Supplemental material for "A bootstrapped test of covariance stationarity based on orthonormal transformations"

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A. Introduction

We list the assumptions for reference in Appendix B. Appendix C presents a method for automatic selection of \mathcal{H}_T and \mathcal{K}_T . Appendix D contains omitted proofs. We provide an empirical study in Appendix E, and Appendix F contains all simulation results.

We use the following notation. [z] rounds z to the nearest integer. L_2 is the space of square integrable random variables; and $L_2[a, b)$ is the class of square integrable functions on [a, b). $|| \cdot ||_p$ and $|| \cdot ||$ are the L_p and l_2 norms respectively, $p \ge 1$. Let $\mathbb{Z} = \{... -2, -1, 0, 1, 2, ...\}$, and $\mathbb{N} = \{0, 1, 1, 2, ...\}$. K > 0is a finite constant whose value may be different in different places. awp1 denotes "asymptotically with probability approaching one". Write $\max_{\mathcal{H}_T} = \max_{0 \le h \le \mathcal{H}_T} \cdot \max_{\mathcal{K}_T} = \max_{1 \le k \le \mathcal{K}_T}$ and $\max_{\mathcal{H}_T, \mathcal{K}_T} = \max_{0 \le h \le \mathcal{H}_T, 1 \le k \le \mathcal{K}_T}$. Similarly, $\max_{\mathcal{H}_T} a(h, \tilde{h}) = \max_{0 \le h, \tilde{h} \le \mathcal{H}_T} a(h, \tilde{h})$, etc.

B. Assumptions

Write

$$z_t(h,k) \equiv \{X_t X_{t+h} b_k(t) - E\left[X_t X_{t+h}\right] b_k(t)\}$$
$$\mathcal{Z}_T(h,k) \equiv \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} z_t(h,k).$$

Define σ -fields

$$\mathcal{F}_{T,t}^{\infty} \equiv \sigma\left(\{X_t X_{t+h} : 0 \le h \le \mathcal{H}_T\}_{\tau \ge t}\right) \text{ and } \mathcal{F}_{T,-\infty}^t \equiv \sigma\left(\{X_t X_{t+h} : 0 \le h \le \mathcal{H}_T\}_{\tau \le t}\right),$$

and α -mixing coefficients $\alpha_l \equiv \limsup_{T \to \infty} \sup_{t \in \mathbb{Z}} \sup_{\mathcal{A} \subset \mathcal{F}_{T, -\infty}^t} \mathcal{B} \subset \mathcal{F}_{T, t+l}^\infty | \mathcal{P}(\mathcal{A} \cap \mathcal{B}) - \mathcal{P}(\mathcal{A}) \mathcal{P}(\mathcal{B}) |$, for l > 0.

Assumption 1.

a. (geometric mixing): $\{X_t\}$ is α -mixing with coefficients $\alpha_l = O(\exp\{-l^{\phi}\})$ for some $\phi > 0$.

b. (subexponential tails): $\max_{1 \le t \le T} P(|X_t| > c) \le \varpi \exp\{-c^{\vartheta_1} \mathcal{E}_T^{-\vartheta_2}\}$ for some $\varpi \ge 1$, $\vartheta_1 \ge 2\vartheta_2$ and $\vartheta_2 \ge 1$, and some sequence of constants $\{\mathcal{E}_T\}$, $\liminf_{T \to \infty} \mathcal{E}_T \ge 1$.

c. (nondegeneracy): $\liminf_{T\to\infty} E[\mathcal{Z}_T^2(h,k)] > 0 \ \forall (h,k).$

d. (orthonormal basis): $\{\mathcal{B}_k(x) : 0 \le k \le \mathcal{K}\}$ forms a complete orthonormal basis on $\mathcal{L}[0,1)$; $\mathcal{B}_k(x) \in \{-1,1\}$ on [0,1); and $|\sum_{t=1}^T B_k(t)| = O(\eta(k))$ for some positive strictly monotonic function $\eta : \mathbb{R}_+ \to \mathbb{R}_+, \eta(k) \nearrow \infty$ as $k \to \infty$

Set a block size b_T such that $1 \le b_T < T$, $b_T/T^{\iota} \to \infty$ and $b_T/T^{1-\iota} \to 0$ for some tiny $\iota > 0$ that may be different in different places.

Assumption 2.

a. (i) $\liminf_{T\to\infty} s_T^2(h,k;\tilde{h},\tilde{k}) > 0 \ \forall (h,\tilde{h},k,\tilde{k}); and (ii) \max_{\mathcal{H}_T,\mathcal{K}_T} |s_T^2(h,k;\tilde{h},\tilde{k}) - s^2(h,k;\tilde{h},\tilde{k})| = O(T^{-\iota})$ for some infinitessimal $\iota > 0$.

b. $b_T/T^{\iota} \to \infty$ and $b_T = o(T^{1/2-\iota})$ for some infinitessimal $\iota > 0$.

C. Automatic $\{\mathcal{H}_T, \mathcal{K}_T\}$ selection

We propose picking $(\mathcal{H}_T, \mathcal{K}_T)$ in a data dependent manner based on the lag selection method in Hill and Motegi (2020, Section 3), an extension of Escanciano and Lobato's (2009) method for lag selection. Cf. Inglot and Ledwina (2006). We consider selecting \mathcal{H}_T for a given \mathcal{K}_T , and selecting \mathcal{K}_T for a given \mathcal{H}_T . Finally, we discuss an empirical approach for selecting both iteratively. Assume the orthonormal basis is comprised of Walsh functions $W_k(x)$ to focus ideas, hence $\mathcal{K}_T = o(\sqrt{T})$ is required, cf. Remark 8 in the main paper.

C.1. $\mathcal{H}^*_T(\mathcal{K}_T)$

The optimal $\mathcal{H}_{T}^{*}(\mathcal{K}_{T})$ for a given \mathcal{K}_{T} is chosen from a set $\{0, ..., \overline{\mathcal{H}}_{T}\}$ for some pre-chosen upperbound $\overline{\mathcal{H}}_{T}$, where $\overline{\mathcal{H}}_{T} \to \infty$ with $\overline{\mathcal{H}}_{T} = O(\sqrt{T})$. Similarly \mathcal{K}_{T} is pre-chosen from $\{1, ..., \overline{\mathcal{K}}_{T}\}$ where $\overline{\mathcal{K}}_{T} \to \infty$ and $\overline{\mathcal{K}}_{T} = o(\sqrt{T})$. We only consider sequences $\{\mathcal{H}_{T}, \mathcal{K}_{T}\}$ that satisfy $\mathcal{H}_{T}/\overline{\mathcal{H}}_{T} \to [0, K]$ and $\mathcal{K}_{T}/\overline{\mathcal{K}}_{T} \to [0, K]$ for finite K > 0. Put K = 1 for ease of notation. Under H_{0} , Escanciano and Lobato's (2009) method leads to $P(\mathcal{H}_{T}^{*}(\mathcal{K}_{T}) = 0) \to 1$ because higher lags do not provide useful information and incur a high penalty (see below). Thus, we need to allow for sequences $\{\mathcal{H}_{T}, \mathcal{K}_{T}\}$ that converge or diverge, e.g. $\mathcal{H}_{T} \to [0, ..., \infty]$.

Similar to Hill and Motegi (2020), cf. Escanciano and Lobato (2009), define a *penalized max-correlation difference*:

$$\mathcal{M}_{T}^{\mathcal{P}}(\mathcal{H},\mathcal{K}) \equiv \mathcal{M}_{T}(\mathcal{H},\mathcal{K}) - \mathcal{P}_{T}(\mathcal{H},\mathcal{K}) \text{ where } \mathcal{M}_{T}(\mathcal{H},\mathcal{K}) \equiv \max_{0 \leq h \leq \mathcal{H}, 1 \leq k \leq \mathcal{K}} \left| \sqrt{T} \left(\hat{\rho}_{h}^{(k)} - \hat{\rho}_{h} \right) \right|,$$

with penalty function $\mathcal{P}_T(\cdot)$:

$$\mathcal{P}_{T}(\mathcal{H},\mathcal{K}) = \begin{cases} \sqrt{(\mathcal{H}+1)\mathcal{K}\ln T} \text{ if } \mathcal{M}_{T}(\mathcal{H},\mathcal{K}) \leq \sqrt{q\ln T} \\ \sqrt{2(\mathcal{H}+1)\mathcal{K}} & \text{if } \mathcal{M}_{T}(\mathcal{H},\mathcal{K}) > \sqrt{q\ln T} \end{cases}$$
(C.1)

where q is a fixed positive constant. We need $\mathcal{H} + 1$ since $\mathcal{H} = 0$ is possible, covering a test exclusively for variance constancy. Notice $(\mathcal{H} + 1)\mathcal{K}$ is the total number of objects being searched over. A small value of q leads to the AIC penalty $\sqrt{2(\mathcal{H} + 1)\mathcal{K}}$ being chosen with high probability, while a large q promotes selection of the BIC penalty $\sqrt{(\mathcal{H} + 1)\mathcal{K} \ln T}$. A low q is therefore akin to the de facto AIC penalty used in Jin, Wang and Wang (2015) applied to h. In their setting, however, they are not choosing $\{\mathcal{H}, \mathcal{K}\}$; rather, they penalize the use of higher lags and systematic samples in a maximized high dimensional Wald statistic. Escanciano and Lobato (2009) find q = 2.4 is suitable for their penalized Qstatistic based on empirical size and power, and based on results in Inglot and Ledwina (2006). Hill and Motegi (2020) find that q = 3 works well for their max-correlation difference statistic. In experiments not reported here we find $q \in [1/4, 3/4]$ lead to competitive empirical size for the max-correlation difference.

Following Escanciano and Lobato (2009) and Hill and Motegi (2020), the chosen optimal lag is for each *T*:

$$\mathcal{H}_{T}^{*}(\mathcal{K}_{T}) = \min\left\{\mathcal{H}: 0 \leq \mathcal{H} \leq \bar{\mathcal{H}}_{T}: \mathcal{M}_{T}^{\mathcal{P}}(\mathcal{H}, \mathcal{K}_{T}) \geq \mathcal{M}_{T}^{\mathcal{P}}(h, \mathcal{K}_{T}) \text{ for } h = 0, ..., \bar{\mathcal{H}}_{T}\right\}.$$
(C.2)

In order to characterize $\mathcal{H}^*_T(\mathcal{K}_T)$ we need

$$\Delta r_T(h,k) \equiv \frac{1}{\frac{1}{T} \sum_{t=1}^T E\left[X_t^2\right]} \frac{1}{T} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t).$$

Under Assumption 1 it is straightforward to prove (see the proof of Theorem C.1):

$$\max_{h,k} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h,k) \right| \xrightarrow{p} 0.$$

Under H_0 and Lemma 3.a in Jin, Wang and Wang (2015),

$$|\Delta r_T(h,k)| = \left|\frac{E\left[X_t X_{t+h}\right]}{E\left[X_t^2\right]} \frac{1}{T} \sum_{t=1}^{T-h} B_k(t)\right| \le |\rho_h| \left|\frac{k+1}{T}\right| = O(1/T).$$

Consider the global alternative $E[X_t X_{t+h}] = \gamma_h + c_h(t/T)$, cf, (16) in the main paper, where $c_h : [0,1] \to \mathbb{R}$ are integrable functions on [0,1] uniformly over *h*. Then $\lim_{T\to\infty} T^{-1} \sum_{t=1}^T E[X_t^2] = \gamma_0 + \int_0^1 c_0(u) \, du > 0$, and

$$\lim_{R \to \infty} \Delta r_T(h,k) = \Delta r(h,k) \equiv \frac{\int_0^1 c_h(u) \mathcal{B}_k(u) du}{\gamma_0 + \int_0^1 c_0(u) du}$$

Now define the smallest lag h at which the largest (in magnitude) asymptotic correlation difference occurs over systematic samples $\{1, ..., \mathring{\mathcal{K}}\}$:

$$h^*(\mathring{\mathcal{K}}) \equiv \min\left\{h : h = \operatorname{argmax}_{h \in \mathbb{N}} \left\{\max_{k \in \{1, \dots, \mathring{\mathcal{K}}\}} |\Delta r(h, k)|\right\}\right\} \text{ where } \mathring{\mathcal{K}} \equiv \lim_{T \to \infty} \mathcal{K}_T \in [1, \infty],$$

Theorem C.1. Let Assumptions 1 and 2 hold. Let $\{\mathcal{K}_T\}$ satisfy $\mathcal{K}_T \in [1, ..., \bar{\mathcal{K}}_T]$ where $\mathcal{K}_T \to [1, \infty]$ and $\bar{\mathcal{K}}_T = o(\sqrt{T})$; let $\bar{\mathcal{H}}_T = O(\sqrt{T})$; and let $\bar{\mathcal{H}}_T \bar{\mathcal{K}}_T = o(T/\ln(T))$.

a. Under H_0 , $P(\mathcal{H}^*_T(\mathcal{K}_T) = 0) \to 1$ for any $\{\mathcal{K}_T\}$. Further $\max_{1 \le k \le \mathcal{K}} \mathcal{H}^*_T(k) \xrightarrow{p} 0$ for any fixed finite $1 \le \mathcal{K} \le \mathring{\mathcal{K}} \equiv \lim_{T \to \infty} \mathcal{K}_T$.

b. Under H_1 , $\mathcal{H}^*_{\mathcal{T}}(\mathcal{K}_T) \xrightarrow{p} h^*(\mathring{\mathcal{K}})$. Further $\max_{1 \le k \le \mathscr{K}} |\mathcal{H}^*_{\mathcal{T}}(k) - h^*(k)| \to 1$.

Remark 1. Under either hypothesis $\mathcal{H}_{T}^{*}(\mathcal{K}_{T})$ converges to a plausibly *most informative and efficient* value. We say both *informative* and *efficient* because we use maximal information under H_0 asymptotically with probability approaching one (no data points are trimmed with $\mathcal{H}_{T}^{*}(\mathcal{K}_{T}) = 0$).

Under H_1 we use the least lag (hence the least amount of trimmed sample points) at which \mathcal{K}_T optimizes the correlation difference. In our simulation study, for example, $h^*(\mathring{\mathcal{K}}) = 0$ or 1 for most models under H_1 because the non-covariance stationary processes used have zero autocovariances at lag $h \ge 2$. In alternative models 2 and 9 we have $h^*(\mathring{\mathcal{K}}) = 6$ and 25 because only at those lags are there non-zero autocovariances.

Remark 2. We require a slightly more restrictive bound $\overline{\mathcal{H}}_T \overline{\mathcal{K}}_T = o(T/\ln(T))$ compared to Theorem 4.1 where implicitly $\overline{\mathcal{H}}_T \overline{\mathcal{K}}_T = o(T)$. This ultimately arises from the logarithmic component in the penalty selection threshold $\sqrt{q \ln T}$.

C.2. $\mathcal{K}_T^*(\mathcal{H}_T)$

An identical procedure applies for selecting \mathcal{K}_T for a given maximum lag \mathcal{H}_T . The chosen optimal $\mathcal{K}_T^*(\mathcal{H}_T)$ is:

$$\mathcal{K}_{T}^{*}(\mathcal{H}_{T}) = \min\left\{\mathcal{K}: 1 \leq \mathcal{K} \leq \bar{\mathcal{K}}_{T}: \mathcal{M}_{T}^{\mathcal{P}}(\mathcal{H}_{T}, \mathcal{K}) \geq \mathcal{M}_{T}^{\mathcal{P}}(\mathcal{H}_{T}, k) \text{ for } k = 1, ..., \bar{\mathcal{K}}_{T}\right\}.$$
(C.3)

Define the smallest systematic sample counter k at which the largest (in magnitude) asymptotic correlation difference occurs:

$$k^{*}(\mathring{\mathcal{H}}) \equiv \min\left\{k : k = \operatorname{argmax}_{k \in \mathbb{N}} \left\{\max_{h \in \{0, \dots, \mathring{\mathcal{H}}\}} |\Delta r(h, k)|\right\}\right\} \text{ where } \mathring{\mathcal{H}} \equiv \lim_{T \to \infty} \mathcal{H}_{T}.$$

The proof of the next theorem is identical to the proof of Theorem C.1 and therefore omitted.

