ is $\tilde{\pi}$ -big. Since \mathcal{M} was chosen to be the pull-back of an ample line bundle \mathcal{M}' on $(X_w^{u,v}(P))_{\mathbb{J}}$, we have that $\mathcal{L}^N(-D)$ is the pull-back of the line bundle $\mathcal{S} := i^*(\epsilon \boxtimes \mathcal{M}')$ on \mathcal{Z}_P via f . But \mathcal{M}' being ample implies \mathcal{S} is π -big. Further, since f is birational, the fibers of $\tilde{\pi}$ for general γ have the same dimension as the fibers of π (use [Sha13, Chap. I, §6.3, Theorem 1.25]). Hence $\mathcal{L}^N(-D)$ is $\tilde{\pi}$ -big.

Since f is proper and birational it is surjective. We also have that π is surjective (proof of [Kum17, Corollary 7.9]), hence so is $\tilde{\pi}$. Now apply [Kum17, Theorem 8.3]. \square

Theorem 9.4. *For the morphism $f : \tilde{\mathcal{Z}} \rightarrow \mathcal{Z}_P$,*

- (a) $R^p f_*(\omega_{\tilde{\mathcal{Z}}}(\partial \tilde{\mathcal{Z}})) = 0$ for all $p > 0$, and
- (b) $f_*(\omega_{\tilde{\mathcal{Z}}}(\partial \tilde{\mathcal{Z}})) = \omega_{\mathcal{Z}_P}(\partial \mathcal{Z}_P)$, where $\omega_{\mathcal{Z}_P}(\partial \mathcal{Z}_P) := \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}_P}}(\mathcal{O}_{\mathcal{Z}_P}(-\partial \mathcal{Z}_P), \omega_{\mathcal{Z}_P})$.

Proof. For (a), note that f is surjective since it is proper and birational. As in the previous proposition, $\mathcal{L}^N(-D)$ is $\tilde{\pi}$ -nef and $\tilde{\pi}$ -big. Now the fibers of $\tilde{\pi}$ contain the fibers of f , so that $\mathcal{L}^N(-D)$ is also f -nef. Since f is birational, we also have that $\mathcal{L}^N(-D)$ is f -big. Then (a) follows from [Kum17, Theorem 8.3].

(b) Since $\partial \tilde{\mathcal{Z}} = f^{-1}(\partial \mathcal{Z}_P)$ is the scheme-theoretic inverse image with the reduced scheme structure on $\partial \mathcal{Z}_P$, the morphism

$$f^*(\mathcal{O}_{\mathcal{Z}_P}(-\partial \mathcal{Z}_P)) \rightarrow \mathcal{O}_{\tilde{\mathcal{Z}}}(-\partial \tilde{\mathcal{Z}})$$

is surjective (cf. [sta, Tag 01HJ, Lemma 25.4.7]) with kernel supported on a proper closed subset of $\tilde{\mathcal{Z}}$ (since f is a desingularization). Thus the kernel is a torsion sheaf and so the dual map

$$\mathcal{O}_{\tilde{\mathcal{Z}}}(\partial \tilde{\mathcal{Z}}) \rightarrow \mathcal{H}om_{\mathcal{O}_{\tilde{\mathcal{Z}}}}(f^*(\mathcal{O}_{\mathcal{Z}_P}(-\partial \mathcal{Z}_P)), \mathcal{O}_{\tilde{\mathcal{Z}}}) \quad (27)$$

is an isomorphism, where $\mathcal{O}_{\tilde{\mathcal{Z}}}(\partial \tilde{\mathcal{Z}}) := \mathcal{H}om_{\mathcal{O}_{\tilde{\mathcal{Z}}}}(\mathcal{O}_{\tilde{\mathcal{Z}}}(-\partial \tilde{\mathcal{Z}}), \mathcal{O}_{\tilde{\mathcal{Z}}})$.

