Supplemental material for "*A bootstrapped test of covariance stationarity based on orthonormal transformations*"

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A. Introduction

We list the assumptions for reference below. Appendix B contains omitted proofs and additional results. We provide an empirical study in Appendix C, and Appendix D contains all simulation results.

We use the following notation. $[z]$ rounds z to the nearest integer.. L_2 is the space of square integrable random variables; $L_2[a, b)$ is the class of square integrable functions on $[a, b)$. $|| \cdot ||_p$ and $|| \cdot ||$ are the L_p and l_2 norms respectively, $p \geq 1$. Let $\mathbb{Z} \equiv \{... - 2, -1, 0, 1, 2, ...\}$, and $\mathbb{N} \equiv \{0, 1, 1, 2, ...\}$. $K > 0$ is a finite constant whose value may be different in different places. $awp1$ denotes "*asymptotically with probability approaching one*". Write $\max_{\mathcal{H}_T} = \max_{0 \leq h \leq \mathcal{H}_T}$. $\max_{\mathcal{K}_T} = \max_{1 \leq k \leq \mathcal{K}_T}$ and $\max_{\mathcal{H}_T, \mathcal{K}_T} = \max_{0 \leq h \leq \mathcal{H}_T, 1 \leq k \leq \mathcal{K}_T}$. Similarly, $\max_{\mathcal{H}_T} a(h, \tilde{h}) = \max_{0 \leq h, \tilde{h} \leq \mathcal{H}_T} a(h, \tilde{h})$, etc. Write $\max_{t,T} = \limsup_{T \to \infty} \max_{1 \leq t \leq T}$, $\max_{t,T,\mathcal{H}_T,\mathcal{K}_T} = \max_{t,T} \max_{\mathcal{H}_T,\mathcal{K}_T}$, etc. $|a|_+ \equiv a \vee 0$.

Write

$$z_t(h, k) \equiv X_t X_{t+h} B_k(t) - E\left[X_t X_{t+h}\right] B_k(t)$$
$$\mathcal{Z}_T(h, k) \equiv \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} z_t(h, k).$$

Define σ-fields

$$\mathcal{F}_{T,t}^{\infty} \equiv \sigma\left(X_\tau : \tau \geq t\right) and \mathcal{F}_{T,-\infty}^{t} \equiv \sigma\left(X_\tau : \tau \leq t\right),$$

and α-mixing coefficients (Rosenblatt, 1956), $\alpha(l) \equiv \sup_{t \in \mathbb{Z}} \sup_{\mathcal{A} \subset \mathcal{F}_{T,-\infty}^{t}, \mathcal{B} \subset \mathcal{F}_{T,t+l}^{\infty}} |\mathcal{P}(\mathcal{A} \cap \mathcal{B}) - \mathcal{P}\left(\mathcal{A}\right) \mathcal{P}\left(\mathcal{B}\right)|$, for $l > 0$.

Assumption 1.

a. (weak dependence)*:* $\alpha(l) \leq K_1 \exp\{-K_2 l^\phi\}$ *for some universal constants* $\phi, K_1, K_2 > 0$.

b. (subexponential tails)*:* $\max_{t,T} P(|X_t| > c) \leq \vartheta_1 \exp\{-\vartheta_2 c^\varpi\}$ *for some universal constants* $\varpi, \vartheta_1, \vartheta_2 > 0$.

c. (nondegeneracy)*:* $\liminf_{T \to \infty} E[\mathcal{Z}_T^2(h, k)] > 0 \; \forall (h, k)$.

d. (orthonormal basis)*:* $\{\mathcal{B}_k(x) : 0 \leq k \leq \mathcal{K}\}$ *forms a complete orthonormal basis on* $\mathcal{L}[0, 1)$*;* $\mathcal{B}_k(x) \in \{-1, 1\}$ *on* $[0, 1)$*;* $|\sum_{t=1}^{T} B_k(t)| = O(\eta(k))$ *for some positive strictly monotonic function* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$*,* $\eta(k) \nearrow \infty$ *as* $k \to \infty$*.*

Let $\{\epsilon_t\}_{t \in \mathbb{Z}}$ be an iid sequence, and assume there exists a measurable $\mathbb{R}$-valued function $g_t(\cdot)$ satisfying

$$X_t = g_t\left(\epsilon_t, \epsilon_{t-1}, \ldots\right).$$

Let $\{\epsilon_t'\}_{t \in \mathbb{Z}}$ be an independent copy of $\{\epsilon_t\}_{t \in \mathbb{Z}}$, and let $X_t'(m)$ be the coupled version based on $\{\epsilon_t'\}_{t \in \mathbb{Z}}$:

$$X_t'(m) = g_t\left(\epsilon_t, \epsilon_{t-1}, \ldots, \epsilon_{t-m}', \epsilon_{t-m-1}, \ldots\right).$$

Define $\mathcal{L}_p$-physical dependence coefficients $\theta_t^{(p)}(m) \equiv ||X_t - X_t'(m)||_p$.

Assumption 1.a*. (*weak dependence*): *X_t is $\mathcal{L}_p$-physical dependent for some $p \geq 8$, with $\theta_t^{(p)}(m) \leq d_t^{(p)}\psi_m$ where $\psi_m = O(m^{-\lambda-\iota})$ for some* size *$\lambda \geq 1$.*

Remark 1. By construction and Minkowski's inequality $\theta_t^{(p)}(m) \leq 2||X_t||_p$ hence logically

$$d_t^{(p)} \leq 2\,\|X_t\|_p\,. \tag{A.1}$$

Set a block size b_T such that $1 \leq b_T < T$, $b_T/T^\iota \to \infty$ and $b_T/T^{1-\iota} \to 0$ for some tiny $\iota > 0$ that may be different in different places.

Assumption 2.

a. *(i)* $\liminf_{T\to\infty} s_T^2(h,k;\tilde{h},\tilde{k}) > 0$ $\forall(h,\tilde{h},k,\tilde{k})$*;* *(ii)* $\max_{\mathcal{H}_T,\mathcal{K}_T} |s_T^2(h,k;\tilde{h},\tilde{k}) - s^2(h,k;\tilde{h},\tilde{k})| = O(T^{-\iota})$ *for some infinitessimal $\iota > 0$.*

b. $b_T/T^\iota \to \infty$ *and* $b_T = o(T^{1/2-\iota})$ *for some infinitessimal $\iota > 0$.*

B. Omitted proofs

The following result shows the composite Haar wavelets $\{\psi_k(x)\}$ form a complete orthonormal basis.

Lemma B.1. *a.* $\{\psi_k(x) : 1 \leq k \leq \mathcal{K}_T\}$ *forms a $\{-1,1\}$-valued complete orthonormal basis in $\mathcal{L}[0,1)$; b.* $|\sum_{t=1}^T \Psi_k(t)| = O(2^k)$*; c.* $\lim_{T\to\infty} 1/T \sum_{t=1}^T \Psi_k(t) = 0$*; d.* $\sum_{t=1}^T \Psi_k(t) = 0$ *if 2^k is a multiple of T.*

Proof.

Claim (a). By construction, for $k = 1, 2, ...,$

$$\psi_k(x) = \sum_{m=0}^{2^{k-1}-1} \psi(2^{k-1}x - m) = \psi(2^{k-1}x) + \psi(2^{k-1}x - 1) + \cdots + \psi(2^{k-1}x - 2^{k-1} + 1),$$

where $\psi(x) \in \{-1, 0, 1\}$, and

$$\psi(2^k x - m) = I\left(\frac{m}{2^k} \leq x < \frac{m+1/2}{2^k}\right) - I\left(\frac{m+1/2}{2^k} \leq x < \frac{m+1}{2^k}\right).$$

For a given couplet (x, k), by mutual exclusivity it follows $\psi(2^k x - m) \in \{-1, 1\}$ for only one $m \in \{0, ..., 2^k - 1\}$. Hence $\psi_k(x) \in \{-1, 1\}$.

Next, by construction of the Haar wavelet functions $\psi(2^k x - m)$:

$$\int_0^1 \psi_k(x)dx = \sum_{m=0}^{2^k-1} \int_0^1 \psi(2^{k-1}x - m)dx = \sum_{m=0}^{2^{k-1}-1} \left(\int_{m/2^{k-1}}^{(m+1/2)/2^{k-1}} dx - \int_{(m+1/2)/2^{k-1}}^{(m+1)/2^{k-1}} dx\right) = 0.$$

Furthermore, $\psi_k(x) \in \{-1,1\}$ implies $\int_0^1 \psi_k^2(x)dx = 1$. Finally, let $k_1 > k_2$ to generate by the orthogonality of $\{\psi_{k_1,m}(x), \psi_{k_2,m}(x) : k_1 \neq k_2\}$:

$$\begin{aligned}\int_0^1 \psi_{k_1}(x)\psi_{k_2}(x)dx &= \sum_{m_1=0}^{2^{k_1-1}-1}\sum_{m_2=0}^{2^{k_2-1}-1}\int_0^1 \psi(2^{k_1-1}x - m_1)\psi(2^{k_2-1}x - m_2)dx \\ &= \frac{1}{2^{k_1-1/2}2^{k_2-1/2}}\sum_{m_1=0}^{2^{k_1-1}-1}\sum_{m_2=0}^{2^{k_2-1}-1}\int_0^1 \psi(2^{k_1-1}x - m_1)\psi(2^{k_2-1}x - m_2)dx \\ &= \sum_{m_1=0}^{2^{k_2-1}-1}\sum_{m_2=0}^{2^{k_2-1}-1}\int_0^1 \psi(2^{k_1-1}x - m_1)\psi(2^{k_2-1}x - m_2)dx \\ &\quad + \sum_{m_1=2^{k_2-1}+1}^{2^{k_1-1}-1}\sum_{m_2=0}^{2^{k_2-1}-1}\int_0^1 \psi(2^{k_1-1}x - m_1)\psi(2^{k_2-1}x - m_2)dx \\ &= \sum_{m=0}^{2^{k_2-1}-1}\int_0^1 \psi(2^{k_1-1}x - m)\psi(2^{k_2-1}x - m)dx \\ &\quad + \sum_{m_1=2^{k_2-1}+1}^{2^{k_1-1}-1}\sum_{m_2=0}^{2^{k_2-1}-1}\int_0^1 \psi(2^{k_1-1}x - m_1)\psi(2^{k_2-1}x - m_2)dx = 0.\end{aligned}$$

Hence $\{\psi_k(x) : 1 \le k \le \mathcal{K}_T\}$ forms a $\{-1,1\}$-valued orthonormal basis. Completeness follows from completeness of $\{\psi_{k,m}(x) : 1 \le k \le \mathcal{K}_T\}$ and the definition $\psi_k(x) \equiv 2^{-(k-1)/2}\sum_{m=0}^{2^{k-1}-1}\psi_{k,m}(x)$.

Claim (b). By construction:

$$\sum_{t=1}^{T}\Psi_k(t) = \sum_{m=0}^{2^k-1}\sum_{t=1}^{T}\psi(2^k(t-1)/T - m) = \sum_{m=0}^{2^k-1}\left(2\left[\left(\frac{m+1/2}{2^k}\right)T\right] - \left[\frac{m}{2^k}T\right] - \left[\left(\frac{m+1}{2^k}\right)T\right]\right) \quad \text{(B.1)}$$

Now use $[aT] - aT \in [-1/2, 1/2]\ \forall a \in [0,1]$ to yield for any $m \in \{0,...,2^{k-1}\}$:

$$\begin{aligned}&\left|2\left[\left(\frac{m+1/2}{2^k}\right)T\right] - \left[\frac{m}{2^k}T\right] - \left[\left(\frac{m+1}{2^k}\right)T\right]\right| \\ &\quad \le \left|2\left(\frac{m+1/2}{2^k}\right)T - \frac{m}{2^k}T - \left(\frac{m+1}{2^k}\right)T\right| + 2\left|\left[\left(\frac{m+1/2}{2^k}\right)T\right] - \left(\frac{m+1/2}{2^k}\right)T\right| \\ &\qquad + \left|\left[\frac{m}{2^k}T\right] - \frac{m}{2^k}T\right| + \left|\left[\left(\frac{m+1}{2^k}\right)T\right] - \left(\frac{m+1}{2^k}\right)T\right| \\ &\quad \le 1 + 1/2 + 1/2 = 2.\end{aligned}$$

Therefore $|\sum_{t=1}^{T}\Psi_k(t)| \le 2^{k+1} = O(2^k)$.

Claim (c). Use (a) with $x_t = (t-1)/T$ and $dx_t = 1/T$ to give $1/T\sum_{t=1}^{T}\Psi_k(t) = \int_0^1 \psi(2^k x_t - m)dx_t \rightarrow$

$\int_0^1 \psi(2^k x - m)dx = 0$ as $T \to \infty$.

Claim (d). The claim follows from identity (B.1), and $[2^{-k}(m+1/2)T] = 2^{-k}(m+1/2)T$ if 2^k divides T (in which case $T/2^k$ is even, hence $2^{-k}(m+1/2)T \in \mathbb{N}$). $\mathcal{QED}$.

B.1. Strong mixing

B.1.1. Lemma 3.1

The proof of Lemma 3.1 relies on an extension of Assumption 1.b to $\prod_{i=1}^r X_{t_i}$ for any r-tuple $\{t_1, ..., t_r\}$, $r \in \mathbb{N}$. This is required here for both couplets $X_t X_{t-h}$ and their cross-products $X_s X_{s-l} X_t X_{t-h}$ for our high dimensional results.

Lemma B.2. *Let* $\max_{1 \leq t \leq T} P(|X_t| > c) \leq \vartheta_1 \exp\{-\vartheta_2 c^{\varpi}\}$ *for some universal constants* $\varpi, \vartheta_1, \vartheta_2 > 0$. *It holds that for some* $\tilde{\varpi} > 0$*:*

$$\max_{1 \leq t_1,...,t_r \leq T} P\left(\left|\prod_{i=1}^r X_{t_i}\right| > c\right) \leq r\vartheta_1 \exp\left\{-\vartheta_2 c^{\tilde{\varpi}}\right\}. \tag{B.2}$$

Proof. We prove (B.2) by induction. If $r = 1$ then $\max_{1 \leq t \leq T} P(|X_t| > c) \leq \vartheta_1 \exp\{-\vartheta_2 c^{\varpi}\}$ by supposition. Now let (B.2) hold for some $r \geq 1$. Then Young and Bonferroni inequalities imply for $\tilde{\varpi} = \varpi/2$:

$$\begin{aligned}
\max_{1 \leq t_1,...,t_{r+1} \leq T} P\left(\left|\prod_{i=1}^{r+1} X_{t_i}\right| > c\right) &\leq \max_{1 \leq t_1,...,t_{r+1} \leq T} P\left(\frac{1}{2}\left(\prod_{i=1}^r X_{t_i}\right)^2 + \frac{1}{2}X_{t_{r+1}}^2 > c\right) \\
&\leq \max_{1 \leq t_1,...,t_r \leq T} P\left(\left|\prod_{i=1}^r X_{t_i}\right| > c^{\frac{1}{2}}\right) + \max_{1 \leq t \leq T} P\left(|X_t| > c^{\frac{1}{2}}\right) \\
&\leq r\vartheta_1 \exp\left\{-\vartheta_2 c^{\varpi/2}\right\} + \vartheta_1 \exp\left\{-\vartheta_2 c^{\varpi/2}\right\} \\
&\leq (r+1)\,\vartheta_1 \exp\left\{-\vartheta_2 c^{\tilde{\varpi}}\right\}. \ \mathcal{QED}.
\end{aligned}$$

In the following we allow for a non-zero mean $E[X_t]$ $\forall t$. Recall

$$\rho_T \equiv \sup_{z \geq 0}\left|P\left(\max_{\mathcal{H}_T,\mathcal{K}_T} |\mathcal{Z}_T(h,k)| \leq z\right) - P\left(\max_{\mathcal{H}_T,\mathcal{K}_T} |\mathbf{Z}_T(h,k)| \leq z\right)\right| \to 0. \tag{B.3}$$

Lemma 3.1. *Under Assumption 1,* $\rho_T \lesssim \mathcal{H}_T^{1/2}(\ln(\mathcal{H}_T))^{7/6}/T^{1/9} \to 0$*, for any sequences* $\{\mathcal{H}_T, \mathcal{K}_T\}$ *with* $0 \leq \mathcal{H}_T \leq T-1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^{\kappa})$ *for some finite* $\kappa > 0$*, and* $\eta(\mathcal{K}_T) = o(\sqrt{T})$ *where* $\eta(\cdot)$ *is the Assumption 1.d discrete basis summand bound. In this case* $\max_{\mathcal{H}_T,\mathcal{K}_T} |\mathcal{Z}_T(h,k)| \xrightarrow{d} \max_{h,k \in \mathbb{N}} |\mathbf{Z}(h,k)|$ *where* $\mathbf{Z}(h,k) \sim N(0, \lim_{T\to\infty} \sigma_T^2(h,k))$ *and* $\lim_{T\to\infty} \sigma_T^2(h,k) < \infty$.

Proof. Allowing for a non-zero mean, recall:

$$z_t(h,k) \equiv (X_t - E[X_t])(X_{t+h} - E[X_{t-h}])B_k(t) - \{E[(X_t - E[X_t])(X_{t+h} - E[X_{t-h}])]B_k(t)\}$$

and $\mathcal{Z}_T(h,k) \equiv 1/\sqrt{T}\sum_{t=1}^{T-h} z_t(h,k)$. Here, and in the sequel, let $\{\zeta_t(i), \mathfrak{Z}_T(i)\}_{i=0}^{\mathcal{H}_T\mathcal{K}_T}$ denote $\{z_t(h,k), \mathcal{Z}_T(h,k)\}_{h=0,k=1}^{\mathcal{H}_T,\mathcal{K}_T}$, stacked h-wise over k:

$$\mathfrak{Z}_T(i) = \mathcal{Z}_T(h,k) \text{ with index correspondence } i = (k-1)\mathcal{H}_T + h. \tag{B.4}$$

Thus $\mathfrak{Z}_T(1),...,\mathfrak{Z}_T(\mathcal{H}_T) = \mathcal{Z}_T(1,1),...,\mathcal{Z}_T(\mathcal{H}_T,1)$; $\mathfrak{Z}_T(\mathcal{H}_T+1),...,\mathfrak{Z}_T(2\mathcal{H}_T) = \mathcal{Z}_T(1,2),...,\mathcal{Z}_T(\mathcal{H}_T,2)$; etc. Define

$$\sigma_T^2(i) \equiv E\left[\mathfrak{Z}_T^2(i)\right]$$

and let $\{\boldsymbol{Z}_T(i) : T \in \mathbb{N}\}_{i\geq 0}$ be normally distributed $\boldsymbol{Z}_T(i) \sim N(0, \sigma_T^2(i))$. It suffice to prove the claim for $\mathfrak{Z}_T(i)$.

Under Assumption 1.a,b,c $\zeta_t(i)$ satisfies AS1-AS3 in Chang, Jiang and Shao (2023, p. 990). AS1 holds from Assumption 1.b and Lemma B.2. AS2 holds as follows: by construction and Assumption 1.a, $[\zeta_t(i)]_{i=0}^{\mathcal{H}_T\mathcal{K}_T}$ is $\sigma(X_\tau : \tau \leq t + \mathcal{H}_T)$-measurable, with mixing coefficients $\mathring{\alpha}(l) \leq \alpha(|l - \mathcal{H}_T|_+) \leq K_1 \exp\{-K_2|l - \mathcal{H}_T|_+^\phi\}$. AS3 is Assumption 1.c.