Theorem C.2. Let Assumption 1 and 2 hold. Let $\{\mathcal{H}_T\}$ be an arbitrary sequence of maximum lags, $\mathcal{H}_T \in [1, ..., \bar{\mathcal{H}}_T], \mathcal{H}_T \to [1, \infty]$ and $\mathcal{H}_T = O(\sqrt{T})$; let $\bar{\mathcal{K}}_T = o(\sqrt{T})$; and let $\bar{\mathcal{H}}_T \bar{\mathcal{K}}_T = o(T/\ln(T))$.

a. Under H_0 , $P(\mathcal{K}^*_T(\mathcal{H}_T) = 1) \to 1$ for any $\{\mathcal{H}_T\}$. Further $\sup_{0 \le h \le \mathcal{H}} \mathcal{K}^*_T(h) \xrightarrow{p} 1$ for any fixed finite $0 \le \mathcal{H} \le \mathring{\mathcal{H}} \equiv \lim_{T \to \infty} \mathcal{H}_T$.

b. Under H_1 , $\mathcal{K}^*_T(\mathcal{H}_T) \xrightarrow{p} k^*(\mathring{\mathcal{H}})$. Further $\sup_{0 \le h \le \mathcal{H}} |\mathcal{K}^*_T(h) - k^*(h)| \to 1$ for any fixed finite $1 \le \mathcal{H} \le \mathring{\mathcal{H}}$.

Remark 3. Under H_0 the variance and autocovariances are constant over time. Since no systematic sample provides information for detecting a break in (co)variance, the least of the set $\{1, ..., \overline{K}_T\}$ is the optimal choice.

C.3. Iterative $\{\mathcal{H}_{T}^{*}, \mathcal{K}_{T}^{*}\}$ Selection and Theorem C.1

We now discuss identifying $(\mathcal{H}_T^*, \mathcal{K}_T^*)$ iteratively for a given T, and present the main result. Write $\tilde{\mathcal{H}}_T^*(\mathcal{H}) \equiv \mathcal{H}_T^*(\mathcal{K}_T^*(\mathcal{H}))$ and $\tilde{\mathcal{K}}_T^*(\mathcal{K}) \equiv \mathcal{K}_T^*(\mathcal{H}_T^*(\mathcal{K}))$, hence $\tilde{\mathcal{H}}_T^* : \mathbb{N} \to \mathbb{N}$ and $\tilde{\mathcal{K}}_T^* : \mathbb{N} \to \mathbb{N}$. Identification of unique $\{\mathcal{H}_T^*, \mathcal{K}_T^*\}$ requires $\tilde{\mathcal{H}}_T^*$ and $\tilde{\mathcal{K}}_T^*$ to be contraction mappings (see below). If we begin with an arbitrary start h_0 , set $h_1 = \tilde{\mathcal{H}}_T^*(h_0)$ and iterate $h_{m+1} = \tilde{\mathcal{H}}_T^*(h_m)$, then by the Banach fixed point theorem $h_m \xrightarrow{p} \mathcal{H}_T^*$ as $m \to \infty$.

The algorithm requires going back and forth between $\mathcal{H}_T^*(\mathcal{K}_T^*(\mathcal{H}))$ and $\mathcal{K}_T^*(\mathcal{H}_T^*(\mathcal{K}))$ due to the cross-embedded arguments. Indeed, the iteration on $\tilde{\mathcal{H}}_T^*(h_m)$ implicitly simultaneously iterates on $\tilde{\mathcal{K}}_T^*(\cdot)$. The steps are as follows:

(i) Pick h_0 and compute $k_0 \equiv \mathcal{K}_T^*(h_0)$ in (C.3);

(ii) Compute $h_1 \equiv \mathcal{H}_T^*(k_0)$ and $\hat{k}_1 \equiv \mathcal{K}_T^*(h_1)$ using (C.2) and (C.3);

(iii) Iterate

$$h_{m+1} \equiv \mathcal{H}_T^*(k_m) \text{ and } k_{m+1} \equiv \mathcal{K}_T^*(h_{m+1}). \tag{C.4}$$

(iv) Cease iterations when $m \ge M$ for some preset maximum iteration $M \in \mathbb{N}$, or

$$|h_{m+1} - h_m| \le \tau_h$$
 and $|k_{m+1} - k_m| \le \tau_k$

where $(\tau_h, \tau_k) > 0$ are pre-chosen tolerances. Clearly $|h_{m+1} - h_m|$ and $|k_{m+1} - k_m|$ are integer valued, so $(\tau_h, \tau_k) \in \{0, 1, ...\}$. In experiments not reported here $\tau_h, \tau_k = 0$ lead to $m \le 25$ in all simulated samples, hence convergence was easily satisfied.

We implicitly have both iterations $h_{m+1} = \tilde{\mathcal{H}}_T^*(h_m) = \mathcal{H}_T^*(\mathcal{K}_T^*(h_m))$ and $k_{m+1} \equiv \tilde{\mathcal{K}}_T^*(k_m) = \mathcal{K}_T^*(\mathcal{H}_T^*(k_m))$. Therefore $h_m \xrightarrow{p} \mathcal{H}_T^*$ and $k_m \xrightarrow{p} \mathcal{K}_T^*$ as $m \to \infty$ provided the fixed point theorem applies. We therefore need $\tilde{\mathcal{H}}_T^*(\mathcal{H})$ and $\tilde{\mathcal{K}}_T^*(\mathcal{K})$ to be contractions mappings awp1. Consider $\tilde{\mathcal{H}}_T^*$, and note that we require for any pair $\{h_0, h_1\}$ and some finite $\delta_{\mathcal{H}} > 0$:

$$\left|\tilde{\mathcal{H}}_{T}^{*}(h_{1}) - \tilde{\mathcal{H}}_{T}^{*}(h_{0})\right| = \left|\mathcal{H}_{T}^{*}(\mathcal{K}_{T}^{*}(h_{1})) - \mathcal{H}_{T}^{*}(\mathcal{K}_{T}^{*}(h_{0}))\right| \le \delta_{\mathcal{H}} \left|h_{1} - h_{0}\right| \quad awp \ 1.$$

This is trivial under H_0 since by Theorem C.1.a. $\sup_{1 \le k \le \mathcal{K}} \mathcal{H}_T^*(k) \xrightarrow{p} 0$ for all finite $1 \le \mathcal{K} \le \mathring{\mathcal{K}} \equiv \lim_{T \to \infty} \mathcal{K}_T$.

Conversely, under H_1 Theorem C.2.b yields $\sup_{0 \le h \le \mathcal{H}} |\mathcal{K}^*_T(h) - k^*(h)| \to 1$ for any finite $1 \le \mathcal{H} \le \mathcal{H}$, where

$$k^*(h_i) \equiv \min\left\{k : k = \operatorname{argmax}_{k \in \mathbb{N}} \left\{\max_{h \in \{0, \dots, h_i\}} |\Delta r(h, k)|\right\}\right\}.$$

Now invoke Theorem C.1.b to deduce:

$$\begin{aligned} \left| \tilde{\mathcal{H}}_{T}^{*}(h_{1}) - \tilde{\mathcal{H}}_{T}^{*}(h_{0}) \right| &= \left| \mathcal{H}_{T}^{*}(\mathcal{K}_{T}^{*}(h_{1})) - \mathcal{H}_{T}^{*}(\mathcal{K}_{T}^{*}(h_{0})) \right| \\ &= \left| \min\left\{ h : h = \operatorname{argmax}_{h \in \mathbb{N}} \left\{ \max_{k \in \{1, \dots, k^{*}(h_{i})\}} \left| \int_{0}^{1} c_{h}\left(u\right) \mathcal{B}_{k}(u) du \right| \right\} \right\} \right| \\ &- \min\left\{ h : h = \operatorname{argmax}_{h \in \mathbb{N}} \left\{ \max_{k \in \{1, \dots, k^{*}(h_{0})\}} \left| \int_{0}^{1} c_{h}\left(u\right) \mathcal{B}_{k}(u) du \right| \right\} \right\} \right| + o_{p}\left(1\right) \\ &\equiv \Delta h(h_{1}, h_{0}) + o_{p}\left(1\right), \end{aligned}$$

where $\Delta h(\cdot)$ is implicitly defined. We therefore need

$$\Delta h(h_1, h_0) \le \delta_{\mathcal{H}} |h_1 - h_0| \ \forall (h_0, h_1).$$
(C.5)

Property (C.5) effectively restricts directions of deviation from H_0 .

Assumption 3. Let $\Delta h(h_1, h_0) \leq \delta_{\mathcal{H}} |h_1 - h_0| \ \forall (h_0, h_1) \text{ and } \Delta k(k_1, k_0) \leq \delta_{\mathcal{K}} |k_1 - k_0| \ \forall (k_0, k_1),$ where $\delta_{\mathcal{H}}, \delta_{\mathcal{K}} > 0$ are fixed constants.

The discussion leading to (C.5) proves the following claim.

Theorem C.3. Let Assumptions 1-3 hold. Then $h_{m+1} \equiv \mathcal{H}_T^*(k_m)$ and $k_{m+1} \equiv \mathcal{K}_T^*(h_{m+1})$ defined in (C.4) satisfy $h_m \xrightarrow{p} \mathcal{H}_T^*$ and $k_m \xrightarrow{p} \mathcal{K}_T^*$ as $m \to \infty$.

Theorems C.1 and C.2 now yield the following.

Theorem C.4. Let Assumptions 1-3 hold. As $T \to \infty$, $(h_m, k_m) \xrightarrow{p} (0, 1)$ under H_0 , and under H_1 we have $(h_m, k_m) \xrightarrow{p} (h^*(\mathring{\mathcal{K}}), k^*(\mathring{\mathcal{H}}))$.

C.4. Proof of Theorem C.1

Let q be any fixed positive constant. Recall the penalty function is $\mathcal{P}_T(\mathcal{H},\mathcal{K}) = \sqrt{(\mathcal{H}+1)\mathcal{K}\ln T}$ if $\mathcal{M}_T(\mathcal{H},\mathcal{K}) \leq \sqrt{q\ln T}$, else $\mathcal{P}_T(\mathcal{H},\mathcal{K}) = \sqrt{2(\mathcal{H}+1)\mathcal{K}}$.

In order to reduce notation we drop the argument \mathcal{K}_T and write, e.g., $\mathcal{M}_T(\mathcal{H}_T) = \mathcal{M}_T(\mathcal{H}_T, \mathcal{K}_T)$, $\mathcal{P}_T(\mathcal{H}_T) = \mathcal{P}_T(\mathcal{H}_T, \mathcal{K}_T)$, $\mathcal{H}_T^* = \mathcal{H}_T^*(\mathcal{K}_T)$.

Claim (a). Let H_0 be true. We will only prove the first claim $P(\mathcal{H}_T^*(\mathcal{K}_T) = 0) \to 1$ for any $\{\mathcal{K}_T\}$. It then follows that $\mathcal{H}_T^*(\mathcal{K}) \xrightarrow{p} 0$ for any fixed finite $1 \le \mathcal{K} \le \mathring{\mathcal{K}}$. The second claim $\max_{1 \le k \le \mathscr{K}} \mathcal{H}_T^*(k) \xrightarrow{p} 0$ requires pointwise convergence and equicontinuity. Pointwise convergence follows from the first claim, and equicontinuity is trivial for integer-valued functions.

It suffices to prove the following. *First*, for any $\mathcal{H}_T, \mathcal{H}_T \to [0, \infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \to [0, 1]$, the penalty term satisfies:

$$P\left(\mathcal{P}_T(\mathcal{H}_T) = \sqrt{(\mathcal{H}_T + 1)\,\mathcal{K}_T \ln T}\right) \to 1.$$
(C.6)

Hence $\mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T) \equiv \mathcal{M}_T(\mathcal{H}_T) - \sqrt{(\mathcal{H}_T + 1)\mathcal{K}_T \ln T}$ asymptotically with probability approaching one. *Second*, for such $\{\mathcal{H}_T\}$ the following holds:

$$P\left(\mathcal{M}_{T}(\mathcal{H}_{T}) - \mathcal{M}_{T}(h) \ge \left(\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right)\sqrt{\ln(T)}\right)$$
(C.7)

$$\rightarrow \begin{cases} 1 \text{ if } h \ge \mathcal{H}_{T} \\ 0 \text{ for fixed } h = 0, ..., \mathcal{H}_{T} - 1 \end{cases}$$

Together (C.6) and (C.7) prove the claim $P(\mathcal{H}_T^* = 0) \to 1$ since the following holds for every $h = 0, ..., \overline{\mathcal{H}}_T$ if and only if $\mathcal{H}_T \to 0$:

$$\lim_{T \to \infty} P\left(\mathcal{M}_{\mathcal{T}}^{(\mathcal{P})}(\mathcal{H}_{T}) \ge \mathcal{M}_{T}^{(\mathcal{P})}(h)\right)$$

$$= \lim_{T \to \infty} P\left(\mathcal{M}_{\mathcal{T}}(\mathcal{H}_{T}) - \mathcal{M}_{\mathcal{T}}(h) \ge \left(\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right)\sqrt{\ln(T)}\right) = 1,$$
(C.8)

while \mathcal{H}_T^* is the least of sequences that satisfy (C.8) for every $h = 0, ..., \overline{\mathcal{H}}_T$.

(C.6). By construction of $\mathcal{P}_T(\mathcal{H}_T)$ it suffices to prove $P(\mathcal{M}_T(\mathcal{H}_T) > \sqrt{q \ln T}) \to 0$. Under H_0 , $\sqrt{T}(\hat{\rho}_h^{(k)} - \hat{\rho}_h) = O_p(1)$ by Theorem 3.4, hence $\sqrt{T}(\hat{\rho}_h^{(k)} - \hat{\rho}_h)/\sqrt{q \ln T} \to 0$ for any fixed $q \in (0, \infty)$. Therefore, by Theorem 3.4 for some non-unique $\{\bar{\mathcal{H}}_T, \bar{\mathcal{K}}_T\}, \bar{\mathcal{H}}_T, \bar{\mathcal{K}}_T \to \infty, \bar{\mathcal{H}}_T = o(T)$:

$$\frac{\mathcal{M}_{\mathcal{T}}(\bar{\mathcal{H}}_T)}{\sqrt{q\ln T}} = \frac{\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \sqrt{T} (\hat{\rho}_h^{(k)} - \hat{\rho}_h) \right|}{\sqrt{q\ln T}} \xrightarrow{p} 0.$$
(C.9)

Moreover $\overline{\mathcal{H}}_T \overline{\mathcal{K}}_T = o(T/\ln(T))$ by assumption. By monotonicity of $\mathcal{M}_T(\cdot) \ge 0$, (C.9) holds for any $\{\mathcal{H}_T, \mathcal{K}_T\}$ where $\mathcal{H}_T \to [0, \infty]$ and $\mathcal{H}_T/\overline{\mathcal{H}}_T \to [0, 1], \mathcal{K}_T \to [1, \infty]$ and $\mathcal{K}_T/\overline{\mathcal{K}}_T \to [0, 1]$. Thus $\mathcal{M}_T(\mathcal{H}_T)/\sqrt{q \ln T} \xrightarrow{p} 0$ for all such $\{\mathcal{H}_T\}$.

(C.7). Suppose $h > \mathcal{H}_T$. By (C.9), $\mathcal{M}_T(\bar{\mathcal{H}}_T)/\sqrt{\ln T} = o_p(1)$ and therefore $\mathcal{M}_T(\mathcal{H}_T) - \mathcal{M}_T(h) = o_p(\sqrt{\ln(T)})$ for any $\{\mathcal{H}_T\}$ where $\mathcal{H}_T \to [0,\infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \to [0,1]$, and any $0 \le h \le \bar{\mathcal{H}}_T$. Now use

(C.6), monotonicity of $\mathcal{M}_T(\cdot)$, and $\inf_{T \ge 1} \{\sqrt{(h+1)\mathcal{K}_T} - \sqrt{(\mathcal{H}_T+1)\mathcal{K}_T}\} > 0$, to yield that as $T \to \infty$:

$$P\left(\mathcal{M}_{\mathcal{T}}(\mathcal{H}_{T}) - \mathcal{M}_{\mathcal{T}}(h) \ge \left(\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right)\sqrt{\ln(T)}\right)$$
$$= P\left(\frac{\mathcal{M}_{\mathcal{T}}(\mathcal{H}_{T}) - \mathcal{M}_{\mathcal{T}}(h)}{\sqrt{\ln(T)}} \ge \sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right)$$
$$= P\left(\sqrt{(h+1)\mathcal{K}_{T}} - \sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} \ge \frac{\mathcal{M}_{\mathcal{T}}(h) - \mathcal{M}_{\mathcal{T}}(\mathcal{H}_{T})}{\sqrt{\ln(T)}}\right) \to 1.$$

Similarly, if $h = \mathcal{H}_T$ then $\sqrt{(h+1)\mathcal{K}_T} - \sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} = 0$ and $\mathcal{M}_T(h) - \mathcal{M}_T(\mathcal{H}_T) = 0$ hence the above limit holds.