Finally,

$$\begin{aligned}
f_*(\omega_{\tilde{Z}}(\partial\tilde{Z})) &= f_*(\omega_{\tilde{Z}} \otimes \mathcal{H}om_{\mathcal{O}_{\tilde{Z}}}(\mathcal{O}_{\tilde{Z}}(-\partial\tilde{Z}), \mathcal{O}_{\tilde{Z}})) \\
&= f_*(\omega_{\tilde{Z}} \otimes \mathcal{H}om_{\mathcal{O}_{\tilde{Z}}}(f^*(\mathcal{O}_{\mathcal{Z}_P}(-\partial\mathcal{Z}_P)), \mathcal{O}_{\tilde{Z}})) \quad \text{by (27)} \\
&= f_*\mathcal{H}om_{\mathcal{O}_{\tilde{Z}}}(f^*(\mathcal{O}_{\mathcal{Z}_P}(-\partial\mathcal{Z}_P)), \omega_{\tilde{Z}}) \\
&= \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}_P}}(\mathcal{O}_{\mathcal{Z}_P}(-\partial\mathcal{Z}_P), f_*\omega_{\tilde{Z}}) \quad \text{by adjunction [Har77, Ch. II, §5]} \\
&= \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}_P}}(\mathcal{O}_{\mathcal{Z}_P}(-\partial\mathcal{Z}_P), \omega_{\mathcal{Z}_P}) \quad \text{by Proposition 8.3 and [KM08, Theorem 5.10]} \\
&= \omega_{\mathcal{Z}_P}(\partial\mathcal{Z}_P). \quad \square
\end{aligned}$$

The following is an immediate consequence of Proposition 9.3, Theorem 9.4 and the Grothendieck spectral sequence [Jan03, Part I, Proposition 4.1] applied to $\tilde{\pi} = \pi \circ f$.

Corollary 9.5. *The morphism $\pi : \mathcal{Z}_P \rightarrow \bar{\Gamma}$ from the diagram satisfies*

$$R^p\pi_*(\omega_{\mathcal{Z}_P}(\partial\mathcal{Z}_P)) = 0 \quad \text{for all } p > 0.$$

10. PROOF OF PART (B) OF THEOREM 5.1

From Proposition 3.6, a similar argument to the proof of part (A) of Theorem 5.1 gives the vanishing

$$\mathcal{F}or_1^{\mathcal{O}(\bar{Y}_P)\mathbb{P}}(\gamma_*\tilde{\Delta}_*\mathcal{O}_{(X_w^P)\mathbb{P}}, \mathcal{O}_{\partial((X_P^{u,v})_{\mathbf{j}})}) = 0 \quad \text{for general } \gamma \in \bar{\Gamma}.$$

This together with the definition that $\xi_P^u := \mathcal{O}_{X_P^u}(-\partial X_P^u)$ implies that part (B) of Theorem 5.1 is equivalent to the following

Theorem 10.1. *Assume $c_{u,v}^w(\mathbf{j}) \neq 0$. For general $\gamma \in \bar{\Gamma}$,*

$$H^p(\bar{N}_\gamma, \mathcal{O}_{\bar{N}_\gamma}(-\bar{M}_\gamma)) = 0 \quad \text{for all } p \neq |\mathbf{j}| + l(w) - l(u) - l(v),$$

where $\bar{M}_\gamma := M_{\gamma^{-1}}$ is the subscheme $(\partial((X_P^{u,v})_{\mathbf{j}})) \cap \gamma\tilde{\Delta}((X_w^P)\mathbb{P})$ and $\mathcal{O}_{\bar{N}_\gamma}(-\bar{M}_\gamma)$ is the ideal sheaf of \bar{M}_γ in $\bar{N}_\gamma := (X_P^{u,v})_{\mathbf{j}} \cap \gamma\tilde{\Delta}((X_w^P)\mathbb{P})$.