Recall $\varpi, \phi > 0$ are respectively the mixing and tail decay exponents. Write, e.g., $\varpi_0 \equiv \varpi \wedge 1$, and set

$$\psi \equiv \frac{\varpi_0\phi_0}{\varpi_0 + \phi_0}.$$

Proposition 3 in Chang, Jiang and Shao (2023), and the mapping theorem, therefore yield $\rho_T \equiv$

$$\sup_{z\geq 0}\left|P\left(\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} |\mathcal{Z}_T(i)| \leq z\right) - P\left(\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} |\boldsymbol{Z}_T(i)| \leq z\right)\right| \lesssim g_T, \tag{B.5}$$

where

$$g_T \equiv \frac{\mathcal{H}_T^{1/3}\left(\ln\left(\mathcal{H}_T\mathcal{K}_T\right)\right)^{2/3}}{T^{1/9}}\left\{\mathcal{H}_T^{1/6}\left(\ln\left(\mathcal{H}_T\mathcal{K}_T\right)\right)^{1/2} + \mathcal{H}_T^{1/3} + \left(\ln\left(\mathcal{H}_T\mathcal{K}_T\right)\right)^{1/(3\varpi_0)}\right\} \to 0 \tag{B.6}$$

provided

$$\mathcal{H}_T \ln(\mathcal{H}_T\mathcal{K}_T) = o(T^{1/6})$$
$$\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$$
$$\ln\left(\mathcal{H}_T\mathcal{K}_T\right) = o\left(\min\left\{\mathcal{H}_T^{3\psi/(6+2\psi)}T^{7\psi/(18+6\psi)}, \mathcal{H}_T^{-3\phi_0/(6+2\phi_0)}T^{7\phi_0/(18+6\phi_0)}, T^{\varpi_0/(9-3\varpi_0)}\right\}\right).$$

The first bound ensures $g_T = o(1)$, and the remaining two suffice for $\rho_T \lesssim g_T$. Then together $\mathcal{K}_T = o(T^\kappa)$ for some $\kappa > 0$ and $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$ suffice, cf. Remark 5 in the main paper.

Finally, (B.6) implies $\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} |\mathcal{Z}_T(i)| \overset{d}{\to} \max_{i\in\mathbb{N}} |\boldsymbol{Z}(i)|$ where $\boldsymbol{Z}(i) \sim N(0, \lim_{T\to\infty}\sigma_T^2(i))$ with $\lim_{T\to\infty}\sigma_T^2(i) < \infty$ shown below. Just note that convergence in distribution follows by construction of $\boldsymbol{Z}(i)$: $\lim_{T\to\infty} P(\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} |\boldsymbol{Z}_T(i)| \leq z) = P(\max_{i\in\mathbb{N}} |\boldsymbol{Z}(i)| \leq z)\ \forall z \geq 0$.

It remains to prove

$$\lim_{T\to\infty} \sigma_T^2(i) < \infty, i = 0, 1, 2, ... \tag{B.7}$$

Under Assumption 1 and by measurability, $\zeta_t(i)$ is for each fixed i uniformly $\mathcal{L}_r$-bounded for any $r > 2$, and α-mixing with coefficients $\alpha^{(z)}(l) \leq K_1 \exp\{-K_2|l - h|_+^\phi\}$ for some universal $\phi, K_1, K_2 > 0$, where h is a unique lag index associated with i via (B.4). Then by Lemma 2.1 in McLeish (1975), $\{z_t(h,k)\}$ forms a zero-mean $\mathcal{L}_2$-mixingale array with *coefficients* $\mathring{\alpha}(l) \equiv \alpha^{(z)}(l)^{\{1/2-2/r\}} \leq K_1 \exp\{-K_2(1/2 - 2/r)|l - h|_+^\phi\}$, and *constants* $K||z_t(h,k)||_r$. The former satisfies $\mathring{\alpha}(l) = O(l^{-\lambda})$ for any $\lambda \geq 1/2$ for each fixed h. The constants are uniformly bounded $\max_{\mathcal{H}_T,\mathcal{K}_T} ||z_t(h,k)||_r \leq K \max_{\mathcal{H}_T} ||X_tX_{t-h}||_r \leq K$ by Minkowski and Jensen inequalities, $|B_k(t)| = 1$, and Assumption 1.b. Then Theorem 1.6 in McLeish (1975) proves (B.7), completing the proof. $\mathcal{QED}$.

B.1.2. Theorem 3.2

We first show that may assume $E[X_t] = 0$ in subsequent proofs to ease notation.

Lemma B.3. *Under Assumption 1, for any sequences $\{\mathcal{H}_T, \mathcal{K}_T\}$ with $0 \leq \mathcal{H}_T \leq T-1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$:*

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\left\{\left(X_t-\bar{X}\right)\left(X_{t-h}-\bar{X}\right)-\left(X_t-\mu\right)\left(X_{t-h}-\mu\right)\right\}B_k(t)\right| = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof. Write $\tilde{X}_t \equiv X_t - \mu$ and $\hat{X}_t \equiv X_t - \bar{X}$. We have:

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\left\{\hat{X}_t\hat{X}_{t-h}-\tilde{X}_t\tilde{X}_{t-h}\right\}B_k(t)\\
&\qquad=\left(\bar{X}^2-\mu^2\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}B_k(t)-2\mu\left(\bar{X}-\mu\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}B_k(t)\\
&\qquad\quad-\left(\bar{X}-\mu\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\left\{X_{t-h}-\mu+X_t-\mu\right\}B_k(t)\\
&\qquad=\mathfrak{A}_T(h,k)+\mathfrak{B}_T(h,k)+\mathfrak{C}_T(h,k).
\end{aligned}$$

By Assumption 1.d $|1/\sqrt{T}\sum_{t=1}^{T} B_k(t)| = O(\eta(k)/\sqrt{T})$. Arguments preceding (B.7) identically imply $\bar{X} - \mu = O_p(1/\sqrt{T})$ by Chebyshev's inequality, hence $\bar{X}^2 - \mu^2 = O_p(1/\sqrt{T})$ by the mapping theorem. Therefore, e.g.,

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\left\{\bar{X}^2-\mu^2\right\}\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}B_k(t)\right| = O\left(\frac{\max_{\mathcal{K}_T}\eta(k)}{T}\right) = O_p\left(\frac{\eta(\mathcal{K}_T)}{T}\right).$$

Now use $\eta(\mathcal{K}_T) = o(\sqrt{T})$ to get $\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathfrak{A}_T(h,k)| = o_p(1/\sqrt{T})$, and $\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathfrak{B}_T(h,k) = o_p(1/\sqrt{T})$.

The remaining term $\mathfrak{C}_T$ is handled by applying arguments in the proof of Lemma 3.1 to deduce for some mean zero Gaussian process $\mathring{\mathbf{Z}}(h,k) \sim N(0, \lim_{T\to\infty}\mathring{\sigma}_T^2(h,k))$ and $\lim_{T\to\infty}\mathring{\sigma}_T^2(h,k) < \infty$:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\left\{X_{t-h}-\mu+X_t-\mu\right\}B_k(t)\right| \stackrel{d}{\to} \max_{h,k\in\mathbb{N}}|\mathbf{Z}(h,k)|.$$

Hence $\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathfrak{C}_T| = O_p(1/\sqrt{T})$, completing the proof. $\mathcal{QED}$.

Recall $\sigma^2(h,k) \equiv \lim_{T\to\infty}\sigma_T^2(h,k)$.

Theorem 3.2. *Let H_0 and Assumption 1 hold, and let $\mathcal{H}_T, \mathcal{K}_T \to \infty$. Let $\{\mathbf{Z}(h,k) : h,k \in \mathbb{N}\}$ be a zero mean Gaussian process with $\mathbf{Z}(h,k) \sim N(0,\sigma^2(h,k))$. Then it holds that $\mathcal{M}_T \stackrel{d}{\to} \gamma_0^{-1}\max_{h,k\in\mathbb{N}}|\mathbf{Z}(h,k)|$ for any $\{\mathcal{H}_T,\mathcal{K}_T\}$ with $0 \leq \mathcal{H}_T \leq T-1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$.*

Proof. $\sigma_T^2(i) = O(1)$ in (B.7) implies $\hat{\gamma}_0 - \gamma_0 = O_p(1/\sqrt{T})$ by Chebyshev's inequality.

Moreover, by construction

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} \{X_t X_{t+h} B_k(t) - E\left[X_t X_{t+h}\right] B_k(t)\} = \sqrt{T}(\hat{\gamma}_h^{(k)} - \hat{\gamma}_h) - \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t). \quad \text{(B.8)}$$

Under covariance stationarity H_0, $|E[X_t X_{t+h}]| < E[X_t^2] < \infty$ for all h and t. Assumption 1.d imply $\forall \{h, k\}$:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} E[X_t X_{t+h} B_k(t)] = \gamma_h \times \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} B_k(t) = O\left(\frac{\eta(k)}{\sqrt{T}}\right). \quad \text{(B.9)}$$

Hence:

$$\begin{aligned} \sqrt{T}\left(\hat{\rho}_h^{(k)} - \hat{\rho}_h\right) &= \frac{1}{\gamma_0 + O_p(1/\sqrt{T})} \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} \{X_t X_{t+h} B_k(t) - E\left[X_t X_{t+h}\right] B_k(t)\} \quad \text{(B.10)} \\ &\quad + \frac{1}{\gamma_0 + O_p(1/\sqrt{T})} \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t) \\ &= \frac{1}{\gamma_0 + O_p(1/\sqrt{T})} \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} \{X_t X_{t+h} B_k(t) - E\left[X_t X_{t+h}\right] B_k(t)\} + O_p\left(\frac{\eta(k)}{\sqrt{T}}\right), \end{aligned}$$

where the $O(\cdot)$ and $O_p(\cdot)$ terms do not depend on $\{h, k\}$. Then $\eta(\mathcal{K}_T) = o(\sqrt{T})$ yields:

$$\begin{aligned} \max_{\mathcal{H}_T, \mathcal{K}_T} \Bigg| \sqrt{T}\left(\hat{\rho}_h^{(k)} - \hat{\rho}_h\right) & \quad \text{(B.11)} \\ - \frac{1}{\gamma_0 + O_p(1/\sqrt{T})} \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} \{X_t X_{t+h} B_k(t) - E\left[X_t X_{t+h}\right] B_k(t)\} \Bigg| &= o_p(1). \end{aligned}$$

The claim now follows from Lemma 3.1. $\mathcal{QED}$.

B.1.3. Theorem 4.1

A weak convergence result for the bootstrapped correlation difference is required. We develop ideas under mixing here, and extend to physical dependence in Appendix B.2.

Let $\Rightarrow^p$ denote weak convergence in probability on l_∞ (the space of bounded functions) as defined in Giné and Zinn (1990, Section 3). Recall bootstrap index blocks $\mathfrak{B}_s = \{(s-1)b_T + 1, \ldots, sb_T\}$, $s = 1, \ldots, T/b_T$, with block size b_T, $1 \leq b_T < T$, $b_T \to \infty$ and $b_T/T^{1-\iota} \to 0$ for some small $\iota > 0$. ξ_i is iid $N(0,1)$, and $\varphi_t = \xi_s$ if $t \in \mathfrak{B}_s$. Recall the number of blocks $\mathcal{N}_T = [T/b_T]$, and

$$\Delta\hat{g}_T^{(dw)}(h, k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t \left\{X_t X_{t+h} B_k(t) - \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t)\right\},$$

and define

$$\mathring{\sigma}_T^2(h, k) \equiv E\left[\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} \varphi_t \left\{X_t X_{t+h} B_k(t) - \frac{1}{T} \sum_{s=1}^{T-h} E\left[X_s X_{s+h}\right] B_k(s)\right\}\right)^2\right].$$

Recall

$$z_t(h,k) \equiv \{X_t X_{t+h} - E[X_t X_{t+h}]\} B_k(t)$$
$$\sigma_T^2(h,k) \equiv E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h} z_t(h,k)\right)^2\right].$$

Lemma B.4. *Let Assumptions 1 and 2 hold. Let $\{b_T, \mathcal{H}_T, \mathcal{K}_T\}$ be any sequences satisfying $b_T/T^{\iota} \to \infty$, $b_T = o(T^{1/2-\iota})$, $0 \leq \mathcal{H}_T < T-1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$.*

a. Let $\{\mathring{\boldsymbol{Z}}_T(h,k) : 0 \leq h \leq \mathcal{H}_T, 1 \leq k \leq \mathcal{K}_T\}_{T\leq 1}$ be a Gaussian process, $\mathring{\boldsymbol{Z}}_T(h,k) \sim N(0, \mathring{\sigma}_T^2(h,k))$, independent of the sample $\{X_t\}_{t=1}^T$. Then:

$$\sup_{c>0}\left|P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k)\right| \leq c|\{X_t\}_{t=1}^T\right) - P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathring{\boldsymbol{Z}}_T(h,k)\right| \leq c\right)\right| \xrightarrow{p} 0.$$

b. Let $\{\mathring{\boldsymbol{Z}}(h,k)\}$ be an independent copy of the Lemma 3.1 Gaussian process $\{\boldsymbol{Z}(k,h) : h,k \in \mathbb{N}\}$, $\boldsymbol{Z}(h,k) \sim N(0, \lim_{T\to\infty}\sigma_T^2(h,k))$, independent of the asymptotic draw $\{X_t\}_{t=1}^{\infty}$. Then:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k)\right| \Rightarrow^p \max_{h,k\in\mathbb{N}}\left|\mathring{\boldsymbol{Z}}(h,k)\right|.$$

The proof exploits two results. First, uniform sample covariance convergence. Write

$$\hat{g}(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) \text{ and } g_T(h,k) \equiv E[\hat{g}(h,k)] = \frac{1}{T}\sum_{t=1}^{T-h} E[X_t X_{t+h}] B_k(t)$$

Lemma B.5. *Under Assumption 1,*

$$\max_{\mathcal{H}_T,\mathcal{K}_T}|\hat{g}(h,k) - g_T(h,k)| = O_p\left(1/\sqrt{T}\right)$$

for any $\{\mathcal{H}_T, \mathcal{K}_T\}$ satisfying $0 \leq \mathcal{H}_T \leq T-1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$.

Proof. Define

$$\mathcal{G}_T(h,k) \equiv \sqrt{T}(\hat{g}(h,k) - g_T(h,k)) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\{X_t X_{t+h} - E[X_t X_{t+h}]\} B_k(t)$$

and $s_T^2(h,k) \equiv E[\mathcal{G}_T^2(h,k)]$. The argument used to prove Lemma 3.1 implies

$$\sup_{z\geq 0}\left|P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathcal{G}_T(h,k)| \leq z\right) - P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}|\boldsymbol{G}_T(h,k)| \leq z\right)\right| \to 0$$

for some sequence of random functions $\{\boldsymbol{G}_T(h,k)\}_{T\geq 1}$ with $\boldsymbol{G}_T(h,k) \sim N(0, s_T^2(h,k))$, for any $\{\mathcal{H}_T, \mathcal{K}_T\}$ satisfying $0 \leq \mathcal{H}_T \leq T-1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$. The claim follows instantly. $\mathcal{QED}$.

Next, define

$$y_t(h,k) \equiv \left\{ X_t X_{t+h} B_k(t) - \frac{1}{T-h} \sum_{s=1}^{T-h} E\left[X_s X_{s+h}\right] B_k(s) \right\}. \tag{B.12}$$

We decompose the following summand into big and little blocks:

$$\Delta \ddot{g}_T^*(h,k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t \left\{ X_t X_{t+h} B_k(t) - \frac{1}{T-h} \sum_{s=1}^{T-h} E\left[X_s X_{s+h}\right] B_k(s) \right\} = \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t y_t(h,k).$$

Let $\tilde{b}_T$ and $\tilde{l}_T$ be block sizes, $(\tilde{b}_T, \tilde{l}_T) \rightarrow \infty$, with $1 < \tilde{b}_T < T$, $\tilde{b}_T = o(T)$, $1 \leq \tilde{l}_T < \tilde{b}_T$, and $\tilde{l}_t = o(\tilde{b}_T)$. In each index set $\{1,...,T-h\}$ the number of blocks is

$$\widetilde{\mathcal{N}}_T(h) = \left[(T-h)/\tilde{b}_T\right].$$

Denote the blocks by $\tilde{\mathfrak{B}}_s = \{(s-1)\tilde{b}_T + 1, \ldots, s\tilde{b}_T\}$ with $s = 1, \ldots, \widetilde{\mathcal{N}}_T(h)$, and $\tilde{\mathfrak{B}}_{\widetilde{\mathcal{N}}_T(h)+1} = \{\widetilde{\mathcal{N}}_T(h)\tilde{b}_T, ..., T+h\}$. Then

$$\Delta \ddot{g}_T^*(h,k) = \frac{1}{T} \sum_{i=1}^{\widetilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\tilde{b}_T+\tilde{l}_T+1}^{i\tilde{b}_T} \varphi_t y_t(h,k) + \frac{1}{T} \sum_{i=1}^{\widetilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\tilde{b}_T+1}^{(i-1)\tilde{b}_T+\tilde{l}_T} \varphi_t y_t(h,k)$$
$$+ \frac{1}{T} \sum_{i=\widetilde{\mathcal{N}}_T(h)\tilde{b}_T+1}^{T-h} \varphi_t y_t(h,k).$$

Lemma B.6. *Under Assumptions 1 and 2, for any* $\{\mathcal{H}_T, \mathcal{K}_T\}$ *satisfying* $0 \leq \mathcal{H}_T \leq T-1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^\kappa)$ *for some finite* $\kappa > 0$*, and* $\eta(\mathcal{K}_T) = o(\sqrt{T})$*:*

$$\left| \max_{\mathcal{H}_T,\mathcal{K}_T} \left|\Delta \ddot{g}_T^*(h,k)\right| - \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T} \sum_{i=1}^{\widetilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\tilde{b}_T+\tilde{l}_T+1}^{i\tilde{b}_T} \varphi_t y_t(h,k) \right| \right| = o_p\left(1/\sqrt{T}\right).$$

Proof. The triangle inequality yields for any real-valued functions $\{a(h,k), b(h,k)\}$

$$\left| \max_{\mathcal{H}_T,\mathcal{K}_T} |a(h,k)| - \max_{\mathcal{H}_T,\mathcal{K}_T} |b(h,k)| \right| \leq \max_{\mathcal{H}_T,\mathcal{K}_T} |a(h,k) - b(h,k)|.$$

We therefore prove for some $\{\mathcal{H}_T, \mathcal{K}_T\}$,

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \left|\Delta \ddot{g}_T^*(h,k)\right| - \left| \frac{1}{T} \sum_{i=1}^{\widetilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\tilde{b}_T+\tilde{l}_T+1}^{i\tilde{b}_T} \varphi_t y_t(h,k) \right| \right| = o_p\left(1/\sqrt{T}\right),$$

Step 1. It suffices to replace $y_t(h,k)$ with $z_t(h,k)$ uniformly *awp1*:

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t \left\{ y_t(h,k) - z_t(h,k) \right\} \right| = o_p(1). \tag{B.13}$$

This follows by noting:

$$\left|\frac{1}{T}\sum_{t=1}^{T-h}\varphi_t y_t(h,k) - \frac{1}{T}\sum_{t=1}^{T-h}\varphi_t z_t(h,k)\right|$$
$$\leq \left|\frac{1}{T}\sum_{t=1}^{T-h}\varphi_t\left\{E\left[X_t X_{t+h}\right]B_k(t) - \frac{1}{T-h}\sum_{s=1}^{T-h}E\left[X_s X_{s+h}\right]B_k(s)\right\}\right|$$
$$\leq \left|\frac{1}{T}\sum_{t=1}^{T-h}\varphi_t E\left[X_t X_{t+h}\right]B_k(t)\right| + O_p\left(1/\sqrt{T}\right)$$

in view of the Assumption 1.b,d implication

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T-h}\sum_{s=1}^{T-h}E\left[X_s X_{s+h}\right]B_k(s)\right| \leq K,$$

and

$$\max_{\mathcal{H}_T}\left|\frac{1}{T}\sum_{t=1}^{T-h}\varphi_t\right| = O_p\left(1/\sqrt{T}\right).$$

The latter follows by construction of φ_t:

$$\frac{1}{T}\sum_{t=1}^{T-h}\varphi_t = \frac{1}{T/b_T}\sum_{i=1}^{[(T-h)/b_T]}\xi_i = \frac{1}{T/b_T}\sum_{i=1}^{[\lambda_h T/b_T]}\xi_i \text{ with } \lambda_h = 1 - \frac{h}{T} \in [0,1),$$

where iid $\xi_i \sim N(0,1)$. By Donsker's theorem extended to $\mathcal{D}[0,1]$, and the mapping theorem, $\sup_{\lambda\in[0,1)}|1/\sqrt{N}\sum_{i=1}^{[\lambda N]}\xi_i| = O_p(1)$ (cf Dudley, 1999).