Conversely, suppose $h \in \{0, ..., \mathcal{H}_T - 1\}$ and $\mathcal{H}_T > 1$. Then from $\mathcal{M}_T(\mathcal{H}_T) = o_p(\sqrt{q \ln T})$ and $1 - \sqrt{(h+1)/(\mathcal{H}_T + 1)} > 0$ it follows:

$$P\left(\mathcal{M}_{T}(\mathcal{H}_{T}) - \mathcal{M}_{T}(h) \ge \left(\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right)\sqrt{\ln(T)}\right)$$
$$= P\left(\frac{\mathcal{M}_{T}(\mathcal{H}_{T}) - \mathcal{M}_{T}(h)}{\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}}\sqrt{\ln(T)}} \ge \left(1 - \sqrt{\frac{h+1}{\mathcal{H}_{T}+1}}\right)\right) \to 0.$$

(C.7) follows directly.

Claim (b). Let H_1 hold. Similar to (a), we need only prove $\mathcal{H}^*_T(\mathcal{K}_T) \xrightarrow{p} h^*(\mathring{\mathcal{K}})$. Recall under H_1 , cf. (16) and (17), both $\gamma_0 + \int_0^1 c_0(u) \, du > 0$ and:

$$\Delta r_T(h,k) = \frac{1}{\frac{1}{T} \sum_{t=1}^T E\left[X_t^2\right]} \frac{1}{T} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t)$$

satisfies $|\hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h,k)| \xrightarrow{p} 0$ and

$$\lim_{T \to \infty} \Delta r_T(h,k) = \Delta r(h,k) \equiv \frac{\int_0^1 c_h(u) \mathcal{B}_k(u) du}{\gamma_0 + \int_0^1 c_0(u) du} \neq 0 \text{ for some } h \ge 0, k \ge 1.$$

Recall $\mathring{\mathcal{K}} \equiv \lim_{T \to \infty} \mathscr{K}_T$, and

$$h^* \equiv h^*(\mathring{\mathcal{K}}) \equiv \min\left\{h: h = \operatorname{argmax}_{h \in \mathbb{N}} \left\{\max_{k \in \{1, \dots, \mathring{\mathcal{K}}\}} |\Delta r(h, k)|\right\}\right\},\$$

and define the finite sample version:

$$h_T^* \equiv \min\left\{h_T : h_T = \operatorname{argmax}_{0 \le h \le \mathcal{H}_T} \max_{1 \le k \le \mathcal{K}_T} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h \right| \right\},\$$

the smallest lag at which the largest sample correlation difference in magnitude occurs.

Define

$$\mathbb{N}_{1}(h) \equiv \left\{ h \in \mathbb{N} : \max_{k \in \mathbb{N}} |\Delta r(h, k)| \neq 0 \right\},\$$

and $\underline{N}_1 \equiv \min_{h \in \mathbb{N}} \mathbb{N}_1(h)$, the smallest lag at which the largest k^{th} Walsh coefficient $\int_0^1 c_h(u) \mathcal{B}_k(u) du \neq 0$.

We prove in Step 1 that for any integer sequence $\{\mathcal{H}_T\}$ such that $\mathcal{H}_T \to [N_1, \infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \to [0, 1]$, the penalty satisfies for any sequence of maximum number of systematic samples $\{\mathcal{K}_T\}$:

$$\mathcal{P}\left(\mathcal{P}_T(\mathcal{H}_T) = \sqrt{2(\mathcal{H}_T + 1)\mathcal{K}_T}\right) \to \infty.$$
 (C.10)

We then prove in Step 2 that *if and only if* $\mathcal{H}_T/h_T^* \xrightarrow{p} [1,\infty]$, for each $0 \le h \le \overline{\mathcal{H}}_T$ we have

$$P\left(\mathcal{M}_{T}(\mathcal{H}_{T}) \geq \mathcal{M}_{T}(h) + 2\left(\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right)\right) \to 1.$$
(C.11)

Moreover, $\mathcal{H}_T, \mathcal{K}_T \to \infty$, $|\hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h, k)| \xrightarrow{p} 0$ (see (C.13) below) and $\Delta r_T(h, k) \to \Delta r(h, k)$ yields:

$$h_T^* \xrightarrow{p} h^* \equiv \min\left\{h : h = \operatorname{argmax}_{h \in \mathbb{N}} \left\{ \max_{1 \le k \le \lim_{T \to \infty} \mathcal{K}_T} |\Delta r(h, k)| \right\} \right\}.$$

Notice $h^* \in [N_1, \infty)$ by construction of N_1 .

The proof of the claim then proceeds as follows. Take any integer sequence $\{\mathcal{H}_T\}, \mathcal{H}_T/h_T^* \xrightarrow{p} [1, \infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \to [0, 1]$. Then (C.10) holds because $h^* \in [N_1, \infty)$, hence $\mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T) \equiv \mathcal{M}_T(\mathcal{H}_T) - \sqrt{2(\mathcal{H}_T+1)\mathcal{K}_T}$ awp1. Since such a sequence implies (C.11), we have $\mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T) \geq \mathcal{M}_T^{\mathcal{P}}(h)$ awp1 for each $h = 0, ..., \bar{\mathcal{H}}_T$. Conversely, if (C.11) holds then $\mathcal{H}_T/h_T^* \xrightarrow{p} [1, \infty]$. This yields (C.10) because $h^* \in [N_1, \infty)$. Therefore $\mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T) \geq \mathcal{M}_T^{\mathcal{P}}(h)$ awp1 for each $h = 0, ..., \bar{\mathcal{H}}_T$ if and only if $\mathcal{H}_T/h_T^* \xrightarrow{p} [1, \infty]$. Since the optimal $\{\mathcal{H}_T^*\}$ is the least of such sequences, the selection \mathcal{H}_T^* satisfies $\mathcal{H}_T/h_T^* \xrightarrow{p} 1$. Together $\mathcal{H}_T/h_T^* \xrightarrow{p} h^*$ prove the claim.

Step 1: Consider (C.10). By the triangle inequality

$$\begin{aligned} \left| \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \hat{\rho}_{h}^{(k)} - \hat{\rho}_{h} \right| &- \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \Delta r(h,k) \right| \\ &\leq \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \hat{\rho}_{h}^{(k)} - \hat{\rho}_{h} - \Delta r(h,k) \right| \\ &\leq \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \hat{\rho}_{h}^{(k)} - \hat{\rho}_{h} - \Delta r_{T}(h,k) \right| + \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \Delta r_{T}(h,k) - \Delta r(h,k) \right|. \end{aligned}$$
(C.12)

Note under H_1 :

$$\frac{1}{T}\sum_{t=1}^{T}E[X_t^2] \to g_0 \equiv \gamma_0 + \int_0^1 c_0(u)du \in (0,\infty)$$

$$\frac{1}{T}\sum_{t=1}^{T-h} E[X_t X_{t+h}] B_k(t) \to w_{h,k} \equiv \int_0^1 c_h(u) \mathcal{B}_k(u) du.$$

Hence, Lemma 3.1 and variance bound (A.5) yield for some integer sequences $\{\bar{\mathcal{H}}_T, \mathcal{K}_T\}, \bar{\mathcal{H}}_T, \mathcal{K}_T \rightarrow \infty$:

$$\begin{split} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \hat{\rho}_{h}^{(k)} - \hat{\rho}_{h} - \Delta r_{T}(h, k) \right| \\ &= \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{\hat{\gamma}_{0}} \frac{1}{T} \sum_{t=1}^{T-h} X_{t} X_{t+h} B_{k}(t) \right| \\ &- \frac{1}{\frac{1}{T} \sum_{t=1}^{T} E\left[X_{t}^{2}\right]} \frac{1}{T} \sum_{t=1}^{T-h} E\left[X_{t} X_{t+h}\right] B_{k}(t) \right| \\ &\leq \frac{1}{\hat{\gamma}_{0}} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{t=1}^{T-h} \left\{ X_{t} X_{t+h} B_{k}(t) - E\left[X_{t} X_{t+h}\right] B_{k}(t) \right\} \right| \\ &+ \left\{ \left| \frac{1}{T} \sum_{t=1}^{T} \left\{ X_{t}^{2} - E\left[X_{t}^{2}\right] \right\} \right| \times \frac{1}{\frac{1}{T} \sum_{t=1}^{T} E\left[X_{t}^{2}\right] \hat{\gamma}_{0}} \right. \\ &\left. \times \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{t=1}^{T-h} E\left[X_{t} X_{t+h}\right] B_{k}(t) \right| \right\} \\ &= O_{p}(1/\sqrt{T}) + O_{p}(1/\sqrt{T}) \times \frac{1}{g_{0}^{2}} \times \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| w_{h,k} \right|. \end{split}$$

By Assumption 1.b,d $\max_{\mathcal{H}_T, \mathcal{K}_T} |w_{h,k}| < \infty$. Hence

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h, k) \right| = O_p(1/\sqrt{T}).$$
(C.13)

Further:

$$\begin{split} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} |\Delta r_{T}(h,k) - \Delta r(h,k)| \\ &\leq \frac{1}{\gamma_{0} + \int_{0}^{1} c_{0}(u) \, du} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{t=1}^{T-h} E\left[X_{t} X_{t+h} \right] B_{k}(t) - \int_{0}^{1} c_{h}(u) \, \mathcal{B}_{k}(u) \, du \right| \\ &+ \left\{ \left| \frac{1}{T} \sum_{t=1}^{T} E\left[X_{t}^{2} \right] - \left(\gamma_{0} + \int_{0}^{1} c_{0}(u) \, du \right) \right| \times \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \int_{0}^{1} c_{h}(u) \, \mathcal{B}_{k}(u) \, du \right| \right. \\ &\left. \times \frac{1}{\frac{1}{T} \sum_{t=1}^{T} E\left[X_{t}^{2} \right] \left(\gamma_{0} + \int_{0}^{1} c_{0}(u) \, du \right)} \right\} \\ &= \mathcal{A}_{T} + \mathcal{B}_{1,T} \mathcal{B}_{2,T} \mathcal{B}_{3,T}, \end{split}$$

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say where $(\mathcal{A}_T, \mathcal{B}_{i,T})$ are implicitly defined. By construction and (3.10)-(3.11) in the main paper, $\mathcal{A}_T \rightarrow 0$, and:

$$\begin{aligned} \mathcal{B}_{1,T} &= \left| \frac{1}{T} \sum_{t=1}^{T} \left(\gamma_0 + c_0(t/T) \right) - \left(\gamma_0 + \int_0^1 c_0(u) \, du \right) \right| \to 0 \\ \mathcal{B}_{2,T} &= \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \int_0^1 c_h(u) \, \mathcal{B}_k(u) \, du \right| = O(1) \\ \mathcal{B}_{3,T} &= \frac{1}{\frac{1}{T} \sum_{t=1}^{T} E\left[X_t^2 \right] \left(\gamma_0 + \int_0^1 c_0(u) \, du \right)} \to \frac{1}{\left(\gamma_0 + \int_0^1 c_0(u) \, du \right)^2} \in (0, \infty) \,. \end{aligned}$$

Hence $\max_{\mathcal{H}_T, \mathcal{K}_T} |\Delta r_T(h, k) - \Delta r(h, k)| \rightarrow 0$. Coupled with (C.12) and (C.13), this yields:

$$\left| \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h \right| - \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \Delta r(h, k) \right| \right| \xrightarrow{P} 0, \tag{C.14}$$

where

$$\lim_{T \to \infty} \max_{\mathcal{H}_T, \mathcal{K}_T} |\Delta r(h, k)| = \max_{h, k \in \mathbb{N}} \left| \frac{\int_0^1 c_h(u) \mathcal{B}_k(u) du}{\gamma_0 + \int_0^1 c_0(u) du} \right| \in (0, \infty).$$
(C.15)

Therefore for any $\{\mathcal{H}_T\}, \mathcal{H}_T \to [N_1, \infty] \text{ and } \mathcal{H}_T / \bar{\mathcal{H}}_T \to [0, 1]:$

$$\frac{\mathcal{M}_{T}(\mathcal{H}_{T})}{\sqrt{q \ln T}} = \frac{\sqrt{T} \max_{\mathcal{H}_{T}, \mathcal{K}_{T}} \left| \hat{\rho}_{h}^{(k)} - \hat{\rho}_{h} \right|}{\sqrt{q \ln T}} \\ = \frac{\sqrt{T} \left(\max_{\mathcal{H}_{T}, \mathcal{K}_{T}} \left| \Delta r(h, k) \right| + o_{p}(1) \right)}{\sqrt{q \ln T}} \xrightarrow{p} \infty.$$

This proves (C.10) by construction of the penalty term $\mathcal{P}_T(\mathcal{H}_T)$, cf. (C.1).

Step 2: Next, (C.11). First, by (C.14) and (C.15) $\mathcal{M}_T(\mathcal{H}_T)/\sqrt{T} \xrightarrow{p} (0,\infty)$ for any $\{\mathcal{H}_T\}, \mathcal{H}_T \rightarrow [\underline{N}_1,\infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \rightarrow [0,1]$. Hence $\mathcal{M}_T(\mathcal{H}_T)/\sqrt{T/\ln(T)} \xrightarrow{p} \infty$ for any $\mathcal{H}_T \rightarrow [\underline{N}_1,\infty]$. Monotonicity ensures $\mathcal{M}_T(\mathcal{H}_T) \geq \mathcal{M}_T(h)$ for each $0 \leq h \leq \mathcal{H}_T$, hence for such h:

$$\frac{\mathcal{M}_T(h)}{\mathcal{M}_T(\mathcal{H}_T)} = \frac{\mathcal{M}_T(h)/\sqrt{T}}{\mathcal{M}_T(\mathcal{H}_T)/\sqrt{T}} \xrightarrow{p} [0,1].$$

Indeed, if both

$$(h, \mathcal{H}_T) \ge h_T^* \equiv \min\left\{h_T : h_T = \operatorname{argmax}_{0 \le h \le \mathcal{H}_T} \max_{1 \le k \le \mathcal{K}_T} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h \right| \right\}$$

then by construction $\mathcal{M}_T(h)/\mathcal{M}_T(\mathcal{H}_T) = 1$.

Now suppose $0 \le h$ and $h/\mathcal{H}_T \to [0, 1)$, and $\mathcal{H}_T/h_T^* \xrightarrow{p} [0, 1)$, hence $0 \le h < \mathcal{H}_T < h_T^*$ as $T \to \infty$ awp1. Then $\mathcal{M}_T(h)/\mathcal{M}_T(\mathcal{H}_T) \xrightarrow{p} [0, 1)$ by monotonicity and the construction of h_T^* . Use $\mathcal{H}_T \le \bar{\mathcal{H}}_T$, $\mathcal{K}_T\mathcal{H}_T = o(T/\ln(T))$, and $\mathcal{M}_T(\mathcal{H}_T)/\sqrt{T/\ln(T)} \xrightarrow{p} \infty$, to yield:

$$P\left(\mathcal{M}_{T}(\mathcal{H}_{T}) \geq \mathcal{M}_{T}(h) + 2\left(\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right)\right)$$
(C.16)
$$= P\left(\mathcal{M}_{T}(\mathcal{H}_{T})\left(1 - \frac{\mathcal{M}_{T}(h)}{\mathcal{M}_{T}(\mathcal{H}_{T})}\right) \geq 2\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}}\left(1 - \sqrt{\frac{h+1}{\mathcal{H}_{T}+1}}\right)\right)$$
$$\geq P\left(\frac{\mathcal{M}_{T}(\mathcal{H}_{T})}{\sqrt{T/\ln(T)}}\left(1 - \frac{\mathcal{M}_{T}(h)}{\mathcal{M}_{T}(\mathcal{H}_{T})}\right) \geq 2\sqrt{\frac{(\mathcal{H}_{T}+1)\mathcal{K}_{T}}{T/\ln(T)}}\right) \rightarrow 1.$$

Next, consider $0 \le h$ and $h/h_T^* \xrightarrow{p} [0,1)$, and $\mathcal{H}_T/h_T^* \xrightarrow{p} [1,\infty]$, hence $0 \le h \le h_T^* - 1$ awp1 and $\mathcal{H}_T \ge h_T^*$ awp1. Then $P(\mathcal{M}_T(h) = \mathcal{M}_T(\mathcal{H}_T) \to 0$ since by construction h_T^* is the smallest lag at which the maximum correlation difference occurs. Monotonicity therefore yields $\mathcal{M}_T(h)/\mathcal{M}_T(\mathcal{H}_T) \xrightarrow{p} [0,1)$, and again we deduce (C.16).