Proof. Since \mathcal{Z}_P and $\partial\mathcal{Z}_P$ are CM and $\partial\mathcal{Z}_P$ is of codimension 1 in \mathcal{Z}_P (cf. Propositions 8.1 and 8.4), we have

$$\mathcal{E}xt_{\mathcal{O}_{\mathcal{Z}_P}}^i(\mathcal{O}_{\mathcal{Z}_P}(-\partial\mathcal{Z}_P), \omega_{\mathcal{Z}_P}) = 0 \quad \text{for all } i \geq 1.$$

Similarly, by the identity (26), we also have for general $\gamma \in \bar{\Gamma}$,

$$\mathcal{E}xt_{\mathcal{O}_{\bar{N}_\gamma}}^i(\mathcal{O}_{\bar{N}_\gamma}(-\bar{M}_\gamma), \omega_{\bar{N}_\gamma}) = 0 \quad \text{for all } i \geq 1.$$

By the Serre duality [Har77, Ch. III, Theorem 7.6] applied to \bar{N}_γ and the local-to-global Ext spectral sequence [God58, Ch. II, Théorème 7.3.3], the theorem is equivalent to the vanishing

$$H^p(\bar{N}_\gamma, \mathcal{H}om_{\mathcal{O}_{\bar{N}_\gamma}}(\mathcal{O}_{\bar{N}_\gamma}(-\bar{M}_\gamma), \omega_{\bar{N}_\gamma})) = 0 \quad \text{for all } p > 0, \quad (28)$$

since \bar{N}_γ is CM (for general $\gamma \in \bar{\Gamma}$) and $\dim(\bar{N}_\gamma) = |\mathbf{j}| + l(w) - l(u) - l(v)$ (cf. Corollary 8.5).

Let $\omega_{\bar{N}_\gamma}(\bar{M}_\gamma) := \mathcal{H}om_{\mathcal{O}_{\bar{N}_\gamma}}(\mathcal{O}_{\bar{N}_\gamma}(-\bar{M}_\gamma), \omega_{\bar{N}_\gamma})$. Then for general $\gamma \in \bar{\Gamma}$,

$$\omega_{\mathcal{Z}_P}(\partial\mathcal{Z}_P)|_{\pi^{-1}(\gamma^{-1})} \simeq \omega_{\pi^{-1}(\gamma^{-1})}(\partial\mathcal{Z}_P \cap \pi^{-1}(\gamma^{-1})) = \omega_{\bar{N}_\gamma}(\bar{M}_\gamma). \quad (29)$$

To prove this, observe that by [Sha13, Ch. I, §6.3, Theorem 1.25] and [Har77, Ch. III, Exercise 10.9], there exists an open nonempty subset $\bar{\Gamma}_o \subset \bar{\Gamma}$ such that $\pi : \pi^{-1}(\bar{\Gamma}_o) \rightarrow \bar{\Gamma}_o$ is a flat morphism. Since $\bar{\Gamma}_o$ is smooth and \mathcal{Z}_P and $\partial\mathcal{Z}_P$ are CM, and since the assertion is local in $\bar{\Gamma}$, it is enough to observe (see [ILL⁺07, Corollary 11.35]) that for a nonzero function θ on $\bar{\Gamma}_o$, there is an isomorphism of $\mathcal{O}_{\mathcal{Z}_P^\theta}$ -modules

$$\mathcal{S}/\theta \cdot \mathcal{S} \simeq \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}_P^\theta}}(\mathcal{O}_{\mathcal{Z}_P}(-\partial\mathcal{Z}_P)/\theta \cdot \mathcal{O}_{\mathcal{Z}_P}(-\partial\mathcal{Z}_P), \omega_{\mathcal{Z}_P^\theta}),$$

where \mathcal{Z}_P^θ is the zero scheme of θ in \mathcal{Z}_P and $\mathcal{S} := \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}_P}}(\mathcal{O}_{\mathcal{Z}_P}(-\partial\mathcal{Z}_P), \omega_{\mathcal{Z}_P})$. Taking θ in a local coordinate system, we can continue and get (29).