Next, by construction:

$$\frac{1}{T}\sum_{t=1}^{T-h}\varphi_t E\left[X_t X_{t+h}\right]B_k(t)$$
$$= \frac{1}{T/b_T}\sum_{i=1}^{[(T-h)/b_T]}\xi_i\frac{1}{b_T}\sum_{t=(i-1)b_T+1}^{ib_T}E\left[X_t X_{t+h}\right]B_k(t) = \frac{1}{T/b_T}\sum_{i=1}^{\mathcal{N}_T(h)}\xi_i\varpi_{T,i}(h,k)$$

say, where $\varpi_{T,i}(h,k) \equiv 1/b_T\sum_{t=(i-1)b_T+1}^{ib_T}E[X_t X_{t+h}]B_k(t)$ and $\mathcal{N}_T(h) \equiv [(T-h)/b_T]$. Given ξ_i is iid $N(0,1)$, a generalization of Nemirovski's $\mathcal{L}_q$-moment bound, $q \geq 1$, for independent sequences yields (see, e.g., Bühlmann and Van De Geer, 2011, Lemma 14.24):

$$E\left[\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T/b_T}\sum_{i=1}^{\mathcal{N}_T(h)}\xi_i\varpi_{T,i}(h,k)\right|^q\right] \leq \left\{\frac{8\ln\left(2\mathcal{H}_T\mathcal{K}_T\right)\max_{\mathcal{H}_T,\mathcal{K}_T}\max_{1\leq i\leq\mathcal{N}_T(h)}\varpi_{T,i}^2(h,k)}{T/b_T}\right\}^{q/2}. \quad \text{(B.14)}$$

Moreover:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\max_{1\leq i\leq\mathcal{N}_T(h)}\varpi^2_{T,i}(h,k)=\max_{\mathcal{H}_T,\mathcal{K}_T}\max_{1\leq i\leq\mathcal{N}_T(h)}\left(\frac{1}{b_T}\sum_{t=(i-1)b_T+1}^{ib_T}E\left[X_tX_{t+h}\right]B_k(t)\right)^2 \quad \text{(B.15)}$$

$$\leq\left(\max_{\mathcal{H}_T}\max_{1\leq t\leq T}\left|E\left[X_tX_{t+h}\right]\right|\right)^2\equiv\bar{\varpi}^2_T.$$

Now combine (B.14) and (B.15), choose $q=2$, and invoke $b_T=o(T^{1/2-\iota})$, $\mathcal{H}_T=o(T)$, $\mathcal{K}_T=o(T^\kappa)$ for some finite $\kappa>0$, and $\bar{\varpi}_T=O(1)$ under Assumption 1.b and the Cauchy-Schwartz inequality, to deduce:

$$E\left[\max_{\mathcal{H}_T,\mathcal{K}_T}\left(\frac{1}{T/b_T}\sum_{i=1}^{\mathcal{N}_T(h)}\xi_i\varpi_{T,i}(h,k)\right)^2\right]\leq K\frac{\ln\left(\mathcal{H}_T\mathcal{K}_T\right)}{T/b_T}\bar{\varpi}^2_T$$

$$=o\left(\frac{\ln(T)}{T^{1/2+\iota}}\right)\times\bar{\varpi}^2_T=o\left(\frac{\ln(T)}{T^{1/2+\iota}}\right)=o(1).$$

This proves (B.13) by Chebyshev's inequality.

Step 2. Now observe that

$$\left|\Delta\ddot{g}^*_T(h,k)-\frac{1}{T}\sum_{i=1}^{(T-h)/b_T}\sum_{t=(i-1)b_T+l_T+1}^{ib_T}\varphi_tz_t(h,k)\right|$$

$$\leq\left|\frac{1}{T}\sum_{i=1}^{\tilde{\mathcal{N}}_T(h)}\sum_{t=(i-1)\tilde{b}_T+1}^{(i-1)\tilde{b}_T+\tilde{l}_T}\varphi_tz_t(h,k)\right|+\left|\frac{1}{T}\sum_{i=\tilde{\mathcal{N}}_T(h)\tilde{b}_T+1}^{T-h}\varphi_tz_t(h,k)\right|.$$

Use Lemma 3.1 and $\tilde{b}_T/\tilde{l}_T=o(1)$ to deduce under the assumed properties for $\{\mathcal{H}_T,\mathcal{K}_T\}$:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{i=1}^{\tilde{\mathcal{N}}_T(h)}\sum_{t=(i-1)\tilde{b}_T+1}^{(i-1)\tilde{b}_T+\tilde{l}_T}\varphi_tz_t(h,k)\right|=.\frac{\tilde{l}_T}{\tilde{b}_T}\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T\tilde{l}_T/\tilde{b}_T}\sum_{i=1}^{\tilde{\mathcal{N}}_T(h)}\sum_{t=(i-1)\tilde{b}_T+1}^{(i-1)\tilde{b}_T+\tilde{l}_T}\varphi_tz_t(h,k)\right|$$

$$=O_p\left(\frac{\tilde{l}_T/\tilde{b}_T}{\sqrt{T\tilde{l}_T/\tilde{b}_T}}\right)=O_p\left(1/\sqrt{T\tilde{b}_T/\tilde{l}_T}\right)=o_p\left(1/\sqrt{T}\right).$$

Similarly, for any (h,k), the integer-valued discrepancy implicit in

$$T-h-\tilde{\mathcal{N}}_T(h)\tilde{b}_T=T-h-\left[(T-h)/\tilde{b}_T\right]\tilde{b}_T$$

yields:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{i=\tilde{\mathcal{N}}_T(h)\tilde{b}_T+1}^{T-h}\varphi_tz_t(h,k)\right|=O_p\left(\max_{\mathcal{H}_T}\frac{\sqrt{(T-h)-\tilde{\mathcal{N}}_T(h)\tilde{b}_T}}{T}\right) \quad \text{(B.16)}$$

$$= O_p\left(\max_{\mathcal{H}_T} \frac{\sqrt{1 - \left[\frac{T}{\tilde{b}_T}(1 - h/T)\right] \frac{\tilde{b}_T}{T(1-h/T)}}}{\sqrt{T}}\right) = o_p(1/T).$$

This completes the proof. $\mathcal{QED}$

We are now ready to prove Lemma B.4. Assume $(T - h)/b_T$ and related ratios are integers to reduce notation. The resulting error is otherwise asymptotically negligible; cf. (B.16).

Proof of Lemma B.4.

Claim (a). Define the sample $\mathfrak{X}_T \equiv \{X_t\}_{t=1}^T$, and define

$$\Delta\hat{g}_T^{(dw)}(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h} \varphi_t \left\{X_t X_{t+h} B_k(t) - \frac{1}{T}\sum_{t=1}^{T-h} X_t X_{t+h} B_k(t)\right\}$$

Let $\{\Delta\hat{g}_T^{(dw)}(i)\}_{i=0}^{\mathcal{H}_T \mathcal{K}_T}$, etc., denote the stacked $\{\Delta\hat{g}_T^{(dw)}(h,k)\}_{h=0,k=1}^{\mathcal{H}_T,\mathcal{K}_T}$:

$$\Delta\hat{g}_T^{(dw)}(i) = \Delta\hat{g}_T^{(dw)}(h,k) \text{ with index correspondence } i = (k-1)\mathcal{H}_T + h. \tag{B.17}$$

and define

$$\hat{s}_T^2(i,j) = TE\left[\Delta\hat{g}_T^{(dw)}(i)\Delta\hat{g}_T^{(dw)}(j)|\mathfrak{X}_T\right] \text{ and } \mathring{s}_T^2(i,j) = TE\left[\Delta g_T^*(i)\Delta g_T^*(j)|\mathfrak{X}_T\right]$$

$$\Delta_T \equiv \max_{0 \le i,j \le \mathcal{H}_T\mathcal{K}_T}\left|\hat{s}_T^2(i,j) - \mathring{s}_T^2(i,j)\right|,$$

hence $\mathring{s}_T^2(i,i) \equiv \mathring{\sigma}_T^2(h,k)$ where $i = (k-1)\mathcal{H}_T + h$.

Let $\{\mathring{\mathbf{Z}}_T(i)\}_{T \le 1}$ be sequences of normal random variables $\mathring{\mathbf{Z}}_T(i) \sim N(0, \mathring{s}_T^2(i,i))$ independent of $\mathfrak{X}_T$. Lemma 3.1 in Chernozhukov, Chetverikov and Kato (2013), cf. Chernozhukov, Chetverikov and Kato (2015, Theorem 2, Proposition 1) and Chen (2018, Lemma C.1), implies:

$$\mathcal{E}_T \equiv \sup_{c>0}\left|P\left(\max_{0\le i\le \mathcal{H}_T\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(i)\right| \le c|\mathfrak{X}_T\right) - P\left(\max_{0\le i\le \mathcal{H}_T\mathcal{K}_T}\left|\mathring{\mathbf{Z}}_T(i)\right| \le c\right)\right| \tag{B.18}$$

$$= O_p\left(\Delta_T^{1/3}\max\left\{1, \ln\left(\mathcal{H}_T\mathcal{K}_T/\Delta_T\right)\right\}^{2/3}\right).$$

It suffices to have $\Delta_T = o_p(\ln(\mathcal{H}_T\mathcal{K}_T)^2)$: see the remark following Theorem 2 in Chernozhukov, Chetverikov and Kato (2015), cf. Chernozhukov, Chetverikov and Kato (2015, Proposition 1). Thus we need only show $\Delta_T = o_p(\ln(\mathcal{T})^2)$ given $\mathcal{H}_T = o(T)$ and $\mathcal{K}_T = o(T^\kappa)$.

We prove below $\Delta_T = O_p(1/T^\iota)$ for some $\iota > 0$. Thus, $\mathcal{H}_T = o(T)$ and $\mathcal{K}_T = o(T^\kappa)$ for some finite $\kappa > 0$ produce:

$$\mathcal{E}_T = O_p\left(\Delta_T^{1/3}\max\left\{1, \ln\left(\mathcal{H}_T\mathcal{K}_T/\Delta_T\right)\right\}^{2/3}\right)$$

$$= O_p\left(\Delta_T^{1/3}\max\left\{1, \ln\left(\sqrt{T}\mathcal{H}_T\mathcal{K}_T\right) + \ln\left(\sqrt{T}\Delta_T\right)\right\}^{2/3}\right) = O_p\left(\frac{1}{T^{\iota/3}}\left\{\ln(T)\right\}^{2/3}\right) \xrightarrow{p} 0.$$

This suffices to prove the claim in view of the correspondence $i = (k-1)\mathcal{H}_T + h$.

We now prove $\Delta_T = O_p(1/T^{\iota})$. Define for any $g \in \mathbb{R}$

$$\mathfrak{E}_{l,T}(h,k;g) \equiv \sum_{t=(l-1)b_T+1}^{lb_T} \left\{X_t X_{t+h} B_k(t) - g\right\},$$

and define

$$\hat{g}(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) \text{ and } g_T(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t).$$

By construction of φ_t via iid $\{\xi_l\}_{l=1}^{(T-h)/b_T}$, $\xi_l \sim N(0,1)$:

$$\Delta\hat{g}_T^{(dw)}(h,k) = \frac{1}{T}\sum_{l=1}^{(T-h)/b_T} \xi_l \mathfrak{E}_{l,T}(h,k;\hat{g}(h,k))$$

$$\Delta g_T^{*}(h,k) = \frac{1}{T}\sum_{l=1}^{(T-h)/b_T} \xi_l \mathfrak{E}_{l,T}(h,k;g_T(h,k)).$$

Serial independence, and independence of $\mathfrak{X}_T$, for ξ_t yield for some couplets (h,k) and $(\tilde{h},\tilde{k})$:

$$\begin{aligned}
\hat{s}_T^2(i,j) &= TE\left[\Delta\hat{g}_T^{(dw)}(i)\Delta\hat{g}_T^{(dw)}(j)|\mathfrak{X}_T\right] \\
&= TE\left[\frac{1}{T}\sum_{l=1}^{(T-h)/b_T} \xi_l \mathfrak{E}_{l,T}(h,k;\hat{g}(h,k))\frac{1}{T}\sum_{m=1}^{(T-\tilde{h})/b_T} \xi_m \mathfrak{E}_{m,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k}))|\mathfrak{X}_T\right] \\
&= \frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \mathfrak{E}_{l,T}(h,k;\hat{g}(h,k))\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k})).
\end{aligned}$$

Similarly:

$$\mathring{s}_T^2(i,j) = \frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \mathfrak{E}_{l,T}(h,k;g_T(h,k))\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_T(\tilde{h},\tilde{k})).$$

Now observe for any (i,j) and some associated couplets (h,k) and $(\tilde{h},\tilde{k})$:

$$\begin{aligned}
&\left|\hat{s}_T^2(i,j) - \mathring{s}_T^2(i,j)\right| \\
&\quad \leq \left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \left\{\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k})) - \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_T(\tilde{h},\tilde{k}))\right\}\right. \\
&\qquad\qquad \left.\times\left\{\mathfrak{E}_{l,T}(h,k;\hat{g}(h,k)) - \mathfrak{E}_{l,T}(h,k;g_T(h,k))\right\}\right| \\
&\qquad + \left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_T(\tilde{h},\tilde{k}))\left\{\mathfrak{E}_{l,T}(h,k;\hat{g}(h,k)) - \mathfrak{E}_{l,T}(h,k;g_T(h,k))\right\}\right|
\end{aligned}$$

$$+\left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\mathfrak{E}_{l,T}(h,k;g_T(h,k))\left\{\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k}))-\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_T(\tilde{h},\tilde{k}))\right\}\right|$$
$$=\mathcal{S}_{1,T}(h,k,\tilde{h},\tilde{k})+\mathcal{S}_{2,T}(h,k,\tilde{h},\tilde{k})+\mathcal{S}_{3,T}(h,k,\tilde{h},\tilde{k}).$$

It follows $\Delta_T = O_p(1/T^{\iota})$ for some tiny $\iota > 0$ if we show each:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathcal{S}_{i,T}(h,k,\tilde{h},\tilde{k})\right| = O_p\left(1/T^{\iota}\right). \tag{B.19}$$

Consider $\mathcal{S}_{2,T}(\cdot)$; $\mathcal{S}_{1,T}(\cdot)$ and $\mathcal{S}_{3,T}(\cdot)$ are similar. Use

$$\{X_tX_{t+h}B_k(t)-\hat{g}(h,k)\}-\{X_tX_{t+h}B_k(t)-g_T(h,k)\}=-\{\hat{g}(h,k)-g_T(h,k)\}$$

with Lemma B.5 to yield:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathcal{S}_{2,T}(h,k,\tilde{h},\tilde{k})\right|$$
$$\leq\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{t=(l-1)b_T+1}^{lb_T}\left\{X_tX_{t+h}B_{\tilde{k}}(t)-g_T(h,k)\right\}\right|\times b_T\max_{\mathcal{H}_T,\mathcal{K}_T}|\hat{g}(h,k)-g_T(h,k)|$$
$$=\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{t=1}^{T-h\vee\tilde{h}}\{X_tX_{t+h}-E[X_tX_{t+h}]\}B_k(t)\right|\times O_p\left(b_T/\sqrt{T}\right).$$

Moreover, by the same argument used to prove (B.5), we have for any $\{\mathcal{H}_T\}$, $0\leq\mathcal{H}_T<T-1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$ and $\eta(\mathcal{K}_T) = o(\sqrt{T})$:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{t=1}^{T-h\vee\tilde{h}}\left\{X_tX_{t+\tilde{h}}-E\left[X_tX_{t+\tilde{h}}\right]\right\}B_{\tilde{k}}(t)\right|=O_p(1/\sqrt{T}).$$

Therefore

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathcal{S}_{2,T}(h,k,\tilde{h},\tilde{k})\right|=O_p\left(1/\sqrt{T}\right)\times O_p\left(b_T/\sqrt{T}\right)=O_p\left(b_T/T\right)=o_p(1/T^{\iota}),$$

given $b_T = o(T^{1-\iota})$, proving (B.19).

Claim (b). Now let $\{\mathring{\mathbf{Z}}(h,k) : 0\leq h\leq\mathcal{H}_T, 1\leq k\leq\mathcal{K}_T\}$ be an independent copy of the Lemma 3.1 law $\mathbf{Z}(h,k)\sim N(0,\lim_{T\to\infty}\sigma_T^2(h,k))$, independent of the asymptotic draw $\{X_t\}_{t=1}^{\infty}$, where

$$\sigma_T^2(h,k)=E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}z_t(h,k)\right)^2\right].$$

Let $[\mathring{\mathbf{Z}}(i)]_{i=0}^{\mathcal{H}_T\mathcal{K}_T}$ be the stacked version of $\mathbf{Z}(h,k)$, cf. (B.17), and define

$$v^2(i,j)\equiv E[\mathring{\mathbf{Z}}(i)\mathring{\mathbf{Z}}(j)],$$

hence $v^2(i,i) \equiv \lim_{T\to\infty} \sigma_T^2(h,k)$ with $i = (k-1)\mathcal{H}_T + h$. We prove below

$$\tilde{\mathcal{E}}_T \equiv \sup_{c>0} \left| P\left(\max_{0\le i\le \mathcal{H}_T\mathcal{K}_T} \left|\mathring{\boldsymbol{Z}}_T(i)\right| \le c|\mathfrak{X}_T \right) - P\left(\max_{0\le i\le \mathcal{H}_T\mathcal{K}_T} \left|\mathring{\boldsymbol{Z}}(i)\right| \le c \right)\right| \xrightarrow{p} 0. \tag{B.20}$$

By claim (a) with (B.20):

$$\sup_{c>0} \left| P\left(\max_{\mathcal{H}_T,\mathcal{K}_T} \left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k)\right| \le c|\mathfrak{X}_T \right) - P\left(\max_{\mathcal{H}_T,\mathcal{K}_T} \left|\mathring{\boldsymbol{Z}}(h,k)\right| \le c \right)\right| \xrightarrow{p} 0.$$

Hence:

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k)\right| \xrightarrow{d} \max_{h,k\in\mathbb{N}} \left|\mathring{\boldsymbol{Z}}(h,k)\right| \ awp1 \text{ with respect to } \{X_t\}_{t=1}^{\infty}.$$

This gives as claimed by definition (cf. Giné and Zinn, 1990, Section 3):

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k)\right| \Rightarrow^p \max_{h,k\in\mathbb{N}} \left|\mathring{\boldsymbol{Z}}(h,k)\right|.$$

We now prove (B.20). With $\mathring{s}_T^2(i,j) = TE[\Delta g_T^*(i)\Delta g_T^*(j)|\mathfrak{X}_T]$ and $v^2(i,j)$ define

$$\tilde{\Delta}_T \equiv \max_{0\le i,j\le \mathcal{H}_T\mathcal{K}_T} \left|\mathring{s}_T^2(i,j) - v^2(i,j)\right|.$$

As above $\tilde{\mathcal{E}}_T = O_p(\tilde{\Delta}_T^{1/3} \times \max\{1, \ln(\mathcal{H}_T\mathcal{K}_T/\tilde{\Delta}_T)\}^{2/3})$. The proof is complete if we show

$$\tilde{\Delta}_T = O(1/T^{\iota}) \text{ for some } \iota > 0, \tag{B.21}$$

since then

$$\tilde{\mathcal{E}}_T = O_p\left(\tilde{\Delta}_T^{1/3} \max\left\{1, \ln(\mathcal{H}_T\mathcal{K}_T/\tilde{\Delta}_T)\right\}^{2/3}\right) = O_p\left(T^{-\iota/3}\left\{\ln(T)\right\}^{2/3}\right) \xrightarrow{p} 0.$$

We now prove (B.21). Define

$$\Delta\ddot{g}_T^*(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h} \varphi_t \left\{ X_t X_{t+h} B_k(t) - \frac{1}{T-h}\sum_{s=1}^{T-h} E\left[X_s X_{s+h}\right] B_k(s) \right\},$$

and let $\Delta\ddot{g}_T^*(i)$ stack $\Delta\ddot{g}_T^*(h,k)$. Define

$$\mathring{s}_T^2(i,j) = TE\left[\Delta g_T^*(i)\Delta g_T^*(j)|\mathfrak{X}_T\right]$$

$$\ddot{s}_T^2(i,j) = TE\left[\Delta\ddot{g}_T^*(i)\Delta\ddot{g}_T^*(j)|\mathfrak{X}_T\right]$$

$$s_T^2(i,j) = TE\left[\Delta\ddot{g}_T^*(i)\Delta\ddot{g}_T^*(j)\right]$$

$$s^2(i,j) = \lim_{T\to\infty} TE\left[\Delta\ddot{g}_T^*(i)\Delta\ddot{g}_T^*(j)\right].$$

We prove (B.21) by showing in order:

$$\max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T}\left|\mathring{s}_T^2(i,j)-\ddot{s}_T^2(i,j)\right| = O_p\left(T^{-\iota}\right) \tag{B.22}$$

$$\max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T}\left|\ddot{s}_T^2(i,j)-s_T^2(i,j)\right| = O_p(T^{-\iota}) \tag{B.23}$$

$$\max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T}\left|s_T^2(i,j)-s^2(i,j)\right| = O(T^{-\iota}) \tag{B.24}$$

$$\max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T}\left|s^2(i,j)-v^2(i,j)\right| = O(T^{-\iota}). \tag{B.25}$$