Now let $(h, \mathcal{H}_T) \ge h_T^* awp1$. Then by construction $\mathcal{M}_T(\mathcal{H}_T) = \mathcal{M}_T(h) awp1$. Trivially if $h < \mathcal{H}_T$ $(h \ge \mathcal{H}_T)$ then $\sqrt{\mathcal{H}_T} - \sqrt{h} > 0$ ($\sqrt{\mathcal{H}_T} - \sqrt{h} \le 0$). Hence

$$P\left(\mathcal{M}_{T}(\mathcal{H}_{T}) \geq \mathcal{M}_{T}(h) + 2\left[\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right]\right) \to 1 \text{ if and if } h \geq \mathcal{H}_{T}.$$

Next, let $\mathcal{H}_T < h_T^* \leq h \ awp1$ such that $\mathcal{M}_T(h) = \mathcal{M}_T(h_T^*) \ awp1$. Use $\mathcal{H}_T/h \to [0,1), \ h = o(T/\ln(T)), \ \mathcal{M}_T(h_T^*)/\sqrt{T/\ln(T)} \xrightarrow{p} \infty$, and $\mathcal{M}_T(\mathcal{H}_T)/\mathcal{M}_T(h_T^*) \xrightarrow{p} [0,1)$ to yield:

$$P\left(\mathcal{M}_{T}(\mathcal{H}_{T}) \geq \mathcal{M}_{T}(h) + 2\left[\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right]\right)$$
$$= P\left(2\left(1 - \sqrt{\frac{\mathcal{H}_{T}+1}{h+1}}\right)\sqrt{\frac{h}{T/\ln(T)}} \geq \frac{\mathcal{M}_{T}(h_{T}^{*})}{\sqrt{T/\ln(T)}}\left(1 - \frac{\mathcal{M}_{T}(\mathcal{H}_{T})}{\mathcal{M}_{T}(h_{T}^{*})}\right)\right) \to 0$$

Finally, generally $\mathcal{M}_T(h) = \mathcal{M}_T(\mathcal{H}_T)$ a.s. for some $\{h, \mathcal{H}_T\}$ and all but a finite number of T is possible. For example, when $h = \mathcal{H}_T$. In this case, *if and only if* $h \ge \mathcal{H}_T$:

$$P\left(\mathcal{M}_{T}(\mathcal{H}_{T}) \geq \mathcal{M}_{T}(h) + 2\left[\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right]\right)$$
$$= P\left(0 \geq 2\left[\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right]\right) \to 1$$

Combining the above results, we deduce for every $0 \le h \le \overline{\mathcal{H}}_T$ if and only if $\mathcal{H}_T \ge h_T^*$:

$$P\left(\mathcal{M}_{T}(\mathcal{H}_{T}) \geq \mathcal{M}_{T}(h) + 2\left[\sqrt{(\mathcal{H}_{T}+1)\mathcal{K}_{T}} - \sqrt{(h+1)\mathcal{K}_{T}}\right]\right) \to 1$$

This proves (C.11). QED.

D. Omitted proofs

D.1. Lemma A.2

Lemma A.2. Let $\max_{1 \le t \le T} P(|X_t| > c) \le \varpi \exp\{-c^{\vartheta_1} \mathcal{E}_T^{-\vartheta_2}\}$ for some $\varpi > 0$, any $\vartheta_1 \ge 2\vartheta_2$ and $\vartheta_2 \ge 1$, and some sequence of constants $\{\mathcal{E}_T\}$, $\liminf_{T \to \infty} \mathcal{E}_T \ge 1$. It holds that

$$\max_{1 \le t_1, \dots, t_r \le T} P\left(\left| \prod_{i=1}^r X_{t_i} \right| > c \right) \le r \varpi \exp\left\{ -c^{\vartheta_2} \mathcal{E}_T^{-\vartheta_2} \right\}.$$
(D.1)

Proof. We prove (D.1) by induction. If r = 1 then $\max_{1 \le t \le T} P(|X_t| > c) \le \varpi \exp\{-c^{\vartheta_1} \mathcal{E}_T^{-\vartheta_2}\}$ $\le \varpi \exp\{-c^{\vartheta_2} \mathcal{E}_T^{-\vartheta_2}\}$ by assumption, given $\vartheta_1 > \vartheta_2 \ge 1$. Now let (D.1) hold for some $r \ge 1$: $\max_{1 \le t_1, \dots, t_r \le T} P(|\Box_{i=1}^r X_{t_i}| > c) \le r \varpi \exp\{-c^{\vartheta_2} \mathcal{E}_T^{-\vartheta_2}\}$. Young and Bonferroni inequalities yield for any $\vartheta_1 \ge 2\vartheta_2$

$$\begin{split} \max_{1 \le t_1, \dots, t_{r+1} \le T} P\left(\left| \prod_{i=1}^{r+1} X_{t_i} \right| > c \right) \le \max_{1 \le t_1, \dots, t_{r+1} \le T} P\left(\frac{1}{2} \left(\prod_{i=1}^r X_{t_i} \right)^2 + \frac{1}{2} X_{t_{r+1}}^2 > c \right) \\ \le \max_{1 \le t_1, \dots, t_r \le T} P\left(\left| \prod_{i=1}^r X_{t_i} \right| > c^{\frac{1}{2}} \right) + \max_{1 \le t \le T} P\left(|X_t| > c^{\frac{1}{2}} \right) \\ \le r \varpi \exp\left\{ -c^{\vartheta_2} \mathcal{E}_T^{-\vartheta_2} \right\} + \varpi \exp\left\{ -c^{\vartheta_1/2} \mathcal{E}_T^{-\vartheta_2} \right\} \\ \le (r+1) \varpi \exp\left\{ -c^{\vartheta_2} \mathcal{E}_T^{-\vartheta_2} \right\}. \end{split}$$

Hence (D.1) holds for r + 1. The proof is complete because r is arbitrary. $Q\mathcal{ED}$.

D.2. Lemma A.4

Let \Rightarrow^p denote weak convergence in probability on l_{∞} (the space of bounded functions) as defined in Giné and Zinn (1990, Section 3). Recall $\{\mathcal{E}_T\}$ is the Assumption 1 exponential moment scale, lim inf $_{T\to\infty} \mathcal{E}_T \ge 1$; the bootstrap index blocks are $\mathfrak{B}_s = \{(s-1)b_T + 1, \ldots, sb_T\}$, $s = 1, \ldots, T/b_T$, with block size b_T , $1 \le b_T < T$, $b_T \to \infty$ and $b_T/T^{1-\iota} \to 0$ for some small $\iota > 0$; ξ_i is iid N(0, 1); and $\varphi_t = \xi_s$ if $t \in \mathfrak{B}_s$. Recall the number of blocks $\mathcal{N}_T = [T/b_T]$, and

$$\Delta \hat{g}_T^{(dw)}(h,k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t \left\{ X_t X_{t+h} B_k(t) - \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) \right\},$$

and define

$$\mathring{\sigma}_T^2(h,k) \equiv E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\varphi_t\left\{X_tX_{t+h}B_k(t) - \frac{1}{T}\sum_{s=1}^{T-h}E\left[X_sX_{s+h}\right]B_k(s)\right\}\right)^2\right].$$

Recall

$$z_t(h,k) \equiv \{X_t X_{t+h} - E[X_t X_{t+h}]\} B_k(t)$$

$$\sigma_T^2(h,k) \equiv E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h} z_t(h,k)\right)^2\right]$$

We refer to the following condition below:

$$\frac{1}{T^{1/9}} \left[\mathcal{E}_T^{2/3} \left\{ \ln \left(\mathcal{H}_T \mathcal{K}_T \right) \right\}^{(1+2\phi)/(3\phi)} + \mathcal{E}_T \left(\ln \mathcal{H}_T \mathcal{K}_T \right)^{7/6} \right] \to 0.$$
(D.2)

Lemma A.4. Let Assumptions 1 and 2 hold.

a. Let $\{\mathbf{\mathring{Z}}_{T}(h,k): 0 \le h \le \mathcal{H}_{T}, 1 \le k \le \mathcal{K}_{T}\}_{T \le 1}$ be a Gaussian process, $\mathbf{\mathring{Z}}_{T}(h,k) \sim N(0, \overset{\circ}{\sigma}_{T}^{2}(h,k))$, independent of the sample $\{X_{t}\}_{t=1}^{T}$. For any sequences $\{\mathcal{E}_{T}, \mathcal{H}_{T}, \mathcal{K}_{T}\}$, where $0 \le \mathcal{H}_{T} < T - 1$, $\mathcal{H}_{T} = o(T), \eta(\mathcal{K}_{T}) = o(\sqrt{T})$ and (D.2) hold:

$$\sup_{c>0} \left| P\left(\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \sqrt{T} \Delta \hat{g}_T^{(dw)}(h, k) \right| \le c \left| \{X_t\}_{t=1}^T \right| - P\left(\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \mathring{Z}_T(h, k) \right| \le c \right) \right| \xrightarrow{p} 0.$$

b. Let $\{\mathring{\mathbf{Z}}(h,k)\}$ be an independent copy of the Lemma 3.1 Gaussian process $\{\mathbf{Z}(k,h) : h, k \in \mathbb{N}\}$, $\mathbf{Z}(h,k) \sim N(0, \lim_{T\to\infty} \sigma_T^2(h,k))$, independent of the asymptotic draw $\{X_t\}_{t=1}^{\infty}$. For any sequences $\{b_T, \mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$, such that $0 \leq \mathcal{H}_T < T - 1$, $b_T/T^\iota \to \infty$, $b_T = o(T^{1/2-\iota})$, $\mathcal{H}_T = O(T^{1-\iota}/b_T)$, $\eta(\mathcal{K}_T) = o(\sqrt{T})$, and (D.2) hold:

$$\max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \sqrt{T} \Delta \hat{g}_{T}^{(dw)}(h,k) \right| \Rightarrow^{p} \max_{h,k \in \mathbb{N}} \left| \mathring{Z}(h,k) \right|$$

The proof requires two preliminary results. We first prove the following uniform sample covariance result. Write

$$\hat{g}(h,k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) \text{ and } g_T(h,k) \equiv E\left[\hat{g}(h,k)\right] = \frac{1}{T} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t)$$

Lemma D.1. Under Assumption 1, for any $\{\mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$ satisfying $0 \leq \mathcal{H}_T < T - 1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, $\eta(\mathcal{K}_T) = o(\sqrt{T})$ and (D.2):

$$\max_{\mathcal{H}_T, \mathcal{K}_T} |\hat{g}(h, k) - g_T(h, k)| = O_p\left(1/\sqrt{T}\right).$$

Proof. Define

$$\mathcal{G}_{T}(h,k) \equiv \sqrt{T} \left(\hat{g}(h,k) - g_{T}(h,k) \right) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} \left\{ X_{t} X_{t+h} - E \left[X_{t} X_{t+h} \right] \right\} B_{k}(t)$$

and $s_T^2(h,k) \equiv E[\mathcal{G}_T^2(h,k)]$. The argument used to prove Lemma 3.1 yields

$$\sup_{z\geq 0} \left| P\left(\max_{\mathcal{H}_T, \mathcal{K}_T} |\mathcal{G}_T(h, k)| \le z \right) - P\left(\max_{\mathcal{H}_T, \mathcal{K}_T} |\mathcal{G}_T(h, k)| \le z \right) \right| \to 0$$

for some sequence of random functions $\{G_T(h,k)\}_{T\geq 1}$ with $G_T(h,k) \sim N(0,s_T^2(h,k))$, for any $\{\mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$ satisfying $0 \leq \mathcal{H}_T < T - 1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, $\eta(\mathcal{K}_T) = o(\sqrt{T})$ and (D.2). The claim follows instantly. $Q\mathcal{ED}$.

Next, define

$$y_t(h,k) \equiv \left\{ X_t X_{t+h} B_k(t) - \frac{1}{T-h} \sum_{s=1}^{T-h} E\left[X_s X_{s+h} \right] B_k(s) \right\}.$$

We decompose the following summand into big and little blocks:

$$\Delta \ddot{g}_{T}^{*}(h,k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} \varphi_{t} \left\{ X_{t} X_{t+h} B_{k}(t) - \frac{1}{T-h} \sum_{s=1}^{T-h} E\left[X_{s} X_{s+h} \right] B_{k}(s) \right\} = \frac{1}{T} \sum_{t=1}^{T-h} \varphi_{t} y_{t}(h,k).$$

Let \tilde{b}_T and \tilde{l}_T be block sizes, $(\tilde{b}_T, \tilde{l}_T) \to \infty$, with $1 < \tilde{b}_T < T$, $\tilde{b}_T = o(T)$, $1 \le \tilde{l}_T < \tilde{b}_T$, and $\tilde{l}_t = o(\tilde{b}_T)$. In each index set $\{1, ..., T - h\}$ the number of blocks is $\tilde{\mathcal{N}}_T(h) = [(T - h)/\tilde{b}_T]$. Denote the blocks by $\tilde{\mathfrak{B}}_s = \{(s-1)\tilde{b}_T + 1, ..., s\tilde{b}_T\}$ with $s = 1, ..., \tilde{\mathcal{N}}_T(h)$, and $\tilde{\mathfrak{B}}_{\tilde{\mathcal{N}}_T(h)+1} = \{\tilde{\mathcal{N}}_T(h)\tilde{b}_T, ..., T + h\}$. Then

$$\begin{split} \Delta \ddot{g}_{T}^{*}(h,k) &= \frac{1}{T} \sum_{i=1}^{\widetilde{N}_{T}(h)} \sum_{t=(i-1)\widetilde{b}_{T}+\widetilde{l}_{T}+1}^{i\widetilde{b}_{T}} \varphi_{t} y_{t}(h,k) + \frac{1}{T} \sum_{i=1}^{\widetilde{N}_{T}(h)} \sum_{t=(i-1)\widetilde{b}_{T}+1}^{(i-1)\widetilde{b}_{T}+\widetilde{l}_{T}} \varphi_{t} y_{t}(h,k) \\ &+ \frac{1}{T} \sum_{i=\widetilde{N}_{T}(h)\widetilde{b}_{T}+1}^{T-h} \varphi_{t} y_{t}(h,k). \end{split}$$

Lemma D.2. Under Assumptions 1 and 2, for any $\{\mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$ satisfying $0 \leq \mathcal{H}_T < T - 1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, $\eta(\mathcal{K}_T) = o(\sqrt{T})$ and (D.2:

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \Delta \ddot{g}_T^*(h,k) \right| - \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T} \sum_{i=1}^{\widetilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\widetilde{b}_T + \widetilde{t}_T + 1}^{i\widetilde{b}_T} \varphi_t y_t(h,k) \right| = o_p \left(1/\sqrt{T} \right).$$

Proof. The triangle inequality yields for any real-valued functions $\{a(h,k), b(h,k)\}$

$$\max_{\mathcal{H}_{T},\mathcal{K}_{T}} |a(h,k)| - \max_{\mathcal{H}_{T},\mathcal{K}_{T}} |b(h,k)| \le \max_{\mathcal{H}_{T},\mathcal{K}_{T}} |a(h,k) - b(h,k)|.$$

We therefore prove

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \left| \Delta \ddot{g}_T^*(h, k) \right| - \left| \frac{1}{T} \sum_{i=1}^{\tilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\tilde{b}_T + \tilde{l}_T + 1}^{i\tilde{b}_T} \varphi_t y_t(h, k) \right| \right| = o_p \left(1/\sqrt{T} \right),$$

Step 1. It suffices to replace $y_t(h, k)$ with $z_t(h, k)$ uniformly *awp1*:

$$\max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{t=1}^{T-h} \varphi_{t} \{ y_{t}(h,k) - z_{t}(h,k) \} \right| = o_{p}(1).$$
(D.3)

This follows by noting:

$$\frac{1}{T}\sum_{t=1}^{T-h}\varphi_{t}y_{t}(h,k) - \frac{1}{T}\sum_{t=1}^{T-h}\varphi_{t}z_{t}(h,k)$$