Now $R^p\pi_*(\omega_{\mathcal{Z}_P}(\partial\mathcal{Z}_P)) = 0$ for $p > 0$ (cf. Corollary 9.5) implies that for general $\gamma \in \bar{\Gamma}$, (28) holds. This follows from the fact that since \mathcal{Z}_P and $\partial\mathcal{Z}_P$ are CM, $\bar{\Gamma}_o$ is smooth, and $\pi : \pi^{-1}(\bar{\Gamma}_o) \rightarrow \bar{\Gamma}_o$ is flat, we have $\omega_{\mathcal{Z}_P}(\partial\mathcal{Z}_P)$ is flat over $\bar{\Gamma}_o$ (see the proof of [Kum17, Theorem 9.1]). Therefore (28) follows from the semicontinuity theorem ([Har77, Ch. III, Theorem 12.8 and Corollary 12.9]). Hence the theorem is proven and therefore so is part (B) of Theorem 5.1. Thus, Corollary 5.2 is proved. \square

11. APPENDIX

The aim of this appendix is to determine the dualizing sheaf $\omega_{X_P^w}$ of the CM scheme $X_P^w \subset \bar{X}_P$ for any $w \in W^P$ (in the thick Kac-Moody flag variety). Even though this result is not used in the paper, we believe that it is interesting on its own for potentially its future use. When $P = B$, this result was proved by Kashiwara (cf. [Kum17, Theorem 10.4]). Moreover, in the finite case (i.e., when G is a semisimple group), this result for any P is obtained in [KRW20, Theorem 3.3].

From [Jan03, Part II, §4.2], the dualizing sheaf of \bar{X}_P in the finite case is given by

$$\omega_{\bar{X}_P} = \mathcal{L}^P(-2\rho + 2\rho^Y),$$

where ρ^Y denotes the half sum of positive roots coming from the Levi component of P .

Given any Cohen-Macaulay subscheme Y of a smooth variety X , its dualizing sheaf ω_Y is given by

$$\omega_Y \simeq \mathcal{E}xt_{\mathcal{O}_X}^{\text{codim } Y}(\mathcal{O}_Y, \mathcal{O}_X) \otimes \omega_X.$$

In particular, by Lemma 7.3, following the analogy with the schemes of finite type, we define the dualizing sheaf of X_P^w by

$$\omega_{X_P^w} := \mathcal{E}xt_{\mathcal{O}_{\bar{X}_P}}^{l(w)}(\mathcal{O}_{X_P^w}, \mathcal{O}_{\bar{X}_P}) \otimes \mathcal{L}^P(-2\rho + 2\rho^Y).$$

Recall the definitions of $\rho_Y = \sum_{i \in Y} \varpi_i$ and $\hat{\rho}_Y = \rho - \rho_Y$ from Section 2. For any $w \in W^P$, let

$$\begin{aligned} \hat{\xi}_P^w &:= \mathbb{C}_{-\rho + w\rho_Y} \otimes \omega_{X_P^w} \otimes \mathcal{L}^P(\rho + \rho_Y - 2\rho^Y) \\ &= \mathbb{C}_{-\rho + w\rho_Y} \otimes \mathcal{E}xt_{\mathcal{O}_{\bar{X}_P}}^{l(w)}(\mathcal{O}_{X_P^w}, \mathcal{O}_{\bar{X}_P}) \otimes \mathcal{L}^P(-\hat{\rho}_Y). \end{aligned}$$

Lemma 11.1. *Restricted to the open cell $C_P^w \subset X_P^w$, we have a B^- -equivariant isomorphism*

$$(\hat{\xi}_P^w)|_{C_P^w} \simeq (\mathcal{O}_{X_P^w})|_{C_P^w}.$$

Proof. Because of the B^- -equivariance of $\hat{\xi}_P^w$, it suffices to show that $i_w^* \hat{\xi}_P^w$ is trivial as a T -module, where $i_w : \{pt\} \rightarrow \bar{X}_P$ is the map sending the point to the fixed point w . First, note that for any character λ of P , $i_w^* \mathcal{L}^P(\lambda) = \mathbb{C}_{-w\lambda}$. Further,

$$i_w^* \left(\mathcal{E}xt_{\mathcal{O}_{\bar{X}_P}}^{l(w)}(\mathcal{O}_{X_P^w}, \mathcal{O}_{\bar{X}_P}) \right) \simeq \det \left(\frac{T_w(\bar{X}_P)}{T_w(X_P^w)} \right) \simeq \mathbb{C}_{\rho - w\rho}.$$