Step 1 **($\mathring{s}_T^2(i,j)$, $\ddot{s}_T^2(i,j)$).** Recall $g_T(h,k) \equiv 1/T\sum_{t=1}^{T-h} E[X_tX_{t+h}]B_k(t)$. After expanding, and cancelling like terms, we have for any (i,j) and some unique couplet $(h,k;\tilde{h},\tilde{k})$, where $i=(k-1)\mathcal{H}_T+h$ and $j=(k-1)\mathcal{H}_T+h$:

$$\begin{aligned}
&\left|\mathring{s}_T^2(i,j)-\ddot{s}_T^2(i,j)\right| \\
&\quad=\Bigg|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}\Big\{-g_T(h,k)X_tX_{t+\tilde{h}}B_{\tilde{k}}(t)-g_T(\tilde{h},\tilde{k})X_tX_{t+h}B_k(t) \\
&\qquad+g_T(h,k)g_T(\tilde{h},\tilde{k})+\frac{T}{T-h}g_T(h,k)X_tX_{t+\tilde{h}}B_{\tilde{k}}(t) \\
&\qquad+\frac{T}{T-\tilde{h}}g_T(\tilde{h},\tilde{k})X_tX_{t+h}B_k(t)-\frac{T}{T-h}\frac{T}{T-\tilde{h}}g_T(h,k)g_T(\tilde{h},\tilde{k})\Big\}\Bigg| \\
&\quad\leq\frac{h}{T-h}\left|g_T(h,k)\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}X_tX_{t+\tilde{h}}B_{\tilde{k}}(t)\right| \\
&\qquad+\frac{\tilde{h}}{T-\tilde{h}}\left|g_T(\tilde{h},\tilde{k})\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}X_tX_{t+h}B_k(t)\right| \\
&\qquad+\frac{T(h+\tilde{h})+h\tilde{h}}{(T-h)\left(T-\tilde{h}\right)}\left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}g_T(h,k)g_T(\tilde{h},\tilde{k})\right| \\
&\quad=\mathcal{D}_{T,1}(h,k;\tilde{h},\tilde{k})+\mathcal{D}_{T,2}(h,k;\tilde{h},\tilde{k})+\mathcal{D}_{T,3}(h,k;\tilde{h},\tilde{k})
\end{aligned}$$

Now twice use the fact that Assumption 1.b implies uniform $\mathcal{L}_r$-boundedness for any $r\geq 1$, with Lemma B.5, and $\mathcal{H}_T=O(T^{1-\iota})$ for any $\iota\in(0,1)$:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\mathcal{D}_{T,1}(h,k;\tilde{h},\tilde{k})\leq\max_{\mathcal{H}_T,\mathcal{K}_T}\left\{\frac{h}{T-h}\left|g_T(h,k)\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}X_tX_{t+\tilde{h}}B_{\tilde{k}}(t)\right|\right\}$$

$$\leq K \frac{\mathcal{H}_T}{T - \mathcal{H}_T} \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} X_t X_{t+\tilde{h}} B_{\tilde{k}}(t) \right|$$

$$= K \frac{\mathcal{H}_T}{T - \mathcal{H}_T} \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{t=(l-1)b_T+1}^{lb_T} X_t X_{t+\tilde{h}} B_{\tilde{k}}(t) \right|$$

$$= O_p\left(\frac{\mathcal{H}_T}{T - \mathcal{H}_T}\right) = O_p\left(\frac{\mathcal{H}_T}{T}\right) = O_p(1/T^{\iota}).$$

Similarly, $\max_{\mathcal{H}_T, \mathcal{K}_T} \mathcal{D}_{T,2}(h, k; \tilde{h}, \tilde{k}) = o_p\left(1/T^{1-a}\right)$. Furthermore, use for any $h \vee \tilde{h} \in \{1, ..., \mathcal{H}_T\}$:

$$\left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} g_T(h,k) g_T(\tilde{h}, \tilde{k}) \right| \leq b_T \left| g_T(h,k) g_T(\tilde{h}, \tilde{k}) \right|$$

with $\mathcal{H}_T = O(T^{1-\iota}/b_T)$, $b_T/T^{\iota} \to \infty$, and $b_T = o(T^{1/2-\iota})$ to arrive at:

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \mathcal{D}_{T,3}(h, k; \tilde{h}, \tilde{k}) \leq \max_{\mathcal{H}_T, \mathcal{K}_T} \left\{ \frac{T(h + \tilde{h}) + h\tilde{h}}{(T - h)(T - \tilde{h})} \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} g_T(h,k) g_T(\tilde{h}, \tilde{k}) \right| \right\}$$

$$\leq b_T \frac{2T\mathcal{H}_T + \mathcal{H}_T^2}{(T - \mathcal{H}_T)^2} \leq K \frac{b_T \mathcal{H}_T}{T} (1 + o(1)) = O\left(T^{-\iota}\right),$$

proving (B.22). For example, if $\mathcal{H}_T = o(T^a)$ for any $a \in (0, 1/2)$ suffices.

Step 2 ($\ddot{s}_T^2(i,j)$, $s_T^2(i,j)$). Write

$$\ddot{g}_T(h,k) \equiv \frac{1}{T - h} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t).$$

For some unique couplet $(h, k; \tilde{h}, \tilde{k})$ with $i = (k-1)\mathcal{H}_T + h$ and $j = (k-1)\mathcal{H}_T + h$, expand terms in $\ddot{s}_T^2(i,j)$, and use

$$\frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} = \left(1 - h \vee \tilde{h}/T\right) b_T$$

to deduce:

$$\ddot{s}_T^2(i,j) = \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} X_s X_{s+h} X_t X_{t+\tilde{h}} B_k(s) B_k(t) \tag{B.26}$$

$$- \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s=(l-1)b_T+1}^{lb_T} X_s X_{s+h} B_k(s) \times \ddot{g}_T(\tilde{h}, \tilde{k})$$

$$- \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{t=(l-1)b_T+1}^{lb_T} X_t X_{t+\tilde{h}} B_{\tilde{k}}(t) \times \ddot{g}_T(h,k)$$
$$+ (1 - \{h \vee \tilde{h}\}/T) b_T \ddot{g}_T(h,k) \ddot{g}_T(\tilde{h},\tilde{k}).$$

Now use $s_T^2(i,j) = E[\ddot{s}_T^2(i,j)]$ to obtain:

$$\max_{0 \leq i,j \leq \mathcal{H}_T \mathcal{K}_T} \left| \ddot{s}_T^2(i,j) - s_T^2(i,j) \right| \leq \mathcal{D}_{1,T} + \mathcal{D}_{2,T},$$

where

$$\mathcal{D}_{1,T} = \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} \left\{ X_s X_{s+h} X_t X_{t+\tilde{h}} - E\left[X_s X_{s+h} X_t X_{t+\tilde{h}} \right] \right\} B_k(s) B_{\tilde{k}}(t) \right|$$
$$\mathcal{D}_{2,T} = 2 \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s=(l-1)b_T+1}^{lb_T} \left\{ X_s X_{s+h} - E\left[X_s X_{s+h} \right] \right\} B_k(s) \times \ddot{g}_T(\tilde{h},\tilde{k}) \right|.$$

Consider $\mathcal{D}_{1,T}$ and write

$$\mathfrak{X}_{T,l}(h,k) \equiv \frac{1}{\sqrt{b_T}} \sum_{t=(l-1)b_T+1}^{lb_T} X_t X_{t+h} B_k(t)$$
$$\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k}) \equiv \mathfrak{X}_{T,l}(h,k) \mathfrak{X}_{T,l}(\tilde{h},\tilde{k}),$$

hence:

$$\mathcal{D}_{1,T} = \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \left(\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k}) - E\left[\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k}) \right] \right) \right|.$$

Let $[\boldsymbol{Y}_{T,l}(i,j)]_{i,j=0}^{\mathcal{H}_T \mathcal{K}_T}$ stack $\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k})$, with correspondence $i = (k-1)\mathcal{H}_T + h$ and $j = (\tilde{k} - 1)\mathcal{H}_T + \tilde{h}$. Similarly $[\mathring{\boldsymbol{Y}}_{T,l}(m)]_{m=0}^{\mathcal{H}_T^2 \mathcal{K}_T^2}$ stacks $[\boldsymbol{Y}_{T,l}(i,j)]_{i,j=0}^{\mathcal{H}_T \mathcal{K}_T}$ with $m = (j-1)\mathcal{H}_T \mathcal{K}_T + i$. Hence

$$\mathcal{D}_{1,T} = \max_{0 \leq m \leq \mathcal{H}_T^2 \mathcal{K}_T^2 + 1} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \left(\mathring{\boldsymbol{Y}}_{T,l}(m) - E\left[\mathring{\boldsymbol{Y}}_{T,l}(m) \right] \right) \right|.$$

We show below that $\mathring{\boldsymbol{Y}}_{T,l}(m)$ satisfies AS1-AS3 in Chang, Jiang and Shao (2023, p. 990). Hence their Gaussian approximation Proposition 3 holds provided $\mathcal{K}_T = o(T^\kappa)$ for some $\kappa > 0$ and $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, similar to (B.5) and (B.6) in the proof of Lemma 3.1. In view of asymptotic Gaussianicity, it follows by standard arguments and $(\mathcal{H}_T, \mathcal{K}_T) = o(T)$,

$$\sqrt{T/b_T} \mathcal{D}_{1,T} = O_p\left(\ln\left((\mathcal{H}_T \mathcal{K}_T)^2 \right) \right) \text{ hence } \mathcal{D}_{1,T} = o_p\left(\frac{1}{T^{1/2+\iota}} \ln(T) \right) = o_p\left(\frac{1}{\sqrt{T}} \right).$$

Now consider $\mathring{Y}_{T,l}(m)$. For sub-exponential tails AS1, Bonferroni's inequality and Lemma A.2 to imply for some $\tilde{\varpi} > 0$:

$$\max_{0\leq m\leq \mathcal{H}_T^2\mathcal{K}_T^2}\max_{1\leq l\leq T} P\left(\left|\mathring{Y}_{T,l}(m)\right| > c\right)$$

$$= \max_{\mathcal{H}_T,\mathcal{K}_T}\max_{1\leq l\leq T} P\left(\left|\frac{1}{b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T} X_s X_{t+h} X_t X_{t+\tilde{h}} B_k(s) B_{\tilde{k}}(t)\right| > c\right)$$

$$\leq b_T^2 \max_{\mathcal{H}_T}\max_{1\leq t\leq T-h} P\left(\left|X_s X_{t+h} X_t X_{t+\tilde{h}}\right| > b_T c\right)$$

$$\leq K b_T^2 \exp\left\{-c^{\tilde{\varpi}} b_T^{\tilde{\varpi}}\right\}.$$

Now exploit $b_T/T^{\iota} \to \infty$ to deduce $\forall c > 0\ \exists \mathcal{T} > 0$ such that

$$b_T^2 \exp\left\{-c^{\tilde{\varpi}} b_T^{\tilde{\varpi}}\right\} \leq \exp\{-c^{\tilde{\varpi}/2} b_T^{\tilde{\varpi}/2}\} \leq \exp\{-c^{\tilde{\varpi}/2}\}\ \forall T \geq \mathcal{T},$$

hence AS1. Mixing AS2 holds by Assumption 1.a and measurability: $[Y_{T,l}(i,j)]_{i,j=0}^{\mathcal{H}_T\mathcal{K}_T}$ is $\sigma(X_\tau : \tau \leq t + \mathcal{H}_T)$-measurable, with mixing coefficients $\mathring{\alpha}(l) \leq \alpha(|l - \mathcal{H}_T|_+) \leq K_1 \exp\{-K_2 |l - \mathcal{H}_T|_+^{\phi}\}$. Nondegeneracy AS3 holds by Assumption 2.a(i).

For $\mathcal{D}_{2,T}$, use Lemma B.5, and $b_T = O(T^{1/2-\iota})$ under Assumption 2.b, to get:

$$\mathcal{D}_{2,T} \leq K \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T/b_T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s=(l-1)b_T+1}^{lb_T}\left\{X_s X_{s+h} - E\left[X_s X_{s+h}\right]\right\} B_k(s)\right|$$

$$= K b_T \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{t=1}^{T-h\vee\tilde{h}}\left\{X_t X_{t+h} - E\left[X_t X_{t+h}\right]\right\} B_k(t)\right| = O_p\left(b_T/T^{1/2}\right) = O_p\left(T^{-\iota}\right).$$

Step 3 ($s_T^2(i,j)$, $s^2(i,j)$). The property holds by Assumption 2.a(ii).

Step 4 ($s^2(i,j), v^2(i,j)$). For some $(h,k;\tilde{h},\tilde{k})$, $s^2(i,j)$ is identically

$$\lim_{T\to\infty}\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T} E\left[\left\{X_s X_{s+h} B_k(s) - \frac{1}{T-h}\sum_{u=1}^{T-h} E\left[X_u X_{u+h}\right] B_k(u)\right\}\right.$$
$$\left.\times\left\{X_t X_{t+\tilde{h}} B_{\tilde{k}}(t) - \frac{1}{T-\tilde{h}}\sum_{u=1}^{T-\tilde{h}} E\left[X_u X_{u+\tilde{h}}\right] B_{\tilde{k}}(u)\right\}\right]$$

and by rearranging terms

$$v^2(i,j) = \lim_{T\to\infty}\frac{1}{T}\sum_{s,t=1}^{T-h\vee\tilde{h}} E\left[\left\{X_s X_{s+h} B_k(s) - \frac{1}{T-h}\sum_{u=1}^{T-h} E\left[X_u X_{u+h}\right] B_k(u)\right\}\right.$$
$$\left.\times\left\{X_t X_{t+\tilde{h}} B_{\tilde{k}}(t) - \frac{1}{T-\tilde{h}}\sum_{u=1}^{T-\tilde{h}} E\left[X_u X_{u+\tilde{h}}\right] B_{\tilde{k}}(u)\right\}\right].$$

Further, block size $b_T \to \infty$. Hence $s^2(i,j) = v^2(i,j)\ \forall i,j$. This completes the proof. $\mathcal{QED}$.

Remark 2. We technically only need the iid random numbers $\{\xi_1,\ldots,\xi_{N_T}\}$ to satisfy $E[\xi_i]=0$, $E[\xi_i^2]$ $= 1$, and $E[\xi_i^4] < \infty$. Thus $\sqrt{T}\Delta\hat{g}_T^{(dw)}(i)|\mathfrak{X}_T$ need not be Gaussian, hence the Gaussian-to-Gaussian result (B.18) may not hold. We will need the added Gaussian approximation step:

$$\sup_{c>0}\left|P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(i)\right|\leq c|\mathfrak{X}_T\right)-P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T}\left|\sqrt{T}\Delta\mathring{\boldsymbol{g}}_T(i)|\mathfrak{X}_T\right|\leq c\right)\right|\xrightarrow{p}0$$

where $\sqrt{T}\Delta\mathring{\boldsymbol{g}}_T(i)|\mathfrak{X}_T \sim N(0, TE[\Delta\hat{g}_T^{(dw)}(i)^2|\mathfrak{X}_T])$. We would then need to alter (B.18), and prove instead

$$\begin{aligned}\mathcal{E}_T &\equiv \sup_{c>0}\left|P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T}\left|\sqrt{T}\Delta\mathring{\boldsymbol{g}}_T(i)\right|\leq c|\mathfrak{X}_T\right)-P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T}\left|\mathring{\boldsymbol{Z}}_T(i)\right|\leq c\right)\right|\\ &= O_p\left(\Delta_T^{1/3}\max\left\{1,\ln\left(\mathcal{H}_T\mathcal{K}_T/\Delta_T\right)\right\}^{2/3}\right)\xrightarrow{p}0.\end{aligned}$$

Recall

$$H_1 : E\left[X_t X_{t+h}\right] = \gamma_h + c_h(t/T) \tag{B.27}$$

where

$$\liminf_{T\to\infty}\max_{h,k\in\mathbb{N}}\left|\int_0^1 c_h(u)\,\mathcal{B}_k(u)du\right| > 0. \tag{B.28}$$

Theorem 4.1. *Let Assumptions 1.b,c,d and 2 hold, let $\mathcal{H}_T,\mathcal{K}_T\to\infty$, and let the number of bootstrap samples $M = M_T \to \infty$ as $T\to\infty$. Let $\{b_T,\mathcal{H}_T\}$ satisfy $b_T\to\infty$ and $b_T = O(T^{1/2-\iota})$, $0\leq\mathcal{H}_T\leq T-1$, and under Assumption 1.a $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, or $\mathcal{H}_T = O(T^{1-\iota}/b_T)$ for tiny $\iota>0$ under Assumption 1.a*. Under H_0, $P(\hat{p}_{T,M}^{(dw)}<\alpha)\to\alpha$ for any sequence $\{\mathcal{K}_T\}$ satisfying $\mathcal{K}_T = o(T^\kappa)$ for some finite $\kappa>0$ and $\eta(\mathcal{K}_T) = o(\sqrt{T})$. Under H_1 in (B.27) where $c_h(\cdot)$ satisfy (B.28), $P(\hat{p}_{T,M}^{(dw)}<\alpha)\to 1$ for any $\{\mathcal{K}_T\}$.*

Proof.

Step 1: Impose mixing Assumption 1.a. Operate conditionally on the sample $\mathfrak{X}_T \equiv \{X_t\}_{t=1}^T$. Define max-*covariance* differences

$$\check{\mathcal{M}}_T \equiv \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}(\hat{\gamma}_h^{(k)}-\hat{\gamma}_h)\right| \text{ and } \check{\mathcal{M}}_T^{(dw)} \equiv \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k)\right|.$$

Compare this to, e.g., the max-*correlation* difference $\mathcal{M}_T^{(dw)} \equiv \hat{\gamma}_0^{-1}\max_{\mathcal{H}_T,\mathcal{K}_T}|\sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k)|$. Thus, by construction:

$$\hat{p}_{T,M}^{(dw)} \equiv \frac{1}{M}\sum_{i=1}^M I\left(\mathcal{M}_{T,i}^{(dw)}\geq\mathcal{M}_T\right) = \frac{1}{M}\sum_{i=1}^M I\left(\check{\mathcal{M}}_{T,i}^{(dw)}\geq\check{\mathcal{M}}_T\right). \tag{B.29}$$

It suffices to prove the claim for the bootstrapped p-value based on $\check{\mathcal{M}}_T$ and $\check{\mathcal{M}}_{T,i}^{(dw)}$.

By the Glivenko-Cantelli theorem, as $M \to \infty$,

$$\hat{p}^{(dw)}_{T,M} \xrightarrow{p} P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}^{(dw)}_T(h,k)\right| \geq \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}(\hat{\gamma}^{(k)}_h - \hat{\gamma}_h)\right| \mid \mathfrak{X}_T\right). \tag{B.30}$$

Further, $\max_{\mathcal{H}_T,\mathcal{K}_T}|\sqrt{T}\Delta\hat{g}^{(dw)}_T(h,k)| \Rightarrow^p \max_{h,k\in\mathbb{N}}|\mathring{\mathbf{Z}}(h,k)|$ by Lemma B.4, hence

$$\sup_{c>0}\left|P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}^{(dw)}_T(h,k)\right| \leq c|\mathfrak{X}_T\right) - P\left(\max_{h,k\in\mathbb{N}}\left|\mathring{\mathbf{Z}}(h,k)\right| \leq c\right)\right| \xrightarrow{p} 0, \tag{B.31}$$

where $\{\mathring{\mathbf{Z}}(h,k) : h,k\in\mathbb{N}\}$ is an independent copy of $\mathbf{Z}(h,k) \sim N(0, \lim_{T\to\infty}\sigma^2_T(h,k))$ from Lemma 3.1, independent of the asymptotic draw $\mathfrak{X}_\infty$. See Giné and Zinn (1990, eq. (3.4)).