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$$\leq \left| \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t \left\{ E\left[X_t X_{t+h} \right] B_k(t) - \frac{1}{T-h} \sum_{s=1}^{T-h} E\left[X_s X_{s+h} \right] B_k(s) \right\} \right|$$

$$\leq \left| \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t E\left[X_t X_{t+h} \right] B_k(t) \right| + O_p\left(1/\sqrt{T} \right)$$

in view of $\max_{\mathcal{H}_T, \mathcal{K}_T} |(T - h)^{-1} \sum_{s=1}^{T-h} E[X_s X_{s+h}] B_k(s)| \leq K$ under Assumption 1.b,d, and $\max_{\mathcal{H}_T} |1/T \sum_{t=1}^{T-h} \varphi_t| = O_p(1/\sqrt{T})$. The latter follows by construction of φ_t :

$$\frac{1}{T}\sum_{t=1}^{T-h}\varphi_t = \frac{1}{T/b_T}\sum_{i=1}^{[(T-h)/b_T]}\xi_i = \frac{1}{T/b_T}\sum_{i=1}^{[\lambda_h T/b_T]}\xi_i \text{ with } \lambda_h = 1 - \frac{h}{T} \in [0, 1),$$

where iid $\xi_i \sim N(0,1)$. By Donsker's theorem extended to $\mathcal{D}[0,1]$, and the mapping theorem, $\sup_{\lambda \in [0,1)} |1/\sqrt{N} \sum_{i=1}^{\lfloor \lambda N \rfloor} \xi_i| = O_p(1) \text{ (cf Dudley, 1999).}$ Next, by construction:

$$\frac{1}{T} \sum_{t=1}^{T-h} \varphi_t E\left[X_t X_{t+h}\right] B_k(t)$$

$$= \frac{1}{T/b_T} \sum_{i=1}^{[(T-h)/b_T]} \xi_i \frac{1}{b_T} \sum_{t=(i-1)b_T+1}^{ib_T} E\left[X_t X_{t+h}\right] B_k(t) = \frac{1}{T/b_T} \sum_{i=1}^{N_T(h)} \xi_i \varpi_{T,i}(h,k)$$

say, where $\varpi_{T,i}(h,k) \equiv 1/b_T \sum_{t=(i-1)b_T+1}^{ib_T} E[X_t X_{t+h}] B_k(t)$ and $\mathcal{N}_T(h) \equiv [(T-h)/b_T]$. Given ξ_i is id N(0,1), a generalization of Nemirovski's \mathcal{L}_q -moment bound, $q \ge 1$, for independent sequences yields (see, e.g., Bühlmann and Van De Geer, 2011, Lemma 14.24):

$$E\left[\max_{\mathcal{H}_{T},\mathcal{K}_{T}}\left|\frac{1}{T/b_{T}}\sum_{i=1}^{N_{T}(h)}\xi_{i}\varpi_{T,i}(h,k)\right|^{q}\right] \leq \left\{\frac{8\ln\left(2\mathcal{H}_{T}\mathcal{K}_{T}\right)\max_{\mathcal{H}_{T},\mathcal{K}_{T}}\max_{1\leq i\leq N_{T}(h)}\infty_{T,i}^{2}(h,k)}{T/b_{T}}\right\}^{q/2}.$$
 (D.4)

Moreover:

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \max_{1 \le i \le \mathcal{N}_T(h)} \varpi_{T,i}^2(h,k) = \max_{\mathcal{H}_T, \mathcal{K}_T} \max_{1 \le i \le \mathcal{N}_T(h)} \left(\frac{1}{b_T} \sum_{t=(i-1)b_T+1}^{ib_T} E\left[X_t X_{t+h}\right] B_k(t) \right)^2 \quad (D.5)$$
$$\leq \left(\max_{\mathcal{H}_T} \max_{1 \le t \le T} |E\left[X_t X_{t+h}\right]| \right)^2 \equiv \bar{\varpi}_T^2.$$

Now combine (D.4) and (D.5), choose q = 2, and invoke $b_T = o(T^{1/2-\iota})$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and $\bar{\varpi}_T = O(1)$ under Assumption 1.b and the Cauchy-Schwartz inequality, to yield:

$$E\left[\max_{\mathcal{H}_{T},\mathcal{K}_{T}}\left(\frac{1}{T/b_{T}}\sum_{i=1}^{N_{T}(h)}\xi_{i}\varpi_{T,i}(h,k)\right)^{2}\right] \leq K\frac{\ln\left(\mathcal{H}_{T}\mathcal{K}_{T}\right)}{T/b_{T}}\bar{\varpi}_{T}^{2}$$
$$= o\left(\frac{\ln\left(T\right)}{T^{1/2+\iota}}\right) \times \bar{\varpi}_{T}^{2} = o\left(\frac{\ln\left(T\right)}{T^{1/2+\iota}}\right) \times o(T^{1/2}) = o(1).$$

This proves (D.3) by Chebyshev's inequality.

Step 2. Now observe that

$$\begin{split} \left| \Delta \ddot{g}_{T}^{*}(h,k) - \frac{1}{T} \sum_{i=1}^{(T-h)/b_{T}} \sum_{t=(i-1)b_{T}+l_{T}+1}^{ib_{T}} \varphi_{t} z_{t}(h,k) \right| \\ \leq \left| \frac{1}{T} \sum_{i=1}^{\tilde{N}_{T}(h)} \sum_{t=(i-1)\tilde{b}_{T}+1}^{(i-1)\tilde{b}_{T}+\tilde{h}_{T}} \varphi_{t} z_{t}(h,k) \right| + \left| \frac{1}{T} \sum_{i=\tilde{N}_{T}(h)\tilde{b}_{T}+1}^{T-h} \varphi_{t} z_{t}(h,k) \right| \end{split}$$

Lemma 3.1 and $\tilde{b}_T/\tilde{l}_T = o(1)$ yield under the assumed properties for $\{\mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$:

$$\begin{split} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{i=1}^{\widetilde{N}_{T}(h)} \sum_{t=(i-1)\widetilde{b}_{T}+1}^{(i-1)\widetilde{b}_{T}+\widetilde{l}_{T}} \varphi_{t} z_{t}(h,k) \right| &= \cdot \frac{\widetilde{l}_{T}}{\widetilde{b}_{T}} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T\widetilde{l}_{T}/\widetilde{b}_{T}} \sum_{i=1}^{\widetilde{N}_{T}(h)} \sum_{t=(i-1)\widetilde{b}_{T}+1}^{(i-1)\widetilde{b}_{T}+\widetilde{l}_{T}} \varphi_{t} z_{t}(h,k) \right| \\ &= O_{p} \left(\frac{\widetilde{l}_{T}/\widetilde{b}_{T}}{\sqrt{T\widetilde{l}_{T}/\widetilde{b}_{T}}} \right) = O_{p} \left(1/\sqrt{T\widetilde{b}_{T}/\widetilde{l}_{T}} \right) = o_{p} \left(1/\sqrt{T} \right). \end{split}$$

Similarly, for any (h, k), the integer-valued discrepancy implicit in $T - h - \tilde{N}_T(h)\tilde{b}_T = T - h - [(T - h)/\tilde{b}_T]\tilde{b}_T$ yields:

$$\begin{split} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{i=\tilde{N}_{T}(h)\tilde{b}_{T}+1}^{T-h} \varphi_{t} z_{t}(h,k) \right| &= O_{p} \left(\max_{\mathcal{H}_{T}} \frac{\sqrt{(T-h) - \tilde{N}_{T}(h)\tilde{b}_{T}}}{T} \right) \end{split} \tag{D.6}$$
$$&= O_{p} \left(\max_{\mathcal{H}_{T}} \frac{\sqrt{1 - \left[\frac{T}{\tilde{b}_{T}} (1-h/T) \right] \frac{\tilde{b}_{T}}{T(1-h/T)}}}{\sqrt{T}} \right) = o_{p} \left(1/T \right). \end{split}$$

This completes the proof. $Q\mathcal{ED}$

We are now ready to prove Lemma A.4. Assume $(T - h)/b_T$ and related ratios are integers to reduce notation. The resulting error otherwise is asymptotically negligible, as in (D.6).

Proof of Lemma A.4.

Claim (a). Define the sample $\mathfrak{X}_T \equiv \{X_t\}_{t=1}^T$, and define

$$\Delta g^*_T(h,k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t \left\{ X_t X_{t+h} B_k(t) - \frac{1}{T} \sum_{s=1}^{T-h} E\left[X_s X_{s+h} \right] B_k(s) \right\}.$$

Let $\{\Delta \hat{g}_T^{(dw)}(i)\}_{i=0}^{\mathcal{H}_T \mathcal{K}_T}$, etc., denote the stacked $\{\Delta \hat{g}_T^{(dw)}(h,k)\}_{h=0,k=1}^{\mathcal{H}_T,\mathcal{K}_T}$:

$$\Delta \hat{g}_T^{(dw)}(i) = \Delta \hat{g}_T^{(dw)}(h,k) \text{ with index correspondence } i = (k-1)\mathcal{H}_T + h. \tag{D.7}$$

and define

$$\begin{split} \hat{s}_{T}^{2}(i,j) &= TE\left[\Delta \hat{g}_{T}^{(dw)}(i)\Delta \hat{g}_{T}^{(dw)}(j) | \mathfrak{X}_{T}\right] \text{ and } \hat{s}_{T}^{2}(i,j) = TE\left[\Delta g_{T}^{*}(i)\Delta g_{T}^{*}(j) | \mathfrak{X}_{T}\right] \\ \Delta_{T} &\equiv \max_{0 \leq i,j \leq \mathcal{H}_{T}\mathcal{K}_{T}} \left| \hat{s}_{T}^{2}(i,j) - \hat{s}_{T}^{2}(i,j) \right|, \end{split}$$

hence $\mathring{s}_T^2(i,i) \equiv \mathring{\sigma}_T^2(h,k)$ where $i = (k-1)\mathcal{H}_T + h$.¹

Let $\{\mathring{Z}_T(i)\}_{T\leq 1}$ be sequences of normal random variables $\mathring{Z}_T(i) \sim N(0, \mathring{s}_T^2(i, i))$ independent of \mathfrak{X}_T . Lemma 3.1 in Chernozhukov, Chetverikov and Kato (2013), cf. Chernozhukov, Chetverikov and Kato (2015, Theorem 2, Proposition 1) and Chen (2018, Lemma C.1), yields:

$$\mathcal{E}_{T} \equiv \sup_{c>0} \left| P\left(\max_{0 \le i \le \mathcal{H}_{T} \mathcal{K}_{T}} \left| \sqrt{T} \Delta \hat{g}_{T}^{(dw)}(i) \right| \le c | \mathfrak{X}_{T} \right) - P\left(\max_{0 \le i \le \mathcal{H}_{T} \mathcal{K}_{T}} \left| \mathring{\mathbf{Z}}_{T}(i) \right| \le c \right) \right|$$

$$= O_{p} \left(\Delta_{T}^{1/3} \max \left\{ 1, \ln \left(\mathcal{H}_{T} \mathcal{K}_{T} / \Delta_{T} \right) \right\}^{2/3} \right).$$
(D.8)

We prove below $\Delta_T = O_p(1/T^{\iota})$ for some $\iota > 0$. Now use $\mathcal{H}_T = o(T)$ and $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$ to reach:

$$\mathcal{E}_T = O_p \left(\Delta_T^{1/3} \max \left\{ 1, \ln \left(\mathcal{H}_T \mathcal{K}_T / \Delta_T \right) \right\}^{2/3} \right)$$
$$= O_p \left(\Delta_T^{1/3} \max \left\{ 1, \ln \left(\sqrt{T} \mathcal{H}_T \mathcal{K}_T \right) + \ln \left(\sqrt{T} \Delta_T \right) \right\}^{2/3} \right) = O_p \left(\frac{1}{T^{\iota/6}} \left\{ \ln \left(T \right) \right\}^{2/3} \right) \xrightarrow{p} 0.$$

This suffices to prove the claim in view of the correspondence $i = (k - 1)\mathcal{H}_T + h$.

We now prove $\Delta_T = O_p(1/T^{\iota})$. Define for any $g \in \mathbb{R}$

$$\mathfrak{E}_{l,T}(h,k;g) \equiv \sum_{t=(l-1)b_T+1}^{lb_T} \{X_t X_{t+h} B_k(t) - g\},\$$

and define

$$\hat{g}(h,k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) \text{ and } g_T(h,k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} E[X_t X_{t+h}] B_k(t).$$

By construction of φ_t via iid $\{\xi_l\}_{l=1}^{(T-h)/b_T}, \xi_l \sim N(0, 1)$:

$$\Delta \hat{g}_{T}^{(dw)}(h,k) = \frac{1}{T} \sum_{l=1}^{(T-h)/b_{T}} \xi_{l} \mathfrak{E}_{l,T}(h,k;\hat{g}(h,k))$$
$$\Delta g_{T}^{*}(h,k) = \frac{1}{T} \sum_{l=1}^{(T-h)/b_{T}} \xi_{l} \mathfrak{E}_{l,T}(h,k;g_{T}(h,k)).$$

¹The correspondence $i = (k - 1)\mathcal{H}_T + h$ is unique. In particular, it is understood that we set k and move through $h \in \{0, ..., \mathcal{H}_T\}$. Thus, (i) set k = 1 and move through $h = 0, ..., \mathcal{H}_T$ for $i = 0, ..., \mathcal{H}_T$; (ii) set k = 2 and then $h = 0, ..., \mathcal{H}_T$ to yield $i = \mathcal{H}_T + 1, ..., 2\mathcal{H}_T$; and so on. Thus, if $\mathcal{H}_T = 100$ then i = 299 is uniquely matched with k = 3 and h = 99.

Serial independence, and independence of \mathfrak{X}_T , for ξ_t yield for some couplets (h, k) and (\tilde{h}, \tilde{k}) :

$$\begin{split} \hat{s}_{T}^{2}(i,j) &= TE\left[\Delta \hat{g}_{T}^{(dw)}(i)\Delta \hat{g}_{T}^{(dw)}(j)|\mathfrak{X}_{T}\right] \\ &= TE\left[\frac{1}{T}\sum_{l=1}^{(T-h)/b_{T}}\xi_{l}\mathfrak{E}_{l,T}(h,k;\hat{g}(h,k))\frac{1}{T}\sum_{m=1}^{(T-\tilde{h})/b_{T}}\xi_{m}\mathfrak{E}_{m,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k}))|\mathfrak{X}_{T}\right] \\ &= \frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}}\mathfrak{E}_{l,T}(h,k;\hat{g}(h,k))\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k})). \end{split}$$

Similarly:

$$\mathring{s}_T^2(i,j) = \frac{1}{T} \sum_{l=1}^{(T-h \vee \tilde{h})/b_T} \mathfrak{E}_{l,T}(h,k;g_T(h,k)) \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_T(\tilde{h},\tilde{k})).$$

Now observe for any (i, j) and some associated couplets (h, k) and (\tilde{h}, \tilde{k}) :

$$\begin{split} \left| \hat{s}_{T}^{2}(i,j) - \hat{s}_{T}^{2}(i,j) \right| \\ &\leq \left| \frac{1}{T} \sum_{l=1}^{(T-h \vee \tilde{h})/b_{T}} \left\{ \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k})) - \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_{T}(\tilde{h},\tilde{k})) \right\} \\ &\qquad \times \left\{ \mathfrak{E}_{l,T}(h,k;\hat{g}(h,k)) - \mathfrak{E}_{l,T}(h,k;g_{T}(h,k)) \right\} \right| \\ &\qquad + \left| \frac{1}{T} \sum_{l=1}^{(T-h \vee \tilde{h})/b_{T}} \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_{T}(\tilde{h},\tilde{k})) \left\{ \mathfrak{E}_{l,T}(h,k;\hat{g}(h,k)) - \mathfrak{E}_{l,T}(h,k;g_{T}(h,k)) \right\} \right| \\ &\qquad + \left| \frac{1}{T} \sum_{l=1}^{(T-h \vee \tilde{h})/b_{T}} \mathfrak{E}_{l,T}(h,k;g_{T}(h,k)) \left\{ \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k})) - \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_{T}(\tilde{h},\tilde{k})) \right\} \right| \\ &= \mathcal{S}_{1,T}(h,k,\tilde{h},\tilde{k}) + \mathcal{S}_{2,T}(h,k,\tilde{h},\tilde{k}) + \mathcal{S}_{3,T}(h,k,\tilde{h},\tilde{k}). \end{split}$$

It follows $\Delta_T = O_p(1/T^{\iota})$ for some tiny $\iota > 0$ if we show each:

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \mathcal{S}_{i,T}(h, k, \tilde{h}, \tilde{k}) \right| = O_p \left(1/T^t \right).$$
(D.9)

Consider $S_{2,T}(\cdot)$; $S_{1,T}(\cdot)$ and $S_{3,T}(\cdot)$ are similar. Use

$$\{X_t X_{t+h} B_k(t) - \hat{g}(h,k)\} - \{X_t X_{t+h} B_k(t) - g_T(h,k)\} = -\{\hat{g}(h,k) - g_T(h,k)\}$$

with Lemma D.1 to yield:

$$\max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| S_{2,T}(h,k,\tilde{h},\tilde{k}) \right|$$

$$\leq \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{l=1}^{(T-h \vee \tilde{h})/b_{T}} \sum_{t=(l-1)b_{T}+1}^{lb_{T}} \frac{\{X_{t}X_{t+h}B_{\tilde{k}}(t) - g_{T}(h,k)\}}{\{X_{t}X_{t+h}B_{\tilde{k}}(t) - g_{T}(h,k)\}} \right| \times b_{T} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} |\hat{g}(h,k) - g_{T}(h,k)|$$

$$= \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \frac{1}{T} \sum_{t=1}^{T-h \lor \bar{h}} \left\{ X_t X_{t+h} - E\left[X_t X_{t+h} \right] \right\} B_k(t) \right| \times O_p(b_T/\sqrt{T}).$$

Moreover, by the same argument used to prove (see eq. (A.4) in the main paper):

$$\sup_{z\geq 0} \left| P\left(\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} |\mathcal{Z}_T(i)| \leq z \right) - P\left(\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} |\mathbf{Z}_T(i)| \leq z \right) \right| \to 0,$$

we have for any $\{\mathcal{H}_T\}$, $0 \leq \mathcal{H}_T < T - 1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$ and $\eta(\mathcal{K}_T) = o(\sqrt{T})$, provided (D.2) holds:

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \frac{1}{T} \sum_{t=1}^{T-h \vee \tilde{h}} \left\{ X_t X_{t+\tilde{h}} - E\left[X_t X_{t+\tilde{h}} \right] \right\} B_{\tilde{k}}(t) \right| = O_p(1/\sqrt{T}).$$

Therefore

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \left| S_{2,T}(h, k, \tilde{h}, \tilde{k}) \right| = O_p(b_T/T) = o_p(1/T^t)$$

given $b_T = o(T^{1-\iota})$, proving (D.9).