The last equality follows from the computation of the tangent spaces:

$$\begin{aligned} T_w(X_P^w) &= T_w(B^-wP/P) = T_w(w(w^{-1}B^-w \cap U_P^-)P/P) = \bigoplus_{\beta \in \Delta^- \cap w(\Delta^- \setminus \Delta_{\bar{Y}}^-)} \mathfrak{g}_\beta \\ T_w(\bar{X}_P) &= \bigoplus_{\beta \in w(\Delta^- \setminus \Delta_{\bar{Y}}^-)} \mathfrak{g}_\beta, \end{aligned}$$

where Δ^- is the set of negative roots, $\Delta_{\bar{Y}}^-$ the set of negative roots of the Levi component of P , and U_P^- is the unipotent radical of the opposite parabolic of P . Then by [Kum02, Corollary 1.3.22],

$$\det \left(\frac{T_w(\bar{X}_P)}{T_w(X_P^w)} \right) = \det \left(\bigoplus_{\beta \in \Delta^+ \cap w(\Delta^- \setminus \Delta_{\bar{Y}}^-)} \mathfrak{g}_\beta \right) = \det \left(\bigoplus_{\beta \in \Delta^+ \cap w\Delta^-} \mathfrak{g}_\beta \right) = \mathbb{C}_{\rho-w\rho}, \text{ since } w \in W^P.$$

Therefore the conclusion of the lemma follows since the weight of $i_w^* \hat{\xi}_P^w$ is given by

$$-\rho + w\rho_Y + (\rho - w\rho) - w(-\hat{\rho}_Y) = w\rho_Y - w\rho + w\rho - w\rho_Y = 0. \quad \square$$

Let $V_P^w := C_P^w \cup \bigcup_{v \leftarrow w} C_P^v$, where the notation $v \leftarrow w$ indicates that $l(w) = l(v) - 1$ and $v > w$. Then, V_P^w is a smooth open subset of X_P^w and $\hat{\xi}_P^w|_{V_P^w}$ is an invertible B^- -equivariant $\mathcal{O}_{V_P^w}$ -module. By Lemma 11.1,

$$\hat{\xi}_P^w|_{V_P^w} \simeq \mathcal{O}_{X_P^w} \left(- \sum_{v \leftarrow w} m_{w,v}^P X_P^v \right) |_{V_P^w}, \quad \text{for some } m_{w,v}^P \in \mathbb{Z}.$$

Lemma 11.2. *The coefficients $m_{w,v}^P$ are given by the formula*

$$m_{w,v}^P = 1 - \langle w\rho_Y, \beta^\vee \rangle,$$

where β is the positive root so that $s_\beta w = v$.

Proof. Let $v \in W^P$ be such that $v \leftarrow w$. We first compute

$$\begin{aligned} i_v^* \left(\mathcal{E}xt_{\mathcal{O}_{\bar{X}_P}}^{l(w)} (\mathcal{O}_{X_P^w}, \mathcal{O}_{\bar{X}_P}) \right) &\simeq \det \left(\frac{T_v(\bar{X}_P)}{T_v(X_P^w)} \right) \\ &\simeq \det \left(\frac{T_v(\bar{X}_P)}{T_v(X_P^v)} \right) \otimes \det \left(\frac{T_v(X_P^w)}{T_v(X_P^v)} \right)^* \\ &\simeq \mathbb{C}_{\rho-v\rho-\beta}, \text{ by [Kum17, Lemma 10.3].} \end{aligned}$$

Hence,

$$\begin{aligned} i_v^* \hat{\xi}_P^w &\simeq \mathbb{C}_{-\rho+w\rho_Y} \otimes \mathbb{C}_{\rho-v\rho-\beta} \otimes \mathbb{C}_{v\hat{\rho}_Y} \\ &= \mathbb{C}_{w\rho_Y - \beta - v\rho_Y} \\ &= \mathbb{C}_{(\langle w\rho_Y, \beta^\vee \rangle - 1)\beta}. \end{aligned} \quad (30)$$