Now impose H_0 and define $\bar{F}^{(0)}_T(c) \equiv P(\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathring{\mathbf{Z}}(h,k)| > c)$. Limit (B.31) implies:

$$P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}^{(dw)}_T(h,k)\right| \geq \check{\mathcal{M}}_T|\mathfrak{X}_T\right) - P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathring{\mathbf{Z}}(h,k)\right| \geq \check{\mathcal{M}}_T\right) \xrightarrow{p} 0.$$

$[\mathring{\mathbf{Z}}(h,k)]^{\mathcal{H}_T,\mathcal{K}_T}_{h=0,k=1}$ is independent of $\mathfrak{X}_T$, hence:

$$P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}^{(dw)}_T(h,k)\right| \geq \check{\mathcal{M}}_T|\mathfrak{X}_T\right) - \bar{F}^{(0)}_T\left(\check{\mathcal{M}}_T\right) \xrightarrow{p} 0. \tag{B.32}$$

$\bar{F}^{(0)}_T$ is continuous by Gaussianicity, thus Lemma 3.1 and Slutsky's theorem yield:

$$\left|\bar{F}^{(0)}_T\left(\check{\mathcal{M}}_T\right) - \bar{F}^{(0)}_T\left(\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathbf{Z}(h,k)|\right)\right| \xrightarrow{p} 0. \tag{B.33}$$

Together, with (B.30), (B.32) and (B.33) we have for any sequence of integers $\{M_T\}$, $M_T \to \infty$:

$$\hat{p}^{(dw)}_{T,M_T} = \bar{F}^{(0)}_T\left(\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathbf{Z}(h,k)|\right) + o_p(1). \tag{B.34}$$

Further, $\bar{F}^{(0)}_T(\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathbf{Z}(h,k)|)$ is distributed uniform on $[0,1]$ since $\{\mathring{\mathbf{Z}}(h,k) : h,k\in\mathbb{N}\}$ is an independent copy of $\{\mathbf{Z}(h,k) : h,k\in\mathbb{N}\}$. Thus $P(\hat{p}^{(dw)}_{T,M_T} < \alpha) = P(\bar{F}^{(0)}_T(\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathbf{Z}(h,k)|) < \alpha) + o(1) = \alpha + o(1) \to \alpha$ from (B.34) as required.

Next, impose H_1 defined by (B.27), with drift/basis property (B.28). Thus

$$\frac{1}{T}\sum_{t=1}^{T-h} E[X_t X_{t+h}]B_k(t) \to \int_0^1 c_h(u)\,\mathcal{B}_k(u)du \neq 0 \text{ for some } h \text{ and } k. \tag{B.35}$$

By the triangle inequality, Lemma 3.1, $\hat{\gamma}^{(k)}_h - \hat{\gamma}_h = 1/T\sum_{t=1}^{T-h} X_t X_{t+h}B_k(t)$ and the definition of $\check{\mathcal{M}}_T$:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right]B_k(t)\right|$$
$$\leq \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\{X_t X_{t+h}B_k(t) - E\left[X_t X_{t+h}\right]B_k(t)\}\right| + \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}(\hat{\gamma}^{(k)}_h - \hat{\gamma}_h)\right| = O_p(1) + \check{\mathcal{M}}_T.$$

Lemma 3.1 and (B.35) therefore yield:

$$\check{\mathcal{M}}_T \geq \sqrt{T} \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \int_0^1 c_h(u) \mathcal{B}_k(u) du + o(1) \right| + O_p(1) \xrightarrow{p} \infty. \tag{B.36}$$

Finally, combine (B.30), (B.31) and (B.36) to deduce $P(\hat{p}^{(dw)}_{T,M_T} < \alpha) \rightarrow 1$ for any $\alpha \in (0, 1)$ because:

$$\begin{aligned}\hat{p}^{(dw)}_{T,M_T} &= P\left(\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \sqrt{T} \Delta \hat{g}^{(dw)}_T(h,k) \right| \geq \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \sqrt{T}(\hat{\gamma}^{(k)}_h - \hat{\gamma}_h) \right| \mid \mathfrak{X}_T \right) + o_p(1) \\ &= P\left(\max_{\mathcal{H}_T, \mathcal{K}_T} |\mathbf{Z}(h,k)| \geq \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \sqrt{T}(\hat{\gamma}^{(k)}_h - \hat{\gamma}_h) \right| \right) + o_p(1) \\ &= \bar{F}^{(0)}_T \left(\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \sqrt{T}(\hat{\gamma}^{(k)}_h - \hat{\gamma}_h) \right| \right) + o_p(1) \xrightarrow{p} 0.\end{aligned}$$

This proves the claim.

Step 2: The proof is identical under physical dependence Assumption 1.a*, except Lemmas 3.1* and B.4* below replace Lemmas 3.1 and B.4. $\mathcal{QED}$.

B.2. Physical dependence

B.2.1. Preliminary results

We first prove sub-exponential tails and physical dependence naturally carry over to:

$$z_t(h,k) \equiv X_t X_{t+h} B_k(t) - E[X_t X_{t+h}] B_k(t).$$

We also derive an upper bound on so-called *dependence adjusted norms*, here combined into one statement (cf. Chang, Chen and Wu, 2024, Wu and Wu, 2016). The latter will be used to exploit a high dimensional central limit theorem due to Chang, Chen and Wu (2024, Theorem 3).

$[\cdot]_+$ rounds to the nearest greater integer.

Lemma B.7. *Let Assumption 1.b hold.*

a. $\max_{t,T,\mathcal{H}_T,\mathcal{K}_T} P(|z_t(h,k)| > c) \leq \tilde{\vartheta}_1 \exp\{-\tilde{\vartheta}_2 c^{\varpi/2}\}$ $\forall c > 0$, *and some universal constants* $\tilde{\vartheta}_1 > 1$ *and* $\tilde{\vartheta}_2 > 0$.

b. $\max_{t,T,\mathcal{H}_T,\mathcal{K}_T} ||z_t(h,k)||_p \leq cp^{\varphi}$ *where* $\varphi = [2/\varpi]_+$, *and c depends only on* $(\tilde{\vartheta}_1, \tilde{\vartheta}_2, \varpi)$.

Proof.

Claim (a). Apply Young and Bonferroni inequalities, $B^2_k(t) = 1$ and Assumption 1.b:

$$\begin{aligned}&\max_{t,T,\mathcal{H}_T,\mathcal{K}_T} P(|X_t X_{t+h} B_k(t)| > c) \qquad \text{(B.37)} \\ &\quad \leq \max_{t,T,\mathcal{H}_T,\mathcal{K}_T} P\left(\frac{1}{2} X^2_t + \frac{1}{2} X^2_{t+h} > c \right)\end{aligned}$$

$$\leq \max_{t,T,\mathcal{H}_T,\mathcal{K}_T} P\left(|X_t| > c^{1/2}\right) + \max_{t,T,\mathcal{H}_T,\mathcal{K}_T} P\left(|X_{t+h}| > c^{1/2}\right)$$
$$\leq 2\vartheta_1 \exp\{-\vartheta_2 c^{\varpi/2}\}.$$

Therefore $\max_{t,T,\mathcal{H}_T} |E[X_t X_{t+h}]| < \infty$.

Now let $\mathcal{K} \geq \max_{t,T,\mathcal{H}_T} \exp\{\tilde{\vartheta}_2(1 + |E[X_t X_{t+h}]|)^a\}$ for some $a > \varpi/2$ and $\tilde{\vartheta}_2 \in (0, \vartheta_2)$. Using (B.37), after tedious work it can be shown that there exist large $a > 0$ and small $\tilde{\vartheta}_2 > 0$ such that $\forall c > 0$:

$$\begin{aligned} &P\left(|X_t X_{t+h} B_k(t) - E\left[X_t X_{t+h}\right] B_k(t)| > c\right) \\ &\quad \leq P\left(|X_t X_{t+h}| > -|E\left[X_t X_{t+h}\right]| + c\right) \\ &\quad \leq \{2\vartheta_1 \vee 1\} \exp\left\{-\vartheta_2 \left[c - |E\left[X_t X_{t+h}\right]|\right]^{\varpi/2} I\left(c > |E\left[X_t X_{t+h}\right]|\right)\right\} \\ &\quad \leq \{2\vartheta_1 \vee 1\} \mathcal{K} \exp\left\{-\tilde{\vartheta}_2 c^{\varpi/2}\right\} \\ &\quad = \tilde{\vartheta}_1 \exp\left\{-\tilde{\vartheta}_2 c^{\varpi/2}\right\} \text{ where } \tilde{\vartheta}_1 > 1 \text{ by construction.} \end{aligned}$$

Claim (b). By a change of variables, (a), and $\tilde{\vartheta}_2 \int_0^\infty v^a \exp\{-\tilde{\vartheta}_2 v\} dv = a!/\tilde{\vartheta}_2^a$ for $a \in \mathbb{N}$:

$$\begin{aligned} \max_{t,T,\mathcal{H}_T,\mathcal{K}_T} \mathbb{E}|z_t(h,k)|^p &= p \max_{t,T,\mathcal{H}_T,\mathcal{K}_T} \int_0^\infty u^{p-1} P\left(|z_t(h,k)| > u\right) du \\ &\leq p\mathcal{K} \int_0^\infty u^{p-1} \exp\{-\tilde{\vartheta}_2 c^{\varpi/2}\} du \\ &= \frac{2p\tilde{\vartheta}_1}{\varpi\tilde{\vartheta}_2} \vartheta_2 \int_0^\infty v^{2p/\varpi - 1} \exp\{-\tilde{\vartheta}_2 v\} dv \\ &\leq \frac{2p\tilde{\vartheta}_1}{\varpi\tilde{\vartheta}_2} \frac{[2p/\varpi - 1]_+!}{\tilde{\vartheta}_2^{[2p/\varpi - 1]_+}} \\ &\leq \tilde{\vartheta}_1 \frac{[2p/\varpi]_+!}{\tilde{\vartheta}_2^{[2p/\varpi]_+} \vee 1} \\ &\leq \tilde{\vartheta}_1 \left(p \left\{\left[\frac{2}{\tilde{\vartheta}_2 \varpi}\right]_+\right\}\right)^{p\{[2/\varpi]_+\}}. \end{aligned}$$

Thus, given $\tilde{\vartheta}_1 > 1$ and $p \geq 1$:

$$\max_{t,T,\mathcal{H}_T,\mathcal{K}_T} \left(\mathbb{E}|z_t(h,k)|^p\right)^{1/p} \leq \tilde{\vartheta}_1 \left(p \left[\frac{2}{\tilde{\vartheta}_2 \varpi}\right]_+\right)^{[2/\varpi]_+} = \tilde{\vartheta}_1 \left[\frac{2}{\tilde{\vartheta}_2 \varpi}\right]_+^{[2/\varpi]_+} p^{[2/\varpi]_+}.$$

$\mathcal{QED}$.

Lemma B.8. *Under Assumption 1.a** $\{z_t(h,k)\}$ *is uniformly* $\mathcal{L}_{p/2}$*-physical dependent for some* $p \geq 4$ *with size* $\lambda \geq 1$*:*

$$\theta_t^{(z)}(h,m) \equiv \left\|z_t(h,k) - z_t'(m,h,k)\right\|_{p/2} \leq d_t^{(z)}(h) m^{-\lambda-\iota}$$

for some $d_t^{(z)}(h)$ that depends only on $||X_t||_p \vee ||X_{t+h}||_p$ up to a universal multiplicative constant. In particular, for some $\varphi > 0$ and tiny $\iota > 0$:

$$\sup_{p\geq 2} p^{-\varphi}\left\{\sup_{m\geq 0}(m+1)^{\lambda-1+\iota/2}\sum_{m=0}^{\infty}\max_{t,T,\mathcal{H}_T,\mathcal{K}_T}\left\|z_t(h,k)-z_t'(m,h,k)\right\|_{p/2}\right\}\leq K.$$

Proof. Note that ϵ_t' are an iid copy of iid ϵ_t, hence the coupled version of $z_t(h,k)$ is

$$z_t'(m,h,k) = X_t'(m)X_{t+h}'(m)B_k(t) - E\left[X_tX_{t+h}\right]B_k(t).$$

By Minkowski and Cauchy- Schwartz inequalities, $|B_k(t)| = 1$, and innovation independence and therefore

$$\left\|X_t'(m)\right\|_p = \|X_t\|_p\,,$$

it follows:

$$\begin{aligned}\left\|z_t(h,k)-z_t'(m,h,k)\right\|_{p/2} &\leq \left\|X_tX_{t+h}-X_t'(m)X_{t+h}'(m)\right\|_{p/2}\\ &\leq \|X_{t+h}\|_p\left\|X_t-X_t'(m)\right\|_p+\|X_t\|_p\left\|X_{t+h}-X_{t+h}'(m)\right\|_p.\end{aligned}$$

Now use Assumption 1.a* to deduce for any k:

$$\left\|z_t(h,k)-z_t'(m,h,k)\right\|_{p/2}\leq K\left(\|X_{t+h}\|_p\,d_t^{(p)}+\|X_t\|_p\,d_{t+h}^{(p)}\right)\psi_m.$$

Using (A.1) and $\psi_m = O(m^{-\lambda-\iota})$ for $m \geq 1$ we therefore have

$$\begin{aligned}\left\|z_t(h,k)-z_t'(m,h,k)\right\|_{p/2} &\leq K\left\{d_t^{(p)}\vee d_{t+h}^{(p)}\right\}\left\{\|X_t\|_p\vee\|X_{t+h}\|_p\right\}m^{-\lambda-\iota}\\ &\leq K\left\{\|X_t\|_p\vee\|X_{t+h}\|_p\right\}^2\{m\vee 1\}^{-\lambda-\iota}\\ &= d_t^{(z)}(h)\{m\vee 1\}^{-\lambda-\iota}.\end{aligned}$$

Next, the argument used to prove Lemma B.7.c yields for some $\varphi > 0$ and every $p \geq 1$:

$$\max_{t,T,\mathcal{H}_T,\mathcal{K}_T}\left\{\|X_t\|_p\vee\|X_{t+h}\|_p\right\}\leq Kp^{\varphi}.$$

Therefore, for tiny $\iota > 0$, and $\lambda \geq 1$:

$$\begin{aligned}&\sup_{p\geq 2}p^{-\varphi}\left\{\sup_{m\geq 0}\left[(m+1)^{\lambda-1+\iota/2}\sum_{l=m}^{\infty}\max_{t,T,\mathcal{H}_T,\mathcal{K}_T}\left\|z_t(h,k)-z_t'(l,h,k)\right\|_{p/2}\right]\right\}\\ &\quad\leq\sup_{p\geq 2}p^{-\varphi}\left\{Kp^{\varphi}\sup_{m\geq 0}\left[(m+1)^{\lambda-1+\iota/2}\sum_{l=m}^{\infty}\frac{1}{l^{\lambda+\iota}}\right]\right\}\\ &\quad\leq\sup_{p\geq 2}p^{-\varphi}\left\{Kp^{\varphi}\sup_{m\geq 0}\left[(m+1)^{\lambda-1+\iota/2}\frac{1}{m^{\lambda-1+\iota/2}}\right]\sum_{l=1}^{\infty}\frac{1}{l^{1+\iota/2}}\right\}\\ &\quad\leq K\sup_{p\geq 2}\{p^{-\varphi}p^{\varphi}\}=K,\end{aligned}$$

as claimed. $\mathcal{QED}$.

Lemma B.9. *Under Assumption 1.a**

$$\left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} z_t(h,k) \right\|_p \leq \sqrt{2p} \sum_{m=0}^{\infty} \max_{1\leq t\leq T} \theta_t^{(z)}(h,m). \tag{B.38}$$

Proof. Write $\mathcal{Z}_{(i),T} \equiv 1/\sqrt{T}\sum_{t=1}^{T} z_t(h,k)$. Wu (2005, Theorem 2.i) provides a truncated proof of (B.38) under stationarity. The following completes the proof in a general setting.

Define $\xi_t \equiv \{\epsilon_t, \epsilon_{,t-1}, ..\}$ and

$$\mathcal{M}_{r,m}(h,k) \equiv \sum_{l=1}^{m} \{E\left[z_l(h,k)|\xi_{l-r}\right] - E\left[z_l(h,k)|\xi_{l-r-1}\right]\}$$

$$y_l^{(r)}(h,k) \equiv \mathbb{E}(z_l(h,k)|\xi_{l-r}) - \mathbb{E}(z_l(h,k)|\xi_{l-r-1}).$$

Then

$$\sum_{t=1}^{T} z_t(h,k) = \sum_{k=0}^{\infty} \mathcal{M}_{r,T}(h,k),$$

hence by Minkowski's inequality

$$\left\| \sum_{t=1}^{T} z_t(h,k) \right\|_p \leq \sum_{r=0}^{\infty} \left\| \sum_{l=1}^{T} y_l^{(r)}(h,k) \right\|_p .$$

Define

$$\mathcal{A}_j^{(r)}(h,k) \equiv \sigma(y_1^{(r)}(h,k), ..., y_j^{(r)}(h,k)),$$

hence $\mathcal{A}_j^{(r)}(h,k) = \sigma(\xi_{j-r})$. Now apply Proposition 4 in Dedecker and Doukhan (2003) to $||\sum_{l=1}^{T} y_l^{(r)}(h,k)||_p$:

$$\begin{aligned} \left\| \sum_{l=1}^{T} y_l^{(r)}(h,k) \right\|_p &\leq \sqrt{2p} \left(\sum_{j=1}^{T} \max_{j\leq l\leq T} \left\| y_j^{(r)}(h,k) \sum_{m=j}^{l} E\left[y_m^{(r)}(h,k)|\mathcal{A}_j^{(r)}(h,k)\right] \right\|_{p/2} \right)^{1/2} \quad \text{(B.39)} \\ &= \sqrt{2p} \left(\sum_{j=1}^{T} \max_{j\leq l\leq T} \left\| y_l^{(r)}(h,k) E\left[y_j^{(r)}(h,k)|\mathcal{A}_j^{(r)}(h,k)\right] \right\|_{p/2} \right)^{1/2} \\ &\leq \sqrt{2p}\sqrt{T} \max_{1\leq t\leq T} \left\| y_t^{(r)}(h,k) \right\|_p . \end{aligned}$$

The equality follows from the martingale difference property and measurability, and iterated expectations:

$$E\left[y_m^{(r)}(h,k)|\mathcal{A}_j^{(r)}(h,k)\right]$$

$$= E\left(E\left[y_m^{(r)}(h,k)|\sigma(\xi_{j-r})\right]|\mathcal{A}_j^{(r)}(h,k)\right)$$
$$= E\left(E\left\{E\left[y_m^{(r)}(h,k)|\xi_{m-r}\right] - E\left[y_m^{(r)}(h,k)|\xi_{m-r-1}\right]|\sigma(\xi_{j-r})\right\}|\mathcal{A}_j^{(r)}(h,k)\right)$$
$$= 0 \; \forall m \geq j+1.$$

The second inequality uses Cauchy-Schwartz and Lyapunov inequalities:

$$\left\|y_l^{(r)}(h,k)E\left[y_j^{(r)}(h,k)|\mathcal{A}_j^{(r)}(h,k)\right]\right\|_{p/2} \leq \left\|y_l^{(r)}(h,k)\right\|_p \left\|E\left[y_j^{(r)}(h,k)|\mathcal{A}_j^{(r)}(h,k)\right]\right\|_p \leq \left\|y_l^{(r)}(h,k)\right\|_p \left\|y_j^{(r)}(h,k)\right\|_p .$$

Finally, we have by definition, and arguments in Wu (2005, proofs of Theorem 1.(i),(ii)),

$$\left\|y_t^{(r)}(h,k)\right\|_p = \|\mathbb{E}(z_t(h,k)|\xi_{t-r}) - \mathbb{E}(z_t(h,k)|\xi_{t-r-1})\|_p \leq \theta_t^{(z)}(h,m).$$

Combining bounds, we have shown $||\mathcal{Z}_T||_p \leq \sqrt{2p}\sum_{m=0}^{\infty}\max_{1\leq t\leq T}\theta_t^{(z)}(h,m)$, completing the proof. $\mathcal{QED}$.

B.2.2. Lemma 3.1*, Theorem 3.2*, Lemma B.4*

We now have versions of high dimensional Gaussian approximation Lemma 3.1 and max-statistic limit Theorem 3.2 under physical dependence. We also develop the required supporting results to prove bootstrap validity Theorem 4.1 under physical dependence. In the former two cases $\mathcal{H}_T = o(T)$ is achieved.

First, Lemma 3.1* replicates Lemma 3.1 under physical dependence, using Kolmogorov distance ρ_T in (B.3).