Claim (b). Now let $\{\mathring{\mathbf{Z}}(h,k): 0 \le h \le \mathcal{H}_T, 1 \le k \le \mathcal{K}_T\}$ be an independent copy of the Lemma 3.1 law $\mathbf{Z}(h,k) \sim N(0, \lim_{T\to\infty} \sigma_T^2(h,k))$, independent of the asymptotic draw $\{X_t\}_{t=1}^{\infty}$, where

$$\sigma_T^2(h,k) = \frac{1}{T} \sum_{s,t=1}^{T-h} E\left[z_s(h,k) z_t(h,k) \right].$$

Let $[\mathring{\mathbf{Z}}(i)]_{i=0}^{\mathcal{H}_T \mathcal{K}_T}$ be the stacked version, cf. (D.7) and footnote 1, and define

$$v^{2}(i,j) \equiv E[\mathbf{\mathring{Z}}(i)\mathbf{\mathring{Z}}(j)],$$

hence $v^2(i,i) \equiv \lim_{T \to \infty} \sigma_T^2(h,k)$ with $i = (k-1)\mathcal{H}_T + h$. We prove below

$$\tilde{\mathcal{E}}_T \equiv \sup_{c>0} \left| P\left(\max_{0 \le i \le \mathcal{H}_T \mathcal{K}_T} \left| \mathring{\mathbf{Z}}_T(i) \right| \le c \left| \mathfrak{X}_T \right) - P\left(\max_{0 \le i \le \mathcal{H}_T \mathcal{K}_T} \left| \mathring{\mathbf{Z}}(i) \right| \le c \right) \right| \xrightarrow{p} 0.$$
(D.10)

Together Claim (a) with (D.10) yield

$$\sup_{c>0} \left| P\left(\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \sqrt{T} \Delta \hat{g}_T^{(dw)}(h, k) \right| \le c |\mathfrak{X}_T\right) - P\left(\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \mathring{\mathbf{Z}}(h, k) \right| \le c \right) \right| \xrightarrow{p} 0.$$

Therefore

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \sqrt{T} \Delta \hat{g}_T^{(dw)}(h, k) \right| \xrightarrow{d} \max_{h, k \in \mathbb{N}} \left| \mathring{Z}(h, k) \right| a w p 1 \text{ with respect to } \{X_t\}_{t=1}^{\infty}$$

This yields as claimed by definition (cf. Giné and Zinn, 1990, Section 3):

$$\max_{\mathcal{H}_{T},\mathcal{H}_{T}} \left| \sqrt{T} \Delta \hat{g}_{T}^{(dw)}(h,k) \right| \Rightarrow^{p} \max_{h,k \in \mathbb{N}} \left| \mathring{Z}(h,k) \right|$$

We now prove (D.10). With $\hat{s}_T^2(i, j) = TE[\Delta g_T^*(i)\Delta g_T^*(j)|\mathfrak{X}_T]$ and $v^2(i, j)$ define

$$\tilde{\Delta}_T \equiv \max_{0 \le i, j \le \mathcal{H}_T \mathcal{K}_T} \left| \mathring{s}_T^2(i, j) - v^2(i, j) \right|$$

As above $\tilde{\mathcal{E}}_T = O_p(\tilde{\Delta}_T^{1/3} \times \max\{1, \ln(\mathcal{H}_T \mathcal{K}_T / \tilde{\Delta}_T)\}^{2/3})$. The proof is complete if we show

$$\tilde{\Delta}_T = O(1/T^{\iota}) \text{ for some } \iota > 0, \tag{D.11}$$

since then

$$\tilde{\mathcal{E}}_T = O_p\left(\tilde{\Delta}_T^{1/3}\max\left\{1,\ln(\mathcal{H}_T\mathcal{K}_T/\tilde{\Delta}_T)\right\}^{2/3}\right) = O_p\left(T^{-\iota/3}\left\{\ln(T)\right\}^{2/3}\right) \xrightarrow{p} 0.$$

We now prove (D.11). Define

$$\Delta \ddot{g}_{T}^{*}(h,k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} \varphi_{t} \left\{ X_{t} X_{t+h} B_{k}(t) - \frac{1}{T-h} \sum_{s=1}^{T-h} E\left[X_{s} X_{s+h} \right] B_{k}(s) \right\},$$

and let $\Delta \ddot{g}_T^*(i)$ stack $\Delta \ddot{g}_T^*(h, k)$. Define

$$\begin{split} \mathring{s}_{T}^{2}(i,j) &= TE\left[\Delta \mathring{g}_{T}^{*}(i)\Delta \mathring{g}_{T}^{*}(j)|\mathfrak{X}_{T}\right] \\ \ddot{s}_{T}^{2}(i,j) &= TE\left[\Delta \dddot{g}_{T}^{*}(i)\Delta \dddot{g}_{T}^{*}(j)|\mathfrak{X}_{T}\right] \\ s_{T}^{2}(i,j) &= TE\left[\Delta \dddot{g}_{T}^{*}(i)\Delta \dddot{g}_{T}^{*}(j)\right] \\ s^{2}(i,j) &= \lim_{T \to \infty} TE\left[\Delta \dddot{g}_{T}^{*}(i)\Delta \dddot{g}_{T}^{*}(j)\right]. \end{split}$$

We prove (D.11) by showing in order:

$$\max_{0 \le i, j \le \mathcal{H}_T \mathcal{K}_T} \left| \mathring{s}_T^2(i, j) - \ddot{s}_T^2(i, j) \right| = O_p \left(T^{-\iota} \right)$$
(D.12)

$$\max_{0 \le i, j \le \mathcal{H}_T \mathcal{K}_T} \left| \ddot{s}_T^2(i, j) - s_T^2(i, j) \right| = O_p(T^{-\iota})$$
(D.13)

$$\max_{0 \le i, j \le \mathcal{H}_T \mathcal{K}_T} \left| s_T^2(i, j) - s^2(i, j) \right| = O(T^{-\iota})$$
(D.14)

$$\max_{0 \le i, j \le \mathcal{H}_T \mathcal{K}_T} \left| s^2(i, j) - v^2(i, j) \right| = O(T^{-\iota}).$$
(D.15)

Step 1 $(\hat{s}_T^2(i, j), \tilde{s}_T^2(i, j))$. Recall $g_T(h, k) \equiv 1/T \sum_{t=1}^{T-h} E[X_t X_{t+h}] B_k(t)$. After expanding, and cancelling like terms, we have for any (i, j) and some unique couplet $(h, k; \tilde{h}, \tilde{k})$, where $i = (k-1)\mathcal{H}_T + i$

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$$\begin{split} h \text{ and } j &= (k-1)\mathcal{H}_{T} + h: \\ \left| \hat{s}_{T}^{2}(i,j) - \overline{s}_{T}^{2}(i,j) \right| \\ &= \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{s,t=(l-1)b_{T}+1}^{lb_{T}} \left\{ -g_{T}(h,k)X_{t}X_{t+\tilde{h}}B_{\tilde{k}}(t) - g_{T}(\tilde{h},\tilde{k})X_{t}X_{t+h}B_{k}(t) \right. \\ &+ g_{T}(h,k)g_{T}(\tilde{h},\tilde{k}) + \frac{T}{T-h}g_{T}(h,k)X_{t}X_{t+\tilde{h}}B_{\tilde{k}}(t) \\ &+ \frac{T}{T-\tilde{h}}g_{T}(\tilde{h},\tilde{k})X_{t}X_{t+h}B_{k}(t) - \frac{T}{T-h}\frac{T}{T-\tilde{h}}g_{T}(h,k)g_{T}(\tilde{h},\tilde{k}) \right\} \right| \\ &\leq \frac{h}{T-h} \left| g_{T}(h,k)\frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{s,t=(l-1)b_{T}+1}^{lb_{T}} X_{t}X_{t+\tilde{h}}B_{\tilde{k}}(t) \right| \\ &+ \frac{\tilde{h}}{T-\tilde{h}} \left| g_{T}(\tilde{h},\tilde{k})\frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{s,t=(l-1)b_{T}+1}^{lb_{T}} X_{t}X_{t+h}B_{k}(t) \right| \\ &+ \frac{T(h+\tilde{h})+h\tilde{h}}{(T-h)(T-\tilde{h})} \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{s,t=(l-1)b_{T}+1}^{lb_{T}} g_{T}(h,k)g_{T}(\tilde{h},\tilde{k}) \right| \\ &= \mathcal{D}_{T,1}(h,k;\tilde{h},\tilde{k}) + \mathcal{D}_{T,2}(h,k;\tilde{h},\tilde{k}) + \mathcal{D}_{T,3}(h,k;\tilde{h},\tilde{k}) \end{split}$$

Now twice wield the fact that uniform exponential tails Assumption 1.b implies uniform \mathcal{L}_r -boundedness for any $r \ge 1$, with uniform law Lemma D.1 and $\mathcal{H}_T = O(T^{1-\iota}/b_T)$, to yield:

$$\begin{split} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \mathcal{D}_{T,1}(h,k;\tilde{h},\tilde{k}) &\leq \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left\{ \frac{h}{T-h} \left| g_{T}(h,k) \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{s,t=(l-1)b_{T}+1}^{lb_{T}} X_{t} X_{t+\tilde{h}} B_{\tilde{k}}(t) \right| \right\} \\ &\leq K \frac{\mathcal{H}_{T}}{T-\mathcal{H}_{T}} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{s,t=(l-1)b_{T}+1}^{lb_{T}} X_{t} X_{t+\tilde{h}} B_{\tilde{k}}(t) \right| \\ &= K \frac{\mathcal{H}_{T}}{T-\mathcal{H}_{T}} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T/b_{T}} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{t=(l-1)b_{T}+1}^{lb_{T}} X_{t} X_{t+\tilde{h}} B_{\tilde{k}}(t) \right| \\ &= O_{p} \left(\frac{\mathcal{H}_{T}}{T-\mathcal{H}_{T}} \right) = O_{p} \left(\frac{\mathcal{H}_{T}}{T} \right) = O_{p} \left(\frac{1}{T^{\iota}b_{T}} \right) = o_{p} \left(1/T^{\iota} \right). \end{split}$$

Similarly, $\max_{\mathcal{H}_T, \mathcal{K}_T} \mathcal{D}_{T,2}(h, k; \tilde{h}, \tilde{k}) = o_p(1/T^{\iota})$. Furthermore, use for any $h \lor \tilde{h} \in \{1, ..., \mathcal{H}_T\}$:

$$\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}g_T(h,k)g_T(\tilde{h},\tilde{k}) \le b_T \left|g_T(h,k)g_T(\tilde{h},\tilde{k})\right|$$

with $\mathcal{H}_T = O(T^{1-\iota}/b_T)$ to arrive at:

$$\begin{split} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \mathcal{D}_{T,3}(h,k;\tilde{h},\tilde{k}) &\leq \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left\{ \frac{T(h+\tilde{h})+h\tilde{h}}{(T-h)\left(T-\tilde{h}\right)} \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{s,t=(l-1)b_{T}+1}^{lb_{T}} g_{T}(h,k)g_{T}(\tilde{h},\tilde{k}) \right| \right\} \\ &\leq b_{T} \frac{2T\mathcal{H}_{T}+\mathcal{H}_{T}^{2}}{\left(T-\mathcal{H}_{T}\right)^{2}} \leq K \frac{b_{T}\mathcal{H}_{T}}{T} \left(1+o(1)\right) = O\left(T^{-\iota}\right), \end{split}$$

proving (D.12).

Step 2 ($\ddot{s}_T^2(i, j), s_T^2(i, j)$). Write

$$\ddot{g}_T(h,k) \equiv \frac{1}{T-h} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t).$$

For some unique couplet $(h, k; \tilde{h}, \tilde{k})$ with $i = (k - 1)\mathcal{H}_T + h$ and $j = (k - 1)\mathcal{H}_T + h$, expand terms in $\ddot{s}_T^2(i, j)$, and use

$$\frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} = (1-h\vee\tilde{h}/T) b_T$$

to deduce:

$$\begin{split} \ddot{s}_{T}^{2}(i,j) &= \frac{1}{T} \sum_{l=1}^{(T-h \vee \tilde{h})/b_{T}} \sum_{s,t=(l-1)b_{T}+1}^{lb_{T}} X_{s} X_{s+h} X_{t} X_{t+\tilde{h}} B_{k}(s) B_{k}(t) \tag{D.16} \\ &- \frac{1}{T/b_{T}} \sum_{l=1}^{(T-h \vee \tilde{h})/b_{T}} \sum_{s=(l-1)b_{T}+1}^{lb_{T}} X_{s} X_{s+h} B_{k}(s) \times \ddot{g}_{T}(\tilde{h}, \tilde{k}) \\ &- \frac{1}{T/b_{T}} \sum_{l=1}^{(T-h \vee \tilde{h})/b_{T}} \sum_{t=(l-1)b_{T}+1}^{lb_{T}} X_{t} X_{t+\tilde{h}} B_{\tilde{k}}(t) \times \ddot{g}_{T}(h, k) \\ &+ (1 - \{h \vee \tilde{h}\}/T) b_{T} \ddot{g}_{T}(h, k) \ddot{g}_{T}(\tilde{h}, \tilde{k}). \end{split}$$

Now use $s_T^2(i, j) = E[\ddot{s}_T^2(i, j)]$ to obtain:

$$\max_{0 \le i, j \le \mathcal{H}_T \mathcal{K}_T} \left| \ddot{s}_T^2(i, j) - s_T^2(i, j) \right| \le \mathcal{D}_{1,T} + \mathcal{D}_{2,T},$$

where

$$\mathcal{D}_{1,T} = \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{s,t=(l-1)b_{T}+1}^{lb_{T}} \left\{ X_{s}X_{s+h}X_{t}X_{t+\tilde{h}} - E\left[X_{s}X_{s+h}X_{t}X_{t+\tilde{h}}\right] \right\} B_{k}(s)B_{\tilde{k}}(t) \right|$$
$$\mathcal{D}_{2,T} = 2 \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T/b_{T}} \sum_{l=1}^{(T-h\vee\tilde{h})/b_{T}} \sum_{s=(l-1)b_{T}+1}^{lb_{T}} \left\{ X_{s}X_{s+h} - E\left[X_{s}X_{s+h}\right] \right\} B_{k}(s) \times \ddot{g}_{T}(\tilde{h},\tilde{k}) \right|.$$