On the other hand

$$\begin{aligned} i_v^* \left(\mathcal{O}_{X_P^w} \left(- \sum_{u \leftarrow w} m_{w,u}^P X_P^u \right) \right) &= \det \left(\frac{T_v(X_P^w)}{T_v(X_P^v)} \right)^{\otimes -m_{w,v}^P} \\ &= \mathbb{C}_\beta^{\otimes -m_{w,v}^P} \\ &= \mathbb{C}_{-m_{w,v}^P \beta}. \end{aligned} \quad (31)$$

Equating the equations (30) and (31), we have the desired result. \square

Theorem 11.3. For any $w \in W^P$, we have a B^- -equivariant isomorphism

$$\hat{\xi}_P^w \simeq \mathcal{O}_{X_P^w} \left(- \sum_{v \leftarrow w} m_{w,v}^P X_P^v \right),$$

where $m_{w,v}^P$ are as in Lemma 11.2. Therefore, the dualizing sheaf of X_P^w is T -equivariantly isomorphic to

$$\mathbb{C}_{\rho - w\rho_Y} \otimes \mathcal{O}_{X_P^w} \left(- \sum_{v \leftarrow w} m_{w,v}^P X_P^v \right) \otimes \mathcal{L}^P(2\rho^Y - \rho - \rho_Y).$$

Proof. Let $j : V_P^w \hookrightarrow X_P^w$ be the inclusion map and set $D := \sum_{v \leftarrow w} m_{w,v}^P X_P^v$. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{X_P^w}(-D) & \longrightarrow & \mathcal{O}_{X_P^w} & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \simeq & & \downarrow & & \\ 0 & \longrightarrow & j_*j^{-1}(\mathcal{O}_{X_P^w}(-D)) & \longrightarrow & j_*j^{-1}(\mathcal{O}_{X_P^w}) & \longrightarrow & j_*j^{-1}\mathcal{O}_D & & \end{array}$$

Here the middle vertical arrow is an isomorphism since X_P^w is normal (cf. Lemma 7.3) and $X_P^w \setminus V_P^w$ is of codimension at least 2 in X_P^w . The right vertical map is injective since $\overline{\text{supp}(D)} \cap V_P^w = \text{supp}(D)$. This implies that the leftmost vertical map is an isomorphism. We also have that $\hat{\xi}_P^w \simeq j_*j^{-1}(\hat{\xi}_P^w)$ since $\hat{\xi}_P^w$ is a CM $\mathcal{O}_{X_P^w}$ -module. Moreover, by Lemmas 7.3 and 11.2,

$$\hat{\xi}_P^w \simeq j_*j^{-1}(\hat{\xi}_P^w) \simeq j_*j^{-1}\mathcal{O}_{X_P^w}(-D) \simeq \mathcal{O}_{X_P^w}(-D). \quad \square$$

For $P = B$, clearly $\rho_Y = 0$ and the coefficients $m_{w,v}^P = 1 - \langle w\rho_Y, \beta^\vee \rangle = 1$. This gives the following theorem of Kashiwara ([Kum17, Theorem 10.4])

Theorem 11.4. For any $w \in W$, we have a B^- -equivariant isomorphism

$$\xi^w \simeq \mathcal{O}_{X^w}(-\partial X^w),$$

where $\xi^w := \mathbb{C}_{-\rho} \otimes \mathcal{L}(-\rho) \otimes \mathcal{E}xt_{\mathcal{O}_{\bar{X}}}^{l(w)}(\mathcal{O}_{X^w}, \mathcal{O}_{\bar{X}})$.

Example 11.5. (a) For the affine Kac-Moody group $G = \widehat{G}_o$, where G_o is a simple simply-connected complex algebraic group, taking the standard maximal parabolic subgroup P , we see that if we take $w = e, v = s_0$, then $m_{w,v}^P = 1$.

(b) Let us take $G = \widehat{SL}_2$ and P the standard maximal parabolic subgroup (as above). Then, for $w = s_0, v = s_1s_0$, $m_{w,v}^P = 0$. If we take $w = s_1s_0, v = s_0s_1s_0$, then $m_{w,v}^P = -1$.

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