Lemma 3.1*. *Under Assumption 1.a*,b,c,d, $\rho_T \rightarrow 0$, for any sequences $\{\mathcal{H}_T, \mathcal{K}_T\}$ with $0 \leq \mathcal{H}_T \leq T - 1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^\kappa)$ for some finite $\kappa > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$ where $\eta(\cdot)$ is the Assumption 1.d discrete basis summand bound. Thus $\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathcal{Z}_T(h,k)| \stackrel{d}{\rightarrow} \max_{h,k\in\mathbb{N}}|\mathbf{Z}(h,k)|$ where $\mathbf{Z}(h,k) \sim N(0, \lim_{T\rightarrow\infty}\sigma_T^2(h,k))$ and $\lim_{T\rightarrow\infty}\sigma_T^2(h,k) < \infty$.*

Proof. Recall from the proof of Lemma 3.1 that $\zeta_t(i)$ stacks $z_t(h,k)$.We show below $\zeta_t(i)$ satisfies Conditions 1 and 3 in Chang, Chen and Wu (2024). By their Theorem 3(ii), therefore, $\rho_T \lesssim g_T$ for some $g_T = o(1)$ to be characterized below.

Condition 3 in Chang, Chen and Wu (2024) holds by Assumption 1.c. Next, by Lemma B.7.a for some $\varpi > 0$, and $\lambda > 0$:

$$\max_{t,T,\mathcal{H}_T,\mathcal{K}_T} E\left[\exp\left\{|z_t(h,k)|^{\varpi/2}/\lambda^{\varpi/2}\right\}\right]$$
$$= 1 + \max_{t,T,\mathcal{H}_T,\mathcal{K}_T}\int_1^\infty P\left(\exp\left\{|z_t(h,k)|^{\varpi/2}/\lambda^{\varpi/2}\right\} > u\right)du$$
$$= 1 + \max_{t,T,\mathcal{H}_T,\mathcal{K}_T}\int_1^\infty P\left(|z_t(h,k)| > \lambda\left(\ln(u)\right)^{2/\varpi}\right)du$$
$$\leq 1 + \tilde{\vartheta}_1\int_1^\infty \exp\{-\tilde{\vartheta}_2\lambda^{\varpi/2}\ln(u)\}du$$

$$= 1 + \tilde{\vartheta}_1 \int_1^\infty u^{-\tilde{\vartheta}_2 \lambda^{\varpi/2}} du.$$

Then Condition 1 holds with their $B_T = 1$ in view of:

$$\begin{aligned}
\inf\left\{\lambda > 0 : E\left[\exp\left\{|z_t(h,k)|^{\varpi/2}/\lambda^{\varpi/2}\right\}\right] \leq 2\right\} \\
&\leq \inf\left\{\lambda \geq \tilde{\vartheta}_2^{-2/\varpi} : \tilde{\vartheta}_1 \int_1^\infty u^{-\tilde{\vartheta}_2\lambda^{\varpi/2}} du \leq 1\right\} \\
&= \inf\left\{\lambda \geq \frac{1}{\tilde{\vartheta}_2^{2/\varpi}} : \frac{1}{\tilde{\vartheta}_2 \lambda^{\varpi/2} - 1} \leq \frac{1}{\tilde{\vartheta}_1}\right\} \\
&= \left(\frac{\tilde{\vartheta}_1 + 1}{\tilde{\vartheta}_2}\right)^{2/\varpi} < \infty \; \forall t \in \{1, \ldots, T\} \text{ and } T \in \mathbb{N}.
\end{aligned}$$

Now consider g_T. Invoke Lemma B.8 under Assumption 1.a* to obtain uniform finite bounds on each aggregated *dependence adjusted norm* in Chang, Chen and Wu (2024, eq. (5)). Their Theorem 3(ii) with their $B_T = 1$, $\alpha = \lambda - 1 + \iota/2$ and $\nu = \varphi$ now gives :

$$\begin{aligned}
\rho_T &\lesssim \frac{(\ln(\mathcal{H}_T\mathcal{K}_T))^{7/6}}{T^{\alpha/(12+6\alpha)}} + \frac{(\ln(\mathcal{H}_T\mathcal{K}_T))^{2/3}}{T^{\alpha/(12+6\alpha)}} + \frac{(\ln(\mathcal{H}_T\mathcal{K}_T))^{1+\varphi}}{T^{\alpha/(4+2\alpha)}} \\
&\lesssim \frac{(\ln(\mathcal{H}_T\mathcal{K}_T))^{(7/6)\vee(1+\varphi)}}{T^{\alpha/(12+6\alpha)}} \to 0
\end{aligned}$$

for any positive integer sequences $\{\mathcal{H}_T, \mathcal{K}_T\}$, with $0 \leq \mathcal{H}_T \leq T - 1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^\kappa)$ for some finite $\kappa > 0$. $\mathcal{QED}$.

We now extend Lemma B.3 to physical dependent cases, allowing us to assume $E[X_t] = 0$.

Lemma B.3*. *Under Assumption 1.a*,b,c,d, for any sequences* $\{\mathcal{H}_T, \mathcal{K}_T\}$ *with* $0 \leq \mathcal{H}_T \leq T - 1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^\kappa)$ *for some finite* $\kappa > 0$, *and* $\eta(\mathcal{K}_T) = o(\sqrt{T})$*:*

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} \left\{ \left(X_t - \bar{X}\right)\left(X_{t-h} - \bar{X}\right) - \left(X_t - \mu\right)\left(X_{t-h} - \mu\right) \right\} B_k(t) \right| = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof. We replicate the proof of Lemma B.3 with a few required changes. Write $\tilde{X}_t \equiv X_t - \mu$ and $\hat{X}_t \equiv X_t - \bar{X}$. We have:

$$\begin{aligned}
\frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} \left\{\hat{X}_t \hat{X}_{t-h} - \tilde{X}_t \tilde{X}_{t-h}\right\} B_k(t) &= \left(\bar{X}^2 - \mu^2\right) \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} B_k(t) - 2\mu\left(\bar{X} - \mu\right) \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} B_k(t) \\
&\quad - \left(\bar{X} - \mu\right) \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} \left\{X_{t-h} - \mu + X_t - \mu\right\} B_k(t) \\
&= \mathfrak{A}_T(h,k) + \mathfrak{B}_T(h,k) + \mathfrak{C}_T(h,k).
\end{aligned}$$

By Assumption 1.d $|1/\sqrt{T}\sum_{t=1}^{T} B_k(t)| = O(\eta(k)/\sqrt{T})$. By Lemma B.9, $\lim_{T\to\infty}\sigma_T^2(i) < \infty$, $i = 0, 1, 2, \ldots$Thus $\bar{X} - \mu = O_p(1/\sqrt{T})$ and therefore $\bar{X}^2 - \mu^2 = O_p(1/\sqrt{T})$ by Chebyshev's inequality and the mapping theorem. Therefore, e.g.,

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\left\{\bar{X}^2 - \mu^2\right\}\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h} B_k(t)\right| = O\left(\frac{\max_{\mathcal{K}_T}\eta(k)}{T}\right) = O_p\left(\frac{\eta(\mathcal{K}_T)}{T}\right).$$

Then $\eta(\mathcal{K}_T) = o(\sqrt{T})$ implies $\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathfrak{A}_T(h,k)| = o_p(1/\sqrt{T})$. Similarly $\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathfrak{B}_T(h,k) = o_p(1/\sqrt{T})$.

The remaining term $\mathfrak{C}_T$ is handled by applying arguments in the proof of Lemma 3.1*, cf. proof of Lemma 3.1, to deduce for some mean zero Gaussian process $\mathring{\mathbf{Z}}(h,k) \sim N(0, \lim_{T\to\infty}\mathring{\sigma}_T^2(h,k))$ and $\lim_{T\to\infty}\mathring{\sigma}_T^2(h,k) < \infty$:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\{X_{t-h} - \mu + X_t - \mu\} B_k(t)\right| \xrightarrow{d} \max_{h,k\in\mathbb{N}}|\mathbf{Z}(h,k)|.$$

Hence $\max_{\mathcal{H}_T,\mathcal{K}_T}|\mathfrak{C}_T| = O_p(1/\sqrt{T})$, completing the proof. $\mathcal{QED}$.

Next, Theorem 3.2* replicates Theorem 3.2 under physical dependence, and instantly implies Theorem 3.3.

Theorem 3.2*. *Let H_0 and Assumption 1.a*,b,c,d hold, and let $\mathcal{H}_T, \mathcal{K}_T \to \infty$. Let $\{\mathbf{Z}(h,k) : h, k \in \mathbb{N}\}$ be a zero mean Gaussian process with $\mathbf{Z}(h,k) \sim N(0, \sigma^2(h,k))$. Then it holds that $\mathcal{M}_T \xrightarrow{d} \gamma_0^{-1}\max_{h,k\in\mathbb{N}}|\mathbf{Z}(h,k)|$ for any $\{\mathcal{H}_T, \mathcal{K}_T\}$ with $0 \le \mathcal{H}_T \le T-1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$.*

Proof. The proof of Theorem 3.2 carries over verbatim, using Lemma 3.1* in lieu of Lemma 3.1. $\mathcal{QED}$.

Step 2 of the proof of Theorem 4.1 under physical dependence requires a version of Lemma B.4 under Assumptions 1.a*.

Lemma B.4*. *Under Assumptions 1.a*,b,c,d and 2 the conclusions of Lemma B.4 hold for sequences $\{b_T, \mathcal{H}_T, \mathcal{K}_T\}$ satisfying $b_T/T^{\iota} \to \infty$, $b_T = o(T^{1/2-\iota})$, $0 \le \mathcal{H}_T < T-1$, $\mathcal{H}_T = O(T^{1-\iota}/b_T)$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$.*

Proof. Replicate the proof of Lemma B.4 with two modifications.

Modification #1: Supporting Lemmas B.10 and B.11 below replace Lemmas B.5 and B.6.

Modification #2: Recall for some $\mathcal{S}_T(h,\tilde{h}) \le T/b_T$,

$$\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k}) \equiv \mathfrak{X}_{T,l}(h,k)\mathfrak{X}_{T,l}(\tilde{h},\tilde{k})$$

$$\mathfrak{X}_{T,l}(h,k) \equiv \frac{1}{\sqrt{b_T}}\sum_{t=(l-1)b_T+1}^{lb_T} X_t X_{t+h} B_k(t),\ l = 1, \ldots, \mathcal{S}_T(h,\tilde{h}).$$

Recall $[\boldsymbol{Y}_{T,l}(i,j)]_{i,j=0}^{\mathcal{H}_T\mathcal{K}_T}$ stacks $\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k})$ and $[\mathring{\boldsymbol{Y}}_{T,l}(m)]_{m=0}^{\mathcal{H}_T^2\mathcal{K}_T^2}$ stacks $[\boldsymbol{Y}_{T,l}(i,j)]_{i,j=0}^{\mathcal{H}_T\mathcal{K}_T}$. We need to show $\mathring{\boldsymbol{Y}}_{T,l}(m)$ is $\mathcal{L}_q$-physical dependent for some $q \geq 2$ with size $\lambda \geq 1$, and satisfies Conditions 1 and 3 in Chang, Chen and Wu (2024). Arguments in Step 2 of the proof of Lemma B.4.b suffice for demonstrating Conditions 1 and 3.

For physical dependence, by the proof of Lemma B.8, and $|B_k(t)| = 1$, $X_t X_{t+h} B_k(t)$ satisfies

$$\left\| X_t X_{t+h} B_k(t) - X_t^{'}(m) X_{t+h}^{'}(m) B_k(t) \right\|_{p/2} \leq d_t^{(z)}(h) \times \{m \vee 1\}^{-\lambda-\iota},$$

where $d_t^{(z)}(h) = \{||X_t||_p \vee ||X_{t+h}||_p\}^2$. A non-sharp bound revealing memory decay is therefore:

$$\begin{aligned}
&\max_{\mathcal{H}_T,\mathcal{K}_T} \max_{1\leq l\leq \mathcal{S}_T(h,\tilde{h})} \left\| \mathfrak{X}_{T,l}(h,k) - \mathfrak{X}'_{T,l}(m,h,k) \right\|_{p/2} \\
&\quad \leq \max_{\mathcal{H}_T,\mathcal{K}_T} \max_{1\leq l\leq \mathcal{S}_T(h,\tilde{h})} \frac{1}{\sqrt{b_T}} \sum_{t=(l-1)b_T+1}^{lb_T} \left\| X_t X_{t+h} B_k(t) - X'_t(m) X'_{t+h}(m) B_k(t) \right\|_{p/2} \\
&\quad \leq \max_{\mathcal{H}_T} \max_{1\leq l\leq \mathcal{S}_T(h,\tilde{h})} \frac{1}{\sqrt{b_T}} \sum_{t=(l-1)b_T+1}^{lb_T} d_t^{(z)}(h) \times \{m \vee 1\}^{-\lambda-\iota} \\
&\quad \leq \sqrt{b_T} \max_{t,\mathcal{H}_T} \left\{ d_t^{(z)}(h) \right\} \times \{m \vee 1\}^{-\lambda-\iota}. \qquad \text{(B.40)}
\end{aligned}$$

Thus, by definition, $\mathfrak{X}_{T,l}(h,k)$ is $\mathcal{L}_{p/2}$-dependent with size λ. A sharper bound on the constants is generated as follows. Apply Lemmas B.8 and B.9 to $\mathfrak{X}_{T,l}(h,k) - E[\mathfrak{X}_{T,l}(h,k)]$, and note $\mathfrak{X}'_{T,l}(m,h,k)$ is a copy of $\mathfrak{X}_{T,l}(h,k)$ by the iid property of the innovations $\{\epsilon_t, \epsilon'_t\}$, to deduce

$$\begin{aligned}
&\max_{m\geq 0} \max_{l,T} \left\| \mathfrak{X}_{T,l}(h,k) - \mathfrak{X}'_{T,l}(m,h,k) \right\|_{p/2} \\
&\quad \leq 2 \max_{m\geq 0} \max_{l,T} \left\| \frac{1}{\sqrt{b_T}} \sum_{t=(l-1)b_T+1}^{lb_T} \left(X_t X_{t+h} B_k(t) E\left[X_t X_{t+h} B_k(t) \right] \right) \right\|_{p/2} \\
&\quad \leq 2\sqrt{2p} \sum_{m=0}^{\infty} \max_{1\leq t\leq T} \theta_t^{(z)}(h,m) \\
&\quad \leq 2\sqrt{2p} \max_{1\leq t\leq T} d_t^{(z)}(h) \sum_{m=0}^{\infty} \{m \vee 1\}^{-1-\iota} \\
&\quad \leq K \max_{t,T} d_t^{(z)}(h). \qquad \text{(B.41)}
\end{aligned}$$

Bounds (B.40) and (B.41) imply we may write

$$\left\| \mathfrak{X}_{T,l}(h,k) - \mathfrak{X}'_{T,l}(m,h,k) \right\|_{p/2} \leq d_{T,l}^{(\mathfrak{X})}(h) \times \{m \vee 1\}^{-\lambda-\iota}$$

for some constants $d_{T,l}^{(\mathfrak{X})}(h) \leq K \max_{t,T} d_t^{(z)}(h)$, hence $\max_{l,T,\mathcal{H}_T} d_{T,l}^{(\mathfrak{X})}(h) \leq K$. Finally, repeat arguments in the proof of Lemma B.8 to conclude $\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k})$, and therefore $\mathring{\boldsymbol{Y}}_{T,l}(m)$, is $\mathcal{L}_{p/4}$-physical dependent for some $p \geq 8$, with size λ and uniformly bounded constants. $\mathcal{QED}$.

Proofs of the following supporting results are identical to the proofs of Lemma B.5 and B.6, using Lemma 3.1* instead of Lemma 3.1.

Lemma B.10. *Under Assumption 1.a*,b,c,d,* $\max_{\mathcal{H}_T,\mathcal{K}_T} |\hat{g}(h,k) - g_T(h,k)| = O_p(1/\sqrt{T})$ *for any* $\{\mathcal{H}_T,\mathcal{K}_T\}$ *satisfying* $0 \leq \mathcal{H}_T \leq T-1$, $\mathcal{H}_T = O(T^{1-\iota}/b_T)$, $\mathcal{K}_T = o(T^{\kappa})$ *for some finite* $\kappa > 0$, *and* $\eta(\mathcal{K}_T) = o(\sqrt{T})$.

Lemma B.11. *Under Assumption 1.a*,b,c,d and 2, for any* $\{\mathcal{H}_T,\mathcal{K}_T\}$ *satisfying* $0 \leq \mathcal{H}_T \leq T-1$, $\mathcal{H}_T = O(T^{1-\iota}/b_T)$, $\mathcal{K}_T = o(T^{\kappa})$ *for some finite* $\kappa > 0$, *and* $\eta(\mathcal{K}_T) = o(\sqrt{T})$*:*

$$\left| \max_{\mathcal{H}_T,\mathcal{K}_T} \left|\Delta \ddot{g}^*_T(h,k)\right| - \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T} \sum_{i=1}^{\tilde{N}_T(h)} \sum_{t=(i-1)\tilde{b}_T+\tilde{l}_T+1}^{i\tilde{b}_T} \varphi_t y_t(h,k) \right| \right| = o_p\left(1/\sqrt{T}\right).$$

B.2.3. Theorem 3.3

Theorem 3.3. *Let* H_0 *and Assumption 1.a*,b,c,d hold, and let* $\mathcal{H}_T,\mathcal{K}_T \to \infty$. *Let* $\{\mathbf{Z}(h,k) : h,k \in \mathbb{N}\}$ *be a zero mean Gaussian process with* $\mathbf{Z}(h,k) \sim N(0,\sigma^2(h,k))$. *Then it holds that* $\mathcal{M}_T \xrightarrow{d} \gamma_0^{-1} \max_{h,k\in\mathbb{N}} |\mathbf{Z}(h,k)|$ *for any* $\{\mathcal{H}_T,\mathcal{K}_T\}$ *with* $0 \leq \mathcal{H}_T \leq T-1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^{\kappa})$ *for some finite* $\kappa > 0$, *and* $\eta(\mathcal{K}_T) = o(\sqrt{T})$.

Proof. See Theorem 3.2*. $\mathcal{QED}$.

C. Empirical study

We now apply our test and the test in Jin, Wang and Wang (2015) to quarterly international (ex post) real interest rates. We analyze 16 countries over the period 1960.Q1 - 2019.Q4. The data were collected from the U.S. Federal Reserve Bank data archive (FRED), which itself is taken from the OECD data archives. The countries are Australia, Austria, Belgium, Canada, Denmark, France, Germany, Ireland Italy, Japan, Netherlands, Norway, Switzerland, UK and US.

Following Rapach and Weber (2004), we use the 10-year government bond yield as our measure of the nominal interest rate $r_{n,t}$, and the Consumer Price Index in order to compute inflation i_t. The (ex post) real bond rate is $r_{r,t} = r_{n,t} - i_t$. See Table A.1 for the exact date range available for each series and subsequent size. Figure 1 contains plots of each series.

Unit root tests have been proposed as a standard for testing for non-stationarity in interest rates. See, e.g., Rose (1988) and Rapach and Weber (2004) and their historical references. In that framework, it is implicitly assumed that real interest rates are unbounded (asymptotically with probability approaching one), in particular if a unit root is present. In the case of a unit root, of course, variance is unbounded asymptotically, and α-mixing fails to hold.

Testing real interest rates is complicated by the fact that nominal rates $r_{n,t}$ and inflation i_t may be nonstationary while real rates $r_{r,t} = r_{n,t} - i_t$ can yet be stationary. In a unit root test setting, it is possible that $r_{n,t} \sim I(1)$ and $i_t \sim I(1)$ yet $(r_{n,t}, i_t)$ are cointegrated with integrating vector $[-1,1]$, hence $r_{r,t}$ are stationary. Conversely, nonstationarity necessarily exists when just $r_{r,t} \sim I(1)$ or just $i_t \sim I(1)$. Rose (1988) finds the latter for each country in our study based on quarterly post-war data and conventional unit root tests, hence Rose (1988) broadly concludes unit root nonstationarity. Rapach and Weber (2004) obtain more nuanced results. They find nonstationarity in nominal rates for all countries except Germany and Switzerland, and mixed results for inflation based on Phillips and Perron (1988) and Ng and Perron (1997, 2001) unit root tests. In order to handle the evident cases $r_{n,t} \sim I(1)$ and $i_t \sim I(1)$ they apply several cointegration tests, including tests by Ng and Perron (2001) and one eventually published in Perron and Rodriguez (2016).

A different approach for studying structural time variation in interest rates couches rates in a parametric regime switching regression model. See, e.g., Garcia and Perron (1996), Bekdache (1000), and Ang and Bekaert (2002). See also Teräsvirtra (1994) and Gray (1996).