Consider $\mathcal{D}_{1,T}$ and write

$$\mathfrak{X}_{T,l}(h,k) \equiv \frac{1}{\sqrt{b_T}} \sum_{t=(l-1)b_T+1}^{lb_T} X_t X_{t+h} B_k(t) \text{ and } \mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k}) \equiv \mathfrak{X}_{T,l}(h,k) \mathfrak{X}_{T,l}(\tilde{h},\tilde{k}),$$

hence:

$$\mathcal{D}_{1,T} = \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h \vee \tilde{h})/b_T} \left(\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k}) - E\left[\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k}) \right] \right) \right|.$$

Let $[\mathbf{Y}_{T,l}(i,j)]_{i,j=0}^{\mathcal{H}_T \mathcal{K}_T}$ stack $\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k})$, with correspondence $i = (k-1)\mathcal{H}_T + h$ and $j = (\tilde{k} - 1)\mathcal{H}_T + \tilde{h}$. Similarly $[\mathring{\mathbf{Y}}_{T,l}(l)]_{l=0}^{\mathcal{H}_T^2 \mathcal{K}_T^2}$ stacks $[\mathbf{Y}_{T,l}(i,j)]_{i,j=0}^{\mathcal{H}_T \mathcal{K}_T}$ with $l = (j-1)\mathcal{H}_T \mathcal{K}_T + i$. Hence

$$\mathcal{D}_{1,T} = \max_{0 \le l \le \mathcal{H}_T^2 \mathcal{K}_T^2} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h \lor \tilde{h})/b_T} \left(\mathring{\mathbf{Y}}_{T,l}(l) - E\left[\mathring{\mathbf{Y}}_{T,l}(l) \right) \right] \right|$$

We show below that $\mathring{Y}_{T,l}(l)$ satisfies Conditions 1-3 in Chang, Chen and Wu (2021). Hence, similar to (A.4)-(A.6) in the main paper, $\mathcal{D}_{1,T} = O_p(b_T^{1/2}/T^{1/2}) = o_p(1)$ provided

$$\frac{1}{T^{1/9}} \left[\mathcal{E}_T^{2/3} \left\{ \ln \left(\mathcal{H}_T \mathcal{K}_T \right) \right\}^{(1+2\phi)/(3\phi)} + \mathcal{E}_T \left(\ln \mathcal{H}_T \mathcal{K}_T \right)^{7/6} \right] \to 0$$
$$(\ln(\mathcal{H}_T \mathcal{K}_T))^{3-\phi} = o(T^{3\phi}).$$

The latter hold since $\mathcal{H}_T = O(T^{1-\iota}/b_T), b_T/T^{\iota} \to \infty$, and $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and therefore $\mathcal{E}_T = o(T^{1/6}/\{\ln(T)\}^{(1+2\phi)/(2\phi)})$. Now $b_T = o(T^{1-\iota})$ yields $\mathcal{D}_{1,T} = o_p(1/T^{\iota})$ for some $\iota > 0$.

We now show $\mathring{Y}_{T,l}(l)$ satisfies Chang, Chen and Wu's (2021) Conditions 1-3. For Condition 1, Bonferroni's inequality and Lemma A.2 yield

$$\begin{aligned} \max_{0 \le l \le \mathcal{H}_T^2 \mathcal{K}_T^2} P\left(\left| \mathring{\mathbf{Y}}_{T,l}(l) \right| > c\right) &= \max_{\mathcal{H}_T, \mathcal{K}_T} P\left(\left| \frac{1}{b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} X_s X_{t+h} X_t X_{t+\tilde{h}} B_k(s) B_{\tilde{k}}(t) \right| > c \right) \\ &\le b_T^2 \max_{\mathcal{H}_T} \max_{1 \le t \le T-h} P\left(\left| X_s X_{t+h} X_t X_{t+\tilde{h}} \right| > b_T c \right) \le 2\varpi b_T^2 \exp\left\{ -\frac{b_T^{\vartheta_2}}{\mathcal{E}_T^{\vartheta_2}} c^{\vartheta_2} \right\} \end{aligned}$$

Use $b_T/T^t \to \infty$ by supposition to deduce for any $c > 0 \exists \mathcal{T} \in \mathbb{N}$ such that

$$b_T^2 \exp\{-b_T^{\vartheta_2/2} \mathcal{E}_T^{-\vartheta_2} c^{\vartheta_2}\} \le \exp\{-c^{\vartheta_2} \mathcal{E}_T^{-\vartheta_2}\} \ \forall T \ge \mathcal{T}.$$

Hence Condition 1 holds:

$$\max_{0 \le l \le \mathcal{H}_T^2 \mathcal{K}_T^2} P\left(\left| \mathring{Y}_{T,l}(l) \right| > c \right) \le \tilde{\varpi} \exp\{-c^{\vartheta_2} \mathcal{E}_T^{-\vartheta_2}\} \ \forall T \ge \mathcal{T} \text{ and some } \tilde{\varpi} \ge 2.$$

Condition 2 holds by Assumption 1.a and measurability. Condition 3 holds by Assumption 2.a(*i*).

For $\mathcal{D}_{2,T}$, use Lemma D.1, and $b_T = O(T^{1/2-\iota})$ under Assumption 2.b, to get:

$$\begin{split} \mathcal{D}_{2,T} &\leq K \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T/b_{T}} \sum_{l=1}^{(T-h \vee \tilde{h})/b_{T}} \sum_{s=(l-1)b_{T}+1}^{lb_{T}} \{X_{s}X_{s+h} - E\left[X_{s}X_{s+h}\right]\} B_{k}(s) \right| \\ &= Kb_{T} \max_{\mathcal{H}_{T},\mathcal{K}_{T}} \left| \frac{1}{T} \sum_{t=1}^{T-h \vee \tilde{h}} \{X_{t}X_{t+h} - E\left[X_{t}X_{t+h}\right]\} B_{k}(t) \right| = O_{p}\left(b_{T}/T^{1/2}\right) = O_{p}\left(T^{-t}\right). \end{split}$$

Step 3 $(s_T^2(i, j), s^2(i, j))$. The property holds by Assumption 2.a(*ii*).

Step 4 ($s^2(i, j), v^2(i, j)$). For some $(h, k; \tilde{h}, \tilde{k}), s^2(i, j)$ is identically

$$\lim_{T \to \infty} \frac{1}{T} \sum_{l=1}^{(T-h \vee \tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} E\left[\left\{ X_s X_{s+h} B_k(s) - \frac{1}{T-h} \sum_{u=1}^{T-h} E\left[X_u X_{u+h} \right] B_k(u) \right\} \\ \times \left\{ X_t X_{t+\tilde{h}} B_{\tilde{k}}(t) - \frac{1}{T-\tilde{h}} \sum_{u=1}^{T-\tilde{h}} E\left[X_u X_{u+\tilde{h}} \right] B_{\tilde{k}}(u) \right\} \right]$$

and by rearranging terms

$$v^{2}(i,j) = \lim_{T \to \infty} \frac{1}{T} \sum_{s,t=1}^{T-h \lor \tilde{h}} E\left[\left\{ X_{s} X_{s+h} B_{k}(s) - \frac{1}{T-h} \sum_{u=1}^{T-h} E\left[X_{u} X_{u+h} \right] B_{k}(u) \right\} \right. \\ \left. \times \left\{ X_{t} X_{t+\tilde{h}} B_{\tilde{k}}(t) - \frac{1}{T-\tilde{h}} \sum_{u=1}^{T-\tilde{h}} E\left[X_{u} X_{u+\tilde{h}} \right] B_{\tilde{k}}(u) \right\} \right].$$

Further, block size $b_T \to \infty$. Hence $s^2(i, j) = v^2(i, j) \forall i, j$. This completes the proof. $Q\mathcal{ED}$.

Remark 4. We technically only need the iid random numbers $\{\xi_1, \ldots, \xi_{N_T}\}$ to satisfy $E[\xi_i] = 0$, $E[\xi_i^2] = 1$, and $E[\xi_i^4] < \infty$. In this general setting $\sqrt{T}\Delta \hat{g}_T^{(dw)}(i) | \mathfrak{X}_T$ is not necessarily normally distributed, hence the Gaussian-to-Gaussian result (D.8) may not hold. We will need the added step:

$$\sup_{c>0} \left| P\left(\max_{0 \le i \le \mathcal{H}_T \mathcal{K}_T} \left| \sqrt{T} \Delta \hat{g}_T^{(dw)}(i) \right| \le c \, |\mathfrak{X}_T\right) - P\left(\max_{0 \le i \le \mathcal{H}_T \mathcal{K}_T} \left| \sqrt{T} \Delta \hat{g}_T(i) \, |\mathfrak{X}_T\right| \le c \right) \right| \xrightarrow{P} 0$$

where $\sqrt{T}\Delta \mathring{g}_T(i)|\mathfrak{X}_T \sim N(0, TE[\Delta \widehat{g}_T^{(dw)}(i)^2|\mathfrak{X}_T])$. We would then need to alter (D.8), and prove instead

$$\mathcal{E}_T \equiv \sup_{c>0} \left| P\left(\max_{0 \le i \le \mathcal{H}_T \mathcal{K}_T} \left| \sqrt{T} \Delta \mathring{\boldsymbol{g}}_T(i) \right| \le c | \mathfrak{X}_T \right) - P\left(\max_{0 \le i \le \mathcal{H}_T \mathcal{K}_T} \left| \mathring{\boldsymbol{Z}}_T(i) \right| \le c \right) \right|$$
$$= O_p \left(\Delta_T^{1/3} \max \left\{ 1, \ln \left(\mathcal{H}_T \mathcal{K}_T / \Delta_T \right) \right\}^{2/3} \right) \xrightarrow{P} 0.$$

E. Empirical study

We now apply our test and the test in Jin, Wang and Wang (2015) to quarterly international (ex post) real interest rates. We analyze 16 countries over the period 1960.Q1 - 2019.Q4. The data were collected from the U.S. Federal Reserve Bank data archive (FRED), which itself is taken from the OECD data archives. The countries are Australia, Austria, Belgium, Canada, Denmark, France, Germany, Ireland Italy, Japan, Netherlands, Norway, Switzerland, UK and US.

Following Rapach and Weber (2004), we use the 10-year government bond yield as our measure of the nominal interest rate $r_{n,t}$, and the Consumer Price Index in order to compute inflation i_t . The (ex post) real bond rate is $r_{r,t} = r_{n,t} - i_t$. See Table A.1 for the exact date range available for each series and subsequent size. Figure 1 contains plots of each series.

Unit root tests have been proposed as a standard for testing for non-stationarity in interest rates. See, e.g., Rose (1988) and Rapach and Weber (2004) and their historical references. In that framework, it is implicitly assumed that real interest rates are unbounded (asymptotically with probability approaching one), in particular if a unit root is present. In the case of a unit root, of course, variance is unbounded asymptotically, and α -mixing fails to hold.

Testing real interest rates is complicated by the fact that nominal rates $r_{n,t}$ and inflation i_t may be nonstationary while real rates $r_{r,t} = r_{n,t} - i_t$ can yet be stationary. In a unit root test setting, it is possible that $r_{n,t} \sim I(1)$ and $i_t \sim I(1)$ yet $(r_{n,t}, i_t)$ are cointegrated with integrating vector [-1, 1], hence $r_{r,t}$ are stationary. Conversely, nonstationarity necessarily exists when just $r_{r,t} \sim I(1)$ or just $i_t \sim I(1)$. Rose (1988) finds the latter for each country in our study based on quarterly post-war data and conventional unit root tests, hence Rose (1988) broadly concludes unit root nonstationarity. Rapach and Weber (2004) obtain more nuanced results. They find nonstationarity in nominal rates for all countries except Germany and Switzerland, and mixed results for inflation based on Phillips and Perron (1988) and Ng and Perron (1997, 2001) unit root tests. In order to handle the evident cases $r_{n,t} \sim I(1)$ and $i_t \sim I(1)$ they apply several cointegration tests, including tests by Ng and Perron (2001) and one eventually published in Perron and Rodriguez (2016).

A different approach for studying structural time variation in interest rates couches rates in a parametric regime switching regression model. See, e.g., Garcia and Perron (1996), Bekdache (1000), and Ang and Bekaert (2002). See also Teräsvirtra (1994) and Gray (1996).

In our setting, under either hypothesis we assume a moment generating function exists uniformly over *t*, and a geometric mixing condition holds. Thus, we implicitly assume a unit root does not exist. The moment conditions can be assured simply by assuming nominal interest rates and inflation are bounded. This is a fairly natural assumption empirically for interest rates which are typically managed by government market actions, and lie in the range [-1,1]. In any case, in our sample range bond yields and inflation never surpass the total range [-.02, .30]. We therefore test for a (non-unit root based) deviation from covariance stationarity. Our setting of course is nonparametric: we do not need to specify a (switching) regression model (e.g. Augmented Dickey Fuller, or Markov Switching), and indeed our test is relevant irrespective of any underlying parametric features.

We report test results for the max-test based on a dependent wild bootstrap, and Jin, Wang and Wang's (2015) test based both on simulated critical values and dependent wild bootstrap. Both tests exploit a Walsh basis in view of simulation evidence suggesting the inferiority of the composite Haar basis. We simulate critical values for each series and each country (hence, 54 simulated sets of critical values), rather than for each sample size. We use $\mathcal{H}_T = [2T^{.49}]$ and $\mathcal{K}_T = [.5T^{.49}]$. See Table A.2 for test results. Tests are performed on nominal and real bond yields, and inflation, but we focus our discussion on real bond yields given is importance in the literature.

Consider the max-correlation difference test. In all countries except one, when the test finds evidence of non-covariance stationarity in nominal rates, the same result applies for real rates. Consider Italy:

the p-values are .024 and .032 for nominal and real rates respectively, while the p-value for inflation is .216. Thus, nominal rates are the driving force for non-stationarity. New Zealand is the sole exception: p-values for nominal and real rates and inflation are .156, .080 and .162. Thus, we reject stationarity at the 10% level for real rates, but *fail to reject* for nominal rates and inflation. It is easily verified, however, that if random variables X_t and Y_t are covariance stationary then so is any linear combination. A deeper study into this is left for future work.

The bootstrapped JWW test, on par with the Monte Carlo study, almost never leads to a rejection of the covariance stationarity null hypothesis. Tests based on simulated critical values, however, match across nominal and real bond yields, with four exceptions: Belgium, Japan, New Zealand and the UK. The JWW test generally yields strong rejections (well under the 1% level) when nonstationarity is detected, while the max-correlation test is more moderate, with rejections variously at the 1%, 5%, and 10% levels.

Finally, in five countries the max-correlation test and JWW test disagree: Australia, France, Italy, New Zealand and Switzerland (denoted by bold in Table A.2). In the first four the max-correlation difference test yielded rejections of covariance stationarity (p-values are .056, .022, .032, and .080), while the JWW test failed to reject. The JWW test with simulated critical value detected non-stationarity for Switzerland at the 1% level ($\hat{D}_T = 58.1$, 1% c.v. = 7.9), but the max-correlation test did not at the 10% (p-value .144). Table A.1. Dates and Sample Sizes

	Nominal Bond	rn	Inflation <i>i</i>		Real Bond <i>r_r</i>	
	Dates	п	Dates	n	Dates	n
Australia	1969.Q3-2021.Q4	210	1960.Q2-2021.Q4	246	1969.Q3-2021.Q4	210
Austria	1990.Q1-2021.Q4	128	1960.Q2-2021.Q4	246	1990.Q1-2021.Q4	128
Belguim	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
Canada	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	244	1960.Q2-2021.Q4	246
Denmark	1987.Q1-2021.Q4	140	1967.Q2-2021.Q4	218	1987.Q1-2021.Q4	140
France	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
Germany	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
Ireland	1971.Q1-2021.Q4	204	1976.Q2-2021.Q4	182	1976.Q2-2021.Q4	182
Italy	1991.Q2-2021.Q4	122	1960.Q2-2021.Q4	246	1991.Q2-2021.Q4	122
Japan	1989.Q1-2021.Q4	132	1960.Q2-2021.Q4	246	1989.Q1-2021.Q4	132
Netherlands	1960.Q1-2021.Q4	248	1960.Q3-2021.Q4	246	1960.Q3-2021.Q4	246
New Zealand	1970.Q1-2021.Q4	208	1960.Q2-2021.Q4	246	1970.Q1-2021.Q4	208
Norway	1985.01-2021.04	148	1960.02-2021.04	246	1985.01-2021.04	148
Switzerlnad	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
UK	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
US	1960.01-2021.04	248	1960.02-2021.04	246	1960.02-2021.04	246

Nominal bond r_n are 10 year government bond yields; inflation *i* is derived from the Consumer Price Index for all goods and services; and real bond yields $r_r = r_n - i$.