In our setting, under either hypothesis we assume a moment generating function exists uniformly over t, and a geometric mixing condition holds. Thus, we implicitly assume a unit root does not exist. The moment conditions can be assured simply by assuming nominal interest rates and inflation are bounded. This is a fairly natural assumption empirically for interest rates which are typically managed by government market actions, and lie in the range $[-1, 1]$. In any case, in our sample range bond yields and inflation never surpass the total range $[-.02, .30]$. We therefore test for a (non-unit root based) deviation from covariance stationarity. Our setting of course is nonparametric: we do not need to specify a (switching) regression model (e.g. Augmented Dickey Fuller, or Markov Switching), and indeed our test is relevant irrespective of any underlying parametric features.

We report test results for the max-test based on a dependent wild bootstrap, and the test in Jin, Wang and Wang (2015) based both on simulated critical values and dependent wild bootstrap. Both tests exploit a Walsh basis in view of simulation evidence suggesting the inferiority of the composite Haar basis. We simulate critical values for each series and each country (hence, 54 simulated sets of critical values), rather than for each sample size. We use $\mathcal{H}_T = [2T^{.49}]$ and $\mathcal{K}_T = [.5T^{.49}]$. See Table A.2 for test results. Tests are performed on nominal and real bond yields, and inflation, but we focus our discussion on real bond yields given is importance in the literature.

Consider the max-correlation difference test. In all countries except one, when the test finds evidence of non-covariance stationarity in nominal rates, the same result applies for real rates. Consider Italy: the p-values are .024 and .032 for nominal and real rates respectively, while the p-value for inflation is .216. Thus, nominal rates are the driving force for non-stationarity. New Zealand is the sole exception: p-values for nominal and real rates and inflation are .156, .080 and .162. Thus, we reject stationarity at the 10% level for real rates, but *fail to reject* for nominal rates and inflation. It is easily verified, however, that if random variables X_t and Y_t are covariance stationary then so is any linear combination. A deeper study into this is left for future work.

The bootstrapped JWW test, on par with the Monte Carlo study, almost never leads to a rejection of the covariance stationarity null hypothesis. Tests based on simulated critical values, however, match across nominal and real bond yields, with four exceptions: Belgium, Japan, New Zealand and the UK. The JWW test generally yields strong rejections (well under the 1% level) when nonstationarity is detected, while the max-correlation test is more moderate, with rejections variously at the 1%, 5%, and 10% levels.

Finally, in five countries the max-correlation test and JWW test disagree: Australia, France, Italy, New Zealand and Switzerland (denoted by bold in Table A.2). In the first four the max-correlation difference test yielded rejections of covariance stationarity (p-values are .056, .022, .032, and .080), while the JWW test failed to reject. The JWW test with simulated critical value detected non-stationarity for Switzerland at the 1% level ($\hat{\mathcal{D}}_T = 58.1$, 1% c.v. = 7.9), but the max-correlation test did not at the 10% (p-value .144).

Table A.1. Dates and Sample Sizes

	Nominal Bond r_n		Inflation i		Real Bond r_r	
	Dates	n	Dates	n	Dates	n
Australia	1969.Q3-2021.Q4	210	1960.Q2-2021.Q4	246	1969.Q3-2021.Q4	210
Austria	1990.Q1-2021.Q4	128	1960.Q2-2021.Q4	246	1990.Q1-2021.Q4	128
Belguim	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
Canada	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	244	1960.Q2-2021.Q4	246
Denmark	1987.Q1-2021.Q4	140	1967.Q2-2021.Q4	218	1987.Q1-2021.Q4	140
France	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
Germany	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
Ireland	1971.Q1-2021.Q4	204	1976.Q2-2021.Q4	182	1976.Q2-2021.Q4	182
Italy	1991.Q2-2021.Q4	122	1960.Q2-2021.Q4	246	1991.Q2-2021.Q4	122
Japan	1989.Q1-2021.Q4	132	1960.Q2-2021.Q4	246	1989.Q1-2021.Q4	132
Netherlands	1960.Q1-2021.Q4	248	1960.Q3-2021.Q4	246	1960.Q3-2021.Q4	246
New Zealand	1970.Q1-2021.Q4	208	1960.Q2-2021.Q4	246	1970.Q1-2021.Q4	208
Norway	1985.Q1-2021.Q4	148	1960.Q2-2021.Q4	246	1985.Q1-2021.Q4	148
Switzerlnad	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
UK	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
US	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246

Nominal bond r_n are 10 year government bond yields; inflation i is derived from the Consumer Price Index for all goods and services; real bond yields $r_r = r_n - i$.

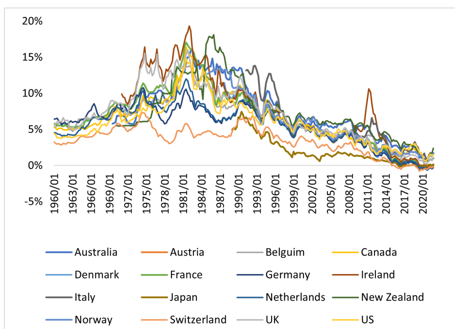
Table A.2. Empirical Study: Covariance Stationarity Tests

	Nominal Bond r_n			Inflation i			Real Bond r_r		
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
Australia	.080	65.5 (3.5, 4.8, 7.5) ***	.729	.174	5.57 (3.8, 5.1, 7.9) **	.605	**.056**	**-2.28 (3.5, 4.8, 7.5)**	**.854**
Austria	.002	12.3 (2.9, 4.9, 6.6) ***	.198	.158	62.7 (3.8, 5.1, 7.9) ***	.134	.000	352 (2.9, 4.1, 6.6) ***	.024
Belguim	.032	2.03 (3.8, 5.1, 7.9)	.876	.236	9.90 (3.8, 5.1, 8.0) ***	.537	.023	44 (3.8, 5.1, 7.9) ***	.919
Canada	.014	24.1 (3.8, 5.1, 7.9) ***	.904	.158	10.4 (3.8, 5.1, 7.9) ***	.361	.018	17.7 (3.8, 5.1, 7.9) ***	.756
Denmark	.000	241 (3.0, 4.1, 6.7) ***	.246	.066	29.1 (3.6, 4.8, 7.6) ***	.319	.000	217 (3.0, 4.1, 6.6) ***	.273
France	.020	-.543 (3.8, 5.1, 7.9)	.661	.174	2.27 (3.8, 5.1, 7.9)	.541	**.022**	**-2.00 (3.8, 5.1, 7.9)**	**.866**
Germany	.180	28.9 (3.8, 5.1, 7.9) ***	.858	.046	109 (3.8, 5.1, 7.9) ***	.170	.090	879 (3.8, 5.1, 7.9) ***	.399
Ireland	.101	6.12 (3.4, 4.7, 7.5) **	.998	.242	30.6 (3.4, 4.6, 7.2) ***	.248	.012	161 (3.4, 4.6, 7.2) ***	.563
Italy	.024	1.92 (2.9, 4.0, 6.5)	.246	.216	218 (3.8, 5.1, 7.8) ***	.076	**.032**	**1.80 (2.9, 4.0, 6.6)**	**.836**
Japan	.054	1.43 (2.9, 4.1, 6.6)	.331	.331	3.34 (3.8, 5.1, 7.9)	.473	.014	92.7 (2.9, 4.0, 6.6) ***	.581
Netherlands	.068	284 (3.8, 5.1, 7.9) ***	.585	.114	12.9 (3.8, 5.1, 7.9) ***	.251	.026	184 (3.8, 5.1, 7.9) ***	.394
New Zealand	.156	11.0 (3.5, 4.8, 7.5) ***	.820	.162	2.13 (3.8, 5.1, 7.9)	.819	**.080**	**-2.86 (3.5, 4.8, 7.5)**	**.982**
Norway	.014	63.2 (3.0, 4.2, 6.8) ***	.273	.042	-3.49 (3.8, 5.1, 7.9)	.719	.006	23.7 (3.0, 4.1, 6.8) ***	.102
Switzerlnad	.136	853 (3.8, 5.1, 7.9) ***	.345	.265	16.2 (3.8, 5.1, 7.9) ***	.371	**.144**	**58.1 (3.8, 5.1, 7.9) *** **	**.334**
UK	.032	-2.02 (3.8, 5.1, 7.9)	.994	.222	54.2 (3.8, 5.1, 7.9) ***	.699	.006	21.5 (3.8, 5.1, 7.9) ***	.890
US	.036	555 (3.8, 5.1, 7.9) ***	.647	.124	9.49 (3.8, 5.1, 7.9) ***	.307	.024	652 (3.8, 5.1, 7.9) ***	.222

$\hat{\mathcal{M}}_T$ is the proposed max-test based on a bootstrapped p-value: reported values are p-values computed by dependent wild bootstrap. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, shown in parentheses: *, **, *** denote rejection at the 10%, 5% and 1% levels. $\hat{\mathcal{D}}_T^{dw}$ is JWW's test based dependent wild bootstrapped p-values.

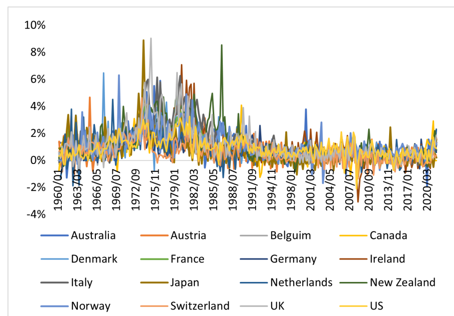
Figure 1: Quarterly Interest Rates and Inflation

(a) Quarterly Interest Rates r_n



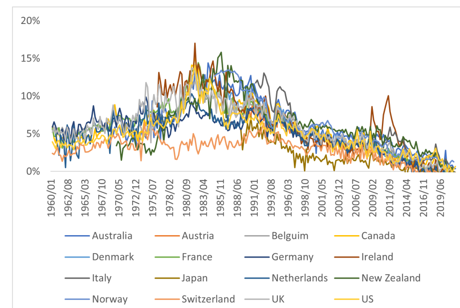
Rates are the nominal 10 year government bond yield.

(b) Quarterly CPI Inflation i



Inflation is computed as the quarterly change in the log-consumer price index.

(c) Real Interest Rates $r_r = r_n - i$



Rates are the nominal 10 year government bond yield minus the inflation rate.

D. Complete simulation results

Table A.3.: Rejection Frequencies under H_0: **Walsh Basis**
Case 1: $\mathcal{H}_T = [\log_2(n)^{.99} - 3.5]$ and $\mathcal{K}_T = [n^{1/3} + .01]$

	$\epsilon_t \overset{iid}{\sim} N(0,1)$							
	$n=64$				$n=128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.005, .025, .093	.002, .024, .092	.012, .041, .077	.304, .548, .675	.001, .036, .103	.003, .041, .111	.002, .027, .066	.118, .376, .596
AR(1)	.006, .043, .106	.006, .045, .105	.066, .105, .162	.263, .427, .504	.006, .052, .130	.005, .054, .132	.064, .109, .159	.141, .271, .380
SETAR	.006, .026, .053	.003, .036, .076	.052, .102, .159	.230, .424, .508	.004, .025, .064	.006, .035, .067	.050, .124, .173	.114, .276, .388
GARCH	.004, .038, .099	.000, .029, .098	.014, .045, .096	.249, .504, .635	.002, .056, .158	.004, .055, .158	.004, .030, .091	.094, .349, .547
	$n=256$				$n=512$			
MA(1)	.002, .024, .090	.003, .022, .096	.005, .041, .082	.051, .281, .526	.006, .038, .105	.006, .041, .106	.010, .053, .097	.045, .227, .445
AR(1)	.006, .036, .103	.005, .037, .107	.045, .093, .146	.067, .173, .279	.005, .052, .132	.004, .061, .132	.035, .076, .140	.042, .100, .190
SETAR	.004, .034, .069	.005, .033, .063	.034, .099, .178	.052, .171, .269	.004, .032, .078	.003, .036, .079	.024, .100, .177	.024, .092, .163
GARCH	.002, .031, .093	.002, .031, .093	.005, .040, .098	.074, .310, .530	.004, .046, .120	.003, .047, .119	.017, .051, .087	.045, .244, .500
	$\epsilon_t \overset{iid}{\sim} t_5$							
	$n=64$				$n=128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.001, .032, .085	.002, .034, .088	.003, .027, .069	.168, .311, .379	.004, .039, .095	.005, .041, .095	.002, .020, .062	.059, .167, .233
AR(1)	.009, .044, .116	.009, .048, .117	.053, .085, .138	.164, .253, .305	.007, .051, .134	.007, .054, .134	.026, .069, .116	.053, .113, .148
SETAR	.004, .019, .050	.004, .021, .053	.032, .080, .131	.143, .249, .292	.004, .013, .035	.001, .013, .039	.025, .080, .158	.042, .107, .142
	$n=256$				$n=512$			
MA(1)	.001, .028, .087	.001, .031, .083	.000, .025, .062	.026, .098, .155	.002, .035, .073	.001, .032, .073	.004, .034, .074	.011, .036, .053
AR(1)	.005, .033, .107	.005, .034, .113	.017, .048, .104	.024, .050, .058	.002, .038, .104	.002, .034, .107	.005, .054, .088	.003, .007, .011
SETAR	.000, .005, .032	.000, .005, .035	.038, .129, .210	.013, .041, .057	.000, .018, .047	.000, .018, .048	.065, .175, .274	.002, .005, .013
	$\epsilon_t \sim$ GARCH							
	$n=64$				$n=128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.001, .030, .088	.001, .034, .091	.014, .045, .096	.258, .498, .642	.002, .042, .119	.002, .047, .130	0.014, .045, .096	.258, .498, .642
AR(1)	.011, .060, .121	.011, .057, .121	.080, .120, .186	.244, .407, .488	.008, .048, .127	.008, .054, .134	.080, .120, .186	.244, .407, .488
SETAR	.005, .021, .052	.004, .022, .056	.064, .122, .174	.242, .417, .512	.003, .018, .051	.003, .017, .054	.064, .122, .174	.242, .417, .512
	$n=256$				$n=512$			
MA(1)	.001, .020, .097	.002, .021, .096	.005, .040, .098	.072, .306, .528	.001, .045, .114	.001, .042, .116	.017, .051, .087	.050, .247, .499
AR(1)	.002, .035, .084	.002, .035, .086	.063, .106, .162	.083, .168, .258	.003, .044, .107	.003, .046, .113	.034, .085, .129	.036, .107, .176
SETAR	.000, .013, .065	.000, .014, .068	.029, .099, .164	.049, .172, .264	.000, .020, .069	.002, .020, .074	.028, .103, .177	.018, .096, .167

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values. The GARCH error is based on an iid $N(0,1)$ innovation.

Table A.4.: Rejection Frequencies under H_0: **Walsh Basis**
Case 2: $\mathcal{H}_T = 2T^{.49}$ and $\mathcal{K}_T = .5T^{.49}$

	$\epsilon_t \overset{iid}{\sim} N(0,1)$							
	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.000, .012, .054	.001. .010, .066	.018, .040, .075	.005, .085, .255	.001, .008, .046	.001, .017, .053	.008, .034, .074	.001, .031, .195
AR(1)	.001, .019, .072	.001, .024, .087	.068, .110, .161	.027, .073, .166	.001, .020, .084	.001, .030, .094	.074, .123, .172	.022, .068, .130
SETAR	.001, .017, .037	.003, .019, .041	.040, .089, .152	.014, .093, .202	.001, .027, .050	.003, .024, .046	.051, .124, .171	.015, .062, .148
GARCH	.000, .021, .095	.000, .025, .109	.014, .047, .103	.004, .077, .247	.003, .044, .113	.004, .047, .128	.008, .034, .085	.000, .023, .156
	$n = 256$				$n = 512$			
MA(1)	.000, .011, .049	.000, .015, .059	.008, .045, .088	.000, .019, .146	.004, .027, .061	.004, .032, .079	.013, .047, .109	.000, .025, .161
AR(1)	.002, .015, .057	.002, .019, .058	.048, .100, .154	.017, .036, .087	.002, .027, .085	.005, .028, .080	.033, .079, .149	.004, .021, .065
SETAR	.004, .033, .059	.003, .032, .058	.030, .100, .174	.003, .024, .108	.004, .031, .067	.005, .033, .069	.026, .097, .176	.000, .016, .069
GARCH	.003, .033, .095	.004, .032, .098	.007, .037, .086	.001, .031, .173	.003, .035, .108	.002, .035, .116	.017, .045, .083	.001, .030, .159
	$\epsilon_t \overset{iid}{\sim} t_5$							
	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.000, .013, .050	.001, .013, .057	.005, .030, .078	.008, .072, .175	.001, .021, .059	.002, .026, .073	.006, .026, .064	.001, .022, .094
AR(1)	.002, .020, .066	.005, .030, .081	.056, .091, .150	.025, .067, .132	.000, .027, .081	.000, .033, .091	.034, .077, .124	.005, .017, .056
SETAR	.001, .018, .057	.001, .023, .057	.026, .067, .118	.012, .061, .155	.001, .014, .038	.001, .014, .045	.020, .083, .152	.002, .018, .048
	$n = 256$				$n = 512$			
MA(1)	.000, .017, .060	.000, .019, .067	.003, .028, .064	.001, .016, .061	.002, .021, .080	.003, .028, .086	.006, .033, .070	.000, .006, .026
AR(1)	.003, .014, .049	.003, .018, .052	.018, .048, .093	.002, .013, .023	.003, .027, .087	.003, .034, .094	.007, .049, .090	.000, .001, .005
SETAR	.001, .014, .033	.002, .012, .037	.038, .115, .191	.001, .007, .024	.003, .014, .049	.002, .015, .047	.056, .165, .263	.000, .002, .008
	$\epsilon_t \sim$ GARCH							
	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.000, .019, .071	.000, .020, .082	.014, .047, .103	.006, .082, .253	.002, .031, .096	.002, .035, .112	.008, .034, .085	.000, .017, .160
AR(1)	.003, .025, .077	.004, .031, .087	.082, .130, .190	.034, .086, .195	.001, .022, .085	.001, .033, .092	.073, .123, .170	.031, .057, .127
SETAR	.004, .019, .057	.004, .022, .063	.063, .122, .179	.028, .094, .226	.001, .014, .050	.001, .016, .056	.040, .106, .155	.01, .041, .108
	$n = 256$				$n = 512$			
MA(1)	.002, .021, .076	.002, .020, .091	.007, .037, .086	.000, .024, .161	.004, .038, .110	.003, .044, .116	.017, .045, .083	.000, .025, .158
AR(1)	.001, .019, .064	.001, .025, .077	.064, .114, .164	.016, .044, .098	.000, .034, .109	.001, .032, .108	.033, .077, .123	.006, .028, .070
SETAR	.000, .013, .042	.000, .013, .044	.023, .092, .159	.002, .017, .091	.000, .023, .071	.001, .025, .082	.027, .100, .174	.001, .009, .064

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values. The GARCH error is based on an iid $N(0,1)$ innovation.

Table A.5.: a. Rejection Frequencies under H_1: **Walsh Basis**
Case 1: $\mathcal{H}_T = [\log_2(n)^{.99} - 3.5]$ and $\mathcal{K}_T = [n^{1/3} + .01]$
$\boldsymbol{\epsilon_t \sim N(0,1)}$

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.122, ,227, .336	.165, .287, .431	.173, .401, .549	.037, .198, .380	.436, .613, .742	.374, .454, .672	.827, .936, .967	.054, .275, .440
alt-2	.031, .147, .268	.032, .151, .274	.019, .048, .093	.275, .535, .679	.088, .398, .424	.090, .409, .541	.010, .044, .089	.116, .379, .602
alt-3	.021, .053, .201	.023, .038, .165	.173, .409, .555	.095, .336, .490	.354, .475, .630	.342, .436, .599	.434, .696, .809	.030, .225, .413
alt-4	.081, .272, .351	.062, .168, .344	.190, .413, .557	.097, .356, .486	.672, .750, .838	.674, .743, .833	.772, .917, .949	.113, .386, .509
alt-5	.102, .228, .342	.086, .149, .363	.081, .140, .191	.260, .415, .504	.494, .713, .888	.428, .739, .932	.160, .336, .471	.061, .171, .296
alt-6	.024, .081, .159	.020, .061, .162	.036, .084, .141	.204, .394, .508	.118, .244, .377	.121, .256, .352	.042, .131, .241	.070, .264, .415
alt-7	.031, .099, .135	.021, .088, .132	.054, .101, .143	.228, .399, .491	.073, .127, .239	.061, .102, .268	.058, .140, .215	.096, .219, .307
alt-8	.001, .121, .147	.003, .085, .123	.081, .140, .191	.260, .415, .504	.043, .131, .408	.027, .150, .437	.069, .110, .147	.115, .255, .352
alt-9	.050, .074, .103	.030, .047, .114	.016, .046, .099	.303, .562, .699	.067, .081, .180	.049, .098, .214	.002, .025, .063	.108, .364, .586
	$n = 256$				$n = 512$			
alt-1	.794, .931, .987	.754, .914, .978	.977, .997, 1.00	.045, .263, .432	1,00, 1.00, 1.00	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.317, .589, .688
alt-2	.108, .401, .488	.102, .419, .560	.004, .035, .087	.056, .283, .524	.214, .457, .512	.212, .451, .517	.008, .040, .081	.039, .230, .448
alt-3	.720, .817, .954	.710, .804, .947	.948, .986, .993	.154, .405, .510	1.00, 1.00, 1.00	.968, .999, 1.00	.988, .999, 1.00	.059, .284, .436
alt-4	.855, .985, 1.00	.876, .987, 1.00	.941, .983, .994	.094, .356, .463	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.999, 1.00, 1.00	.138, .402, .501
alt-5	.864, .988, .998	.723, .993, .999	.915, .977, .987	.060, .314, .433	1.00, 1.00, 1.00	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.697, .755, .777
alt-6	.259, .320, .533	.263, .338, .556	.069, .207, .296	.044, .213, .360	.681, .810, .904	.605, .829, .914	.153, .357, .469	.037, .234, .393
alt-7	.112, .341, .444	.115, .362, .468	.164, .375, .530	.032, .100, .166	.421, .695, .818	.449, .638, .834	.715, .878, .939	.084, .182, .291
alt-8	.162, .217, .467	.152, .240, .503	.044, .097, .148	.076, .163, .262	.585, .918, .996	.210, .935, .997	.070, .189, .318	.053, .100, .128
alt-9	.162, .218, .360	.141, .230, .368	.009, .032, .088	.065, .276, .515	.285, .466, .620	.188, .468, .606	.009, .050, .103	.041, .218, .461

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.5.: b. Rejection Frequencies under H_1: **Walsh Basis**
Case 1: $\mathcal{H}_T = [\log_2(n)^{.99} - 3.5]$ and $\mathcal{K}_T = [n^{1/3} + .01]$
$\boldsymbol{\epsilon_t \sim t_5}$

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.064, .267, .339	.072, .286, .433	.160, .360, .503	.021, .127, .223	.342, .351, .449	.331, .377, .385	.799, .921, .951	.021, .130, .303
alt-2	.007, .041, .157	.008, .047, .165	.002, .024, .061	.206, .336, .397	.024, .079, .214	.047, .083, .225	.007, .031, .060	.062, .194, .268
alt-3	.034, .066, .141	.031, .065, .124	.173, .386, .513	.051, .206, .276	.124, .297, .365	.117, .274, .340	.413, .643, .753	.015, .114, .207
alt-4	.005, .248, .386	.042, .244, .279	.171, .386, .535	.050, .199, .288	.566, .698, .824	.467, .581, .706	.737, .899, .947	.050, .190, .358
alt-5	.110, .270, .294	.101, .183, .239	.056, .096, .138	.175, .270, .308	.295, .464, .684	.214, .494, .697	.130, .297, .411	.025, .084, .139
alt-6	.020, .087, .119	.014, .066, .120	.021, .050, .091	.147, .247, .297	.094, .162, .231	.082, .151, .234	.031, .099, .182	.025, .103, .157
alt-7	.009, .046, .096	.005, .041, .071	.020, .053, .094	.129, .221, .267	.084, .126, .249	.072, .102, .211	.059, .142, .231	.047, .098, .134
alt-8	.021, .082, .125	.021, .069, .114	.056, .096, .138	.180, .269, .313	.046, .102, .311	.008, .111, .320	.029, .068, .107	.053, .124, .163
alt-9	.002, .025, .084	.002, .024, .094	.007, .032, .067	.203, .361, .426	.039, .077, .178	.013, .087, .201	.001, .025, .070	.071, .171, .247
	$n = 256$				$n = 512$			
alt-1	.624, .763, .846	.689, .734, .821	.978, .996, .998	.011, .116, .373	.905, .950, .976	.920, .945, .971	1.00, 1.00, 1.00	.237, .721, .877
alt-2	.024, .058, .168	.033, .062, .171	.004, .026, .069	.030, .093, .144	.050, .161, .307	.049, .166, .306	.005, .029, .061	.010, .039, .063
alt-3	.612, .725, .847	.805, .796, .823	.935, .981, .991	.035, .239, .468	.890, .984, 1.00	.828, .928, .998	.984, .999, .999	.013, .158, .419
alt-4	.690, .803, .910	.697, .801, .904	.920, .974, .992	.029, .165, .361	.989, .997, 1.00	.928, .974, .985	.997, 1.00, 1.00	.050, .303, .518
alt-5	.858, .878, .944	.802, .868, .947	.888, .961, .983	.018, .127, .288	.934, .977, .987	.938, .977, .987	1.00, 1.00, 1.00	.381, .740, .816
alt-6	.119, .245, .304	.118, .252, .314	.069, .199, .294	.011, .044, .080	.332, .418, .592	.341, .440, .605	.164, .350, .468	.006, .034, .073
alt-7	.092, .217, .372	.084, .207, .297	.182, .388, .515	.007, .038, .099	.481, .505, .768	.498, .425, .690	.697, .882, .935	.029, .129, .266
alt-8	.093, .190, .368	.064, .147, .388	.012, .056, .100	.012, .042, .051	.404, .654, .884	.418, .670, .920	.016, .091, .187	.003, .004, .006
alt-9	.071, .192, .232	.041, .124, .235	.006, .025, .073	.033, .090, .127	.181, .370, .428	.141, .269, .433	.004, .040, .084	.010, .038, .059

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.5.: c. Rejection Frequencies under H_1: **Walsh Basis**

Case 1: $\mathcal{H}_T = [\log_2(n)^{.99} - 3.5]$ and $\mathcal{K}_T = [n^{1/3} + .01]$

$\boldsymbol{\epsilon_t \sim}$ **GARCH**

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.122, .247, .357	.154, .317, .459	.177, .376, .533	.037, .206, .369	.312, .435, .507	.409, .412, .573	.845, .946, .972	.048, .243, .408
alt-2	.071, .095, .150	.071, .094, .159	.017, .051, .099	.307, .536, .660	.014, .100, .249	.017, .113, .261	.005, .033, .085	.097, .353, .575
alt-3	.020, .057, .121	.020, .055, .113	.190, .444, .581	.079, .339, .458	.163, .202, .359	.158, .293, .448	.419, .682, .803	.030, .214, .370
alt-4	.033, .227, .337	.020, .126, .335	.188, .424, .564	.089, .341, .468	.229, .308, .513	.227, .307, .505	.779, .911, .955	.111, .360, .505
alt-5	.091, .142, .270	.061, .137, .235	.069, .124, .159	.261, .386, .449	.330, .364, .464	.033, .282, .483	.158, .343, .472	.049, .154, .281
alt-6	.030, .071, .139	.024, .066, .129	.041, .089, .147	.215, .391, .486	.082, .165, .250	.061, .157, .245	.037, .119, .231	.072, .240, .381
alt-7	.041, .087, .127	.041, .076, .130	.047, .103, .165	.216, .375, .476	.043, .111, .207	.033, .118, .216	.055, .120, .201	.077, .173, .261
alt-8	.014, .088, .144	.014, .039, .138	.069, .124, .159	.265, .390, .450	.047, .112, .276	.028, .116, .290	.050, .086, .156	.109, .230, .351
alt-9	.022, .096, .136	.013, .047, .136	.012, .045, .085	.262, .525, .675	.046, .101, .202	.010, .106, .220	.007, .038, .085	.106, .350, .556
	$n = 256$				$n = 512$			
alt-1	.557, .641, .744	.546, .616, .701	.975, .998, .999	.038, .266, .437	.739, .843, .894	.716, .827, .878	1.00, 1.00, 1.00	0.270, .545, .657
alt-2	.021, .176, .276	.027, .182, .280	.011, .049, .105	.059, .286, .490	.032, .214, .302	.038, .252, .321	.010, .047, .100	.044, .243, .472
alt-3	.422, .525, .611	.418, .510, .693	.933, .984, .991	.152, .380, .515	.699, .737, .892	.691, .772, .882	.991, 1.00, 1.00	.064, .258, .395
alt-4	.528, .561, .673	.530, .555, .674	.931, .986, .992	.107, .306, .430	.719, .838, .901	.719, .837, .901	.997, 1.00, 1.00	.115, .343, .452
alt-5	.641, .705, .815	.637, .720, .831	.907, .968, .986	.076, .310, .425	.893, .925, .961	.805, 912, .924	1.00, 1.00, 1.00	.633, .712, .745
alt-6	.087, .189, .296	.071, .196, .303	.092, .224, .337	.041, .189, .328	.157, .322, .487	.159, .339, .521	.190, .378, .505	.046, .227, .375
alt-7	.126, .369, .382	.121, .305, .372	.200, .408, .533	.033, .096, .145	.333, .428, .654	.337, .443, .667	.737, .904, .945	.093, .178, .269
alt-8	.026, .088, .287	.006, .096, .303	.042, .091, .164	.059, .160, .247	.246, .422, .702	.251, .451, .719	.069, .198, .318	.050, .085, .110
alt-9	.018, .114, .271	.010, .115, .275	.011, .050, .098	.054, .292, .527	.181, .245, .436	.164, .249, .440	.016, .047, .102	.045, .231, .473

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.6.: a. Rejection Frequencies under H_1: **Walsh Basis**
Case 2: $\mathcal{H}_T = 2T^{.49}$ and $\mathcal{K}_T = .5T^{.49}$
$\boldsymbol{\epsilon_t \sim N(0,1)}$

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.121, .234, .326	.142, .276, .431	.146, .395, .552	.003, .031, .129	.583, .685, .702	.563, .628, .667	.803, .934, .967	.001, .065, .211
alt-2	.020, .059, .190	.022, .067, .202	.025, .058, .089	.010, .066, .233	.033, .197, .410	.034, .213, .427	.009, .043, .086	.002, .025, .177
alt-3	.200, .313, .417	.200, .303, .422	.150, .406, .553	.006, .060, .195	.332, .579, .687	.333, .518, .661	.382, .674, .790	.000, .022, .127
alt-4	.073, .292, .399	.065, .284, .397	.160, .402, .548	.011, .050, .190	.661, .678, .828	.668, .681, .837	.739, .906, .942	.001, .097, .292
alt-5	.121, .207, .335	.101, .145, .383	.080, .129, .178	.033, .085, .188	.403, .688, .869	.455, .731, .895	.143, .323, .471	.012, .040, .106
alt-6	.041, .123 .166	.031, .113, .179	.064, .121, .190	.017, .062, .156	.072, .134, .278	.076, .155, .308	.053, .137, .252	.010, .024, .089
alt-7	.021, .073, .137	.017, .063, .137	.050, .096, .145	.022, .072, .180	.058, .115, .211	.045, .103, .219	.066, .135, .220	.018, .052, .104
alt-8	.020, .115, .164	.021, .082, .133	.080, .129, .178	.033, .085, .188	.052, .106, .365	.032, .122, .414	.075, .115, .157	.030, .067, .119
alt-9	.011, .039, .102	.012, .027, .088	.014, .051, .099	.010, .095, .287	.043, .038, .123	.025, .060, .168	.003, .024, .070	.000, .036, .159
	$n = 256$				$n = 512$			
alt-1	.815, .932, .987	.889, .908, .987	.972, .998, 1.00	.001, .059, .220	1.00, 1.00, 1.00	.991, 1.00, 1.00	1.00, 1.00, 1.00	.079, .401, .585
alt-2	.483, .946, .994	.518, .942, .989	.012, .033, .074	.000, .016, .146	1.00, 1.00, 1.00	.999, 1.00, 1.00	.010, .037, .085	.000, .020, .127
alt-3	.883, .891, .939	.878, .879, .937	.931, .982, .990	.003, .132, .325	1.00 1.00, 1.00	.965, 1.00, 1.00	.984, .999, 1.00	.003, .086, .249
alt-4	.893, .977, .996	.809, .979, .995	.929, .981, .992	.005, .096, .273	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.997, 1.00, 1.00	.013, .209, .384
alt-5	.886, .983, .999	.867, .989, .999	.908, .977, .991	.003, .058, .223	1.00, 1.00, 1.00	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.363, .720, .753
alt-6	.131, .250, .428	.141, .276, .478	.067, .189, .293	.002, .012, .091	.403, .762, .877	.449, .805, .907	.131, .306, .449	.000, .024, .136
alt-7	.121, .247, .331	.119, .224, .314	.151, .349, .503	.007, .022, .058	.514, .666, .709	.557, .636, .768	.675, .850, .920	.035, .087, .171
alt-8	.083, .252, .358	.042, .174, .410	.049, .106, .158	.018, .035, .091	.537, .851, .975	.516, .870, .983	.060, .173, .285	.001, .020, .067
alt-9	.702, .973, .998	.673, .951, .989	.004, .034, .081	1.00, .026, .145	.994, 1.00, 1.00	.994, 1.00, 1.00	.009, .048, .101	1.00, .029, .147

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.6.: b. Rejection Frequencies under H_1: **Walsh Basis**
Case 2: $\mathcal{H}_T = 2T^{.49}$ and $\mathcal{K}_T = .5T^{.49}$
$\boldsymbol{\epsilon_t \sim t_5}$

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.102, .214, .333	.143, .314, .433	.134, .357, .512	.000, .021, .091	.407, .432, .538	.428, .490, .597	.777, .919, .948	.001, .029, .131
alt-2	.013, .044, .146	.013, .049, .161	.010, .033, .070	.007, .071, .175	.035, .112, .264	.037, .131, .265	.004, .036, .065	.000, .025, .122
alt-3	.213, .317, .446	.200, .315, .453	.126, .379, .505	.001, .035, .114	.316, .468, .567	.317, .453, .542	.375, .628, .738	.000, .019, .077
alt-4	.082, .264, .317	.052, .157, .298	.142, .385, .524	.003, .042, .124	.464, .493, .696	.468, .500, .698	.717, .887, .933	.002, .049, .164
alt-5	.045, .192, .264	.023, .117, .295	.056, .099, .137	.021, .068, .143	.282, .430, .669	.214, .465, .673	.114, .281, .407	.005, .015, .043
alt-6	.031, .056, .138	.028, .041, .146	.050, .095, .139	.008, .038, .111	.055, .094, .187	.047, .084, .206	.036, .108, .194	.004, .016, .047
alt-7	.012, .055, .086	.010, .042, .100	.022, .063, .094	.005, .041, .108	.092, .121, .213	.082, .115, .187	.060, .140, .216	.008, .022, .057
alt-8	.021, .081, .118	.012, .043, .115	.056, .099, .137	.023, .066, .150	.036, .074, .266	.007, .087, .291	.040, .071, .114	.011, .025, .057
alt-9	.020, .038, .072	.014, .024, .081	.011, .036, .069	.006, .087, .204	.023, .056, .135	.007, .070, .173	.004, .024, .074	.000, .023, .097
	$n = 256$				$n = 512$			
alt-1	.727, .792, .894	.773, .795, .874	.976, .995, .998	.000, .022, .111	.963, .974, .986	.949, .977, .983	1.00, 1.00, 1.00	0.035, .407, .722
alt-2	.441, .678, .833	.455, .658, .826	.004, .027, .059	.000, .019, .061	.841, .958, .975	.821, .954, .971	.005, .035, .065	.001, .010, .026
alt-3	.783, .803, .824	.781, .784, .842	.923, .976, .987	.002, .036, .190	.929, .969, .998	.725, .909, .958	.979, .998, .999	.000, .026, .153
alt-4	.840, .896, .906	.753, .798, .906	.909, .972, .988	.000, .034, .150	.914, .979, .989	.915, .979, .989	.997, 1.00, 1.00	.001, .108, .326
alt-5	.753, .816, .912	.707, .833, .917	.867, .956, .976	.000, .023, .104	.927, .972, .985	.936, .975, .987	1.00, 1.00, 1.00	.102, .527, .740
alt-6	.107, .154, .316	.103, .124, .306	.072, .186, .278	.000, .006, .022	.214, .387, .561	.230, .409, .579	.146, .307, .430	.000, .003, .026
alt-7	.132, .261, .294	.106, .284, .247	.165, .361, .480	.000, .005, .025	.472, .521, .659	.426, .510, .623	.647, .850, .914	.012, .032, .120
alt-8	.082, .275, .367	.080, .187, .308	.020, .062, .098	.003, .008, .023	.472, .597, .869	.486, .637, .887	.019, .081, .159	.001, .003, .004
alt-9	.405, .795, .902	.354, .736, .866	.007, .023, .060	.000, .008, .057	.828, .954, .973	.771, .950, .967	.003, .039, .084	.000, .007, .027

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.6.: c. Rejection Frequencies under H_1: **Walsh Basis**
Case 2: $\mathcal{H}_T = 2T^{.49}$ and $\mathcal{K}_T = .5T^{.49}$
$\epsilon_t \sim$ **GARCH**

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.065, .211, .360	.085, .277, .367	.139, .378, .546	.002, .030, .128	.308, .422, .591	.305, .407, .560	.820, .939, .970	.000, .057, .183
alt-2	.021, .054, .162	.021, .062, .173	.022, .051, .106	.008, .079, .257	.044, .099, .266	.043, .104, .277	.005, .033, .088	.001, .032, .168
alt-3	.085, .217, .371	.078, .216, .379	.156, .418, .588	.007, .046, .170	.218, .311, .467	.209, .300, .453	.371, .657, .778	.001, .017, .114
alt-4	.120, .339, .389	.087, .242, .291	.152, .408, .554	.007, .046, .171	.421, .496, .590	.325, .400, .593	.758, .909, .953	.001, .086, .281
alt-5	.056, .156, .223	.043, .127, .216	.073, .120, .163	.027, .071, .150	.157, .368, .522	.136, .394, .506	.148, .324, .455	.015, .028, .078
alt-6	.041, .116, .195	.041, .101, .181	.066, .133, .203	.019, .040, .141	.082, .172, .278	.081, .164, .270	.046, .149, .245	.004, .017, .081
alt-7	.052, .123, .199	.030, .105, .135	.056, .117, .159	.018, .061, .163	.092, .117, .208	.094, .106, .196	.062, .129, .210	.018, .035, .092
alt-8	.046, .117, .159	.035, .103, .147	.073, .120, .163	.028, .076, .158	.043, .095, .248	.034, .089, .269	.055, .090, .148	.020, .051, .115
alt-9	.020, .038, .098	.016, .029, .096	.012, .049, .083	.004, .067, .271	.057, .087, .162	.049, .080, .189	.009, .038, .081	.000, .031, .168
	$n = 256$				$n = 512$			
alt-1	.447, .589, .677	.442, .567, .647	.974, .998, .999	.005, .051, .228	.557, .665, .807	.539, .657, .803	1.00, 1.00, 1.00	.076, .375, .540
alt-2	.282, .370, .576	.287, .353, .551	.009, .042, .102	.000, .029, .142	.517, .689, .803	.590, .667, .798	.009, .051, .099	.000, .031, .155
alt-3	.421, .498, .600	.418, .489, .688	.927, .982, .992	.003, .117, .311	.635, .767, .865	.636, .766, .868	.986, .999, 1.00	.004, .080, .247
alt-4	.518, .580, .675	.527, .583, .675	.916, .979, .989	.003, .087, .241	.746, .794, .888	.755, .808, .889	.995, 1.00, 1.00	.015, .181, .339
alt-5	.424, .596, .721	.449, .518, .709	.892, .962, .983	.002, .056, .229	.829, .916, .963	.587, .828, .909	1.00, 1.00, 1.00	.347, .674, .719
alt-6	.107, .174, .296	.108, .179, .295	.085, .214, .325	.000, .017, .085	.135, .384, .428	.143, .302, .434	.169, .350, .491	1.00, .030, .137
alt-7	.102, .247, .357	.102, .227, .377	.171, .378, .507	.006, .027, .055	.331, .468, .660	.321, .428, .622	.686, .880, .928	.031, .093, .178
alt-8	.103, .275, .350	.104, .240, .321	.049, .100, .166	.010, .029, .082	.335, .415, .638	.252, .392, .649	.062, .181, .284	.006, .028, .063
alt-9	.197, .577, .754	.168, .499, .711	.012, .046, .105	.001, .025, .160	.503, .775, .884	.424, .740, .859	.009, .045, .090	.000, .028, .150

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

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