		Nominal Bond r_n			Inflation <i>i</i>			Real Bond r_r	
	$ \hat{\mathcal{M}}_T$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
Australia	080	65.5 (3.5, 4.8, 7.5) ***	.729	.174	5.57 (3.8, 5.1, 7.9) **	.605	.056	-2.28 (3.5, 4.8, 7.5)	.854
Austria	.002	12.3 (2.9, 4.9, 6.6) ***	.198	.158	62.7 (3.8, 5.1, 7.9) ***	.134	.000	352 (2.9, 4.1, 6.6) ***	.024
Belguim	.032	2.03 (3.8, 5.1, 7.9)	.876	.236	9.90 (3.8, 5.1, 8.0) ***	.537	.023	44 (3.8, 5.1, 7.9) ***	.919
Canada	.014	24.1 (3.8, 5.1, 7.9) ***	.904	.158	10.4 (3.8, 5.1, 7.9) ***	.361	.018	17.7 (3.8, 5.1, 7.9) ***	.756
Denmark	.000	241 (3.0, 4.1, 6.7) ***	.246	.066	29.1 (3.6, 4.8, 7.6) ***	.319	.000	217 (3.0, 4.1, 6.6) ***	.273
France	.020	543 (3.8, 5.1, 7.9)	.661	.174	2.27 (3.8, 5.1, 7.9)	.541	.022	-2.00 (3.8, 5.1, 7.9)	.866
Germany	.180	28.9 (3.8, 5.1, 7.9) ***	.858	.046	109 (3.8, 5.1, 7.9) ***	.170	.090	879 (3.8, 5.1, 7.9) ***	.399
Ireland	.101	6.12 (3.4, 4.7, 7.5) **	.998	.242	30.6 (3.4, 4.6, 7.2) ***	.248	.012	161 (3.4, 4.6, 7.2) ***	.563
Italy	.024	1.92 (2.9, 4.0, 6.5)	.246	.216	218 (3.8, 5.1, 7.8) ***	.076	.032	1.80 (2.9, 4.0, 6.6)	.836
Japan	.054	1.43 (2.9, 4.1, 6.6)	.331	.331	3.34 (3.8, 5.1, 7.9)	.473	.014	92.7 (2.9, 4.0, 6.6) ***	.581
Netherlands	.068	284 (3.8, 5.1, 7.9) ***	.585	.114	12.9 (3.8, 5.1, 7.9) ***	.251	.026	184 (3.8, 5.1, 7.9) ***	.394
New Zealand	.156	11.0 (3.5, 4.8, 7.5) ***	.820	.162	2.13 (3.8, 5.1, 7.9)	.819	.080	-2.86 (3.5, 4.8, 7.5)	.982
Norway	.014	63.2 (3.0, 4.2, 6.8) ***	.273	.042	-3.49 (3.8, 5.1, 7.9)	.719	.006	23.7 (3.0, 4.1, 6.8) ***	.102
SwitzerInad	.136	853 (3.8, 5.1, 7.9) ***	.345	.265	16.2 (3.8, 5.1, 7.9) ***	.371	.144	58.1 (3.8, 5.1, 7.9) ***	.334
UK	.032	-2.02 (3.8, 5.1, 7.9)	.994	.222	54.2 (3.8, 5.1, 7.9) ***	.699	.006	21.5 (3.8, 5.1, 7.9) ***	.890
US	.036	555 (3.8, 5.1, 7.9) ***	.647	.124	9.49 (3.8, 5.1, 7.9) ***	.307	.024	652 (3.8, 5.1, 7.9) ***	.222

 Table A.2. Empirical Study: Covariance Stationarity Tests

 $\hat{\mathcal{M}}_T$ is the proposed max-test based on a bootstrapped p-value: reported values are p-values computed by dependent wild bootstrap. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, shown in parentheses: *, **, *** denote rejection at the 10%, 5% and 1% levels. $\hat{\mathcal{D}}_T^{dw}$ is JWW's test based dependent wild bootstrapped p-values.



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F. Complete simulation results

Table A.3.: Rejection Frequencies under H_0 : Walsh Basis Case 1: $\mathcal{H}_T = [\log_2(n).^{99} - 3.5]$ and $\mathcal{K}_T = [n^{1/3} + .01]$

		$\hat{\mathcal{D}}_{T}^{d_{\mathcal{W}}}$.118, .376, .596 .141, .271, .380 .114, .276, .388 .094, .349, .547		.045, .227, .445 .042, .100, .190 .024, .092, .163 .045, .244, .500		$\hat{\mathcal{D}}_{T}^{d_{W}}$.059, .167, .233 .053, .113, .148 .042, .107, .142		.011, .036, .053 .003, .007, .011 .002, .005, .013		$\hat{\mathcal{D}}_{T}^{d_{W}}$.258, .498, .642 .244, .407, .488 .242, .417, .512		.050, .247, .499 .036, .107, .176 .018, .096, .167
	128	$\hat{\mathcal{D}}_{T}^{cv}$.002, .027, .066 .064, .109, .159 .050, .124, .173 .004, .030, .091	512	.010, .053, .097 .035, .076, .140 .024, .100, .177 .017, .051, .087	128	$\hat{\mathcal{D}}_T^{cv}$.002, .020, .062 .026, .069, .116 .025, .080, .158	512	.004, .034, .074 .005, .054, .088 .065, .175, .274	128	$\hat{\mathcal{D}}_T^{cv}$	0.014, .045, .096 .080, .120, .186 .064, .122, .174	512	.017, .051, .087 .034, .085, .129 .028, .103, .177
	= <i>u</i>	$\hat{\mathcal{M}}_{T}^{(p)}$.003, .041, .111 .005, .054, .132 .006, .035, .067 .004, .055, .158	= <i>u</i>	.006, .041, .106 .004, .061, .132 .003, .036, .079 .003, .047, .119	= u	$\hat{\mathcal{M}}_{T}^{(p)}$.005, .041, .095 .007, .054, .134 .001, .013, .039	= <i>u</i>	.001, .032, .073 .002, .034, .107 .000, .018, .048	= u	$\hat{\mathcal{M}}_{T}^{(p)}$.002, .047, .130 .008, .054, .134 .003, .017, .054	= <i>u</i>	.001, .042, .116 .003, .046, .113 .002, .020, .074
N(0,1)		$\hat{\mathcal{M}}_T$.001, .036, .103 .006, .052, .130 .004, .025, .064 .002, .056, .158		.006, .038, .105 .005, .052, .132 .004, .032, .078 .004, .046, .120	id t ₅	$\hat{\mathcal{M}}_T$.004, .039, .095 .007, .051, .134 .004, .013, .035		.002, .035, .073 .002, .038, .104 .000, .018, .047	ARCH	$\hat{\mathcal{M}}_T$.002, .042, .119 .008, .048, .127 .003, .018, .051		.001, .045, .114 .003, .044, .107 .000, .020, .069
$\epsilon_t \stackrel{iid}{\sim} l$		$\hat{\mathcal{D}}_{T}^{dw}$.304, .548, .675 .263, .427, .504 .230, .424, .508 .249, .504, .635		.051, .281, .526 .067, .173, .279 .052, .171, .269 .074, .310, .530	€t ji	$\hat{\mathcal{D}}_T^{dw}$.168, .311, .379 .164, .253, .305 .143, .249, .292		.026, .098, .155 .024, .050, .058 .013, .041, .057	εt ~ G	$\hat{\mathcal{D}}_T^{dw}$.258, .498, .642 .244, .407, .488 .242, .417, .512		.072, .306, .528 .083, .168, .258 .049, .172, .264
	04	$\hat{\mathcal{D}}_{T}^{cv}$.012, .041, .077 .066, .105, .162 .052, .102, .159 .014, .045, .096	256	.005, .041, .082 .045, .093, .146 .034, .099, .178 .005, .040, .098	64	$\hat{\mathcal{D}}_{T}^{cv}$.003, .027, .069 .053, .085, .138 .032, .080, .131	256	.000, .025, .062 .017, .048, .104 .038, .129, .210	64	$\hat{\mathcal{D}}_{T}^{cv}$.014, .045, .096 .080, .120, .186 .064, .122, .174	256	.005, .040, .098 .063, .106, .162 .029, .099, .164
	= u	$\hat{\mathcal{M}}_{T}^{(p)}$.002, .024, .092 .006, .045, .105 .003, .036, .076 .000, .029, .098	= <i>u</i>	.003, .022, .096 .005, .037, .107 .005, .033, .063 .002, .031, .093	= u	$\hat{\mathcal{M}}_T^{(p)}$.002, .034, .088 .009, .048, .117 .004, .021, .053	= <i>u</i>	.001, .031, .083 .005, .034, .113 .000, .005, .035	= u	$\hat{\mathcal{M}}_T^{(p)}$.001, .034, .091 .011, .057, .121 .004, .022, .056	= <i>u</i>	.002, .021, .096 .002, .035, .086 .000, .014, .068
		$\hat{\mathcal{M}}_T$.005, .025, .093 .006, .043, .106 .006, .026, .053 .004, .038, .099		.002, .024, .090 .006, .036, .103 .004, .034, .069 .002, .031, .093		$\hat{\mathcal{M}}_T$.001, .032, .085 .009, .044, .116 .004, .019, .050		.001, .028, .087 .005, .033, .107 .000, .005, .032		$\hat{\mathcal{M}}_T$.001, .030, .088 .011, .060, .121 .005, .021, .052		.001, .020, .097 .002, .035, .084 .000, .013, .065
			MA(1) AR(1) SETAR GARCH		MA(1) AR(1) SETAR GARCH			MA(1) AR(1) SETAR		MA(1) AR(1) SETAR			MA(1) AR(1) SETAR		MA(1) AR(1) SETAR

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 $\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values. The GARCH error is based on an iid N(0, 1) innovation.

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	- Âr	$\hat{\mathcal{M}}(p)$ =	64 ற <u>с</u> у	$\hat{\sigma}_{dw} = \hat{i}_{i} \hat{d}_{i} \hat{d}_{i}$	<	(0, 1) Ŵr	$\frac{n}{\hat{M}r} \qquad \frac{n}{\hat{M}(p)}$	$(0,1) \qquad n = 128$ $\hat{\mathcal{M}}_{T} \qquad \hat{\mathcal{M}}_{L}(p) \qquad \qquad \hat{\mathcal{D}}_{L}^{cv}$
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	\hat{D}_T^{dw}	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$		\hat{D}_T^{cv}
MA(1) AR(1)	.000, .012, .054	.001010, .066	.018, .040, .075	.005, .085, .255	.001, .008, .046 .001, .020, .084	.001, .0	017, .053 030 .094	017, .053 .008, .034, .074 030 .094 .074 .123
SETAR GARCH	.001, .017, .037 .000, .021, .095	.003, .019, .041 .000, .025, .109	.040, .089, .152 .014, .047, .103	.014, .093, .202 .004, .077, .247	.001, .027, .050 .003, .044, .113	.00	3, .024, .046 4, .047, .128	3, .024, .046 .051, .124, .171 4, .047, .128 .008, .034, .085
		<i>n</i> =	256				<i>n</i> = :	<i>n</i> = 512
MA(1)	.000, .011, .049	.000, .015, .059	.008, .045, .088	.000, .019, .146	.004, .027, .061		.004, .032, .079	.004, .032, .079 .013, .047, .109
AR(1)	.002, .015, .057	.002, .019, .058	.048, .100, .154	.017, .036, .087	.002, .027, .085		.005, .028, .080	.005, .028, .080 $.033, .079, .149$
GARCH	.003, .033, .095	.003, .032, .038	.007, .037, .086	.001, .027, .100	.003, .035, .108		.002, .035, .116	.002, .035, .116 .017, .045, .083
		<i>n</i> =	64	et ii	\tilde{d}_{t_5}		<i>n</i> =	<i>n</i> = 128
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_{T}^{cv}$	\hat{D}_T^{dw}	$\hat{\mathcal{M}}_T$		$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{M}}_T^{(p)} = \hat{\mathcal{D}}_T^{cv}$
MA(1) AR(1) SETAR	.000, .013, .050 .002, .020, .066 .001, .018, .057	.001, .013, .057 .005, .030, .081 .001, .023, .057	.005, .030, .078 .056, .091, .150 .026, .067, .118	.008, .072, .175 .025, .067, .132 .012, .061, .155	.001, .021, .059 .000, .027, .089 .001, .014, .038	~ - ~	 .002, .026, .073 .000, .033, .091 .001, .014, .045 	9 .002, .026, .073 .006, .026, .064 1 .000, .033, .091 .034, .077, .124 3 .001, .014, .045 .020, .083, .152
		<i>n</i> = <i>n</i>	256				<i>n</i> = 1	<i>n</i> = 512
MA(1) AR(1) SETAR	.000, .017, .060 .003, .014, .049 .001, .014, .033	.000, .019, .067 .003, .018, .052 .002, .012, .037	.003, .028, .064 .018, .048, .093 .038, .115, .191	.001, .016, .061 .002, .013, .023 .001, .007, .024	.002, .021, .08 .003, .027, .08 .003, .014, .04	08 07 08	30 .003, .028, .086 37 .003, .034, .094 19 .002, .015, .047	30 .003, .028, .086 .006, .033, .070 37 .003, .034, .094 .007, .049, .090 49 .002, .015, .047 .056, .165, .263
		<i>n</i> =	64	$\epsilon_t \sim G$	ARCH		<i>n</i> =	<i>n</i> = 128
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{_{CV}}$	\hat{D}_T^{dw}	$\hat{\mathcal{M}}_T$		$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{M}}_T^{(p)} = \hat{\mathfrak{D}}_T^{c u}$
MA(1) AR(1) SETAR	.000, .019, .071 .003, .025, .077 .004, .019, .057	.000, .020, .082 .004, .031, .087 .004, .022, .063	.014, .047, .103 .082, .130, .190 .063, .122, .179	.006, .082, .253 .034, .086, .195 .028, .094, .226	.002, .031, .0 .001, .022, .0 .001, .014, .0)96)85)50	096 .002, .035, .112 085 .001, .033, .092 050 .001, .016, .056	96 .002, .035, .112 .008, .034, .085 085 .001, .033, .092 .073, .123, .170 050 .001, .016, .056 .040, .106, .155
		<i>n</i> =	256				<i>n</i> = 1	<i>n</i> = 512
MA(1) AR(1)	.002, .021, .076	.002, .020, .091	.007, .037, .086	.000, .024, .161 .016, .044, .098	.004, .038, .11	- 90	0 .003, .044, .116 9 .001, .032, .108	0 .003, .044, .116 .017, .045, .083 9 .001, .032, .108 .033, .077, .123
	.000, .020, .0			.002,.011,.075	.000,	;		

Table A.4.: Rejection Frequencies under H_0 : Walsh Basis Case 2: $\mathcal{H}_T = 2T^{.49}$ and $\mathcal{K}_T = .5T^{.49}$

			Table A.5.: a. Re Case 1: $\mathcal{H}_T =$	jection Frequenci $[\log_2(n)^{.99} - 3.5]$ $\epsilon_t \sim N(0)$	ies under H_1 : Wa 5] and $\mathcal{K}_T = [n^{1/2}]$	lish Basis ³ + .01]		
		= <i>u</i>	= 64			= <i>u</i>	128	
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_{T}^{(p)}$	$\hat{\mathcal{D}}_{T}^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_{T}^{dw}$
alt-1	.122, ,227, .336	.165, .287, .431	.173, .401, .549	.037, .198, .380	.436, .613, .742	.374, .454, .672	.827, .936, .967	.054, .275, .440
alt-2	.031, .147, .268	.032, .151, .274	.019, .048, .093	.275, .535, .679	.088, .398, .424	.090, .409, .541	.010, .044, .089	.116, .379, .602
alt-3	.021, .053, .201	.023, .038, .165	.173, .409, .555	.095, .336, .490	.354, .475, .630	.342, .436, .599	.434, .696, .809	.030, .225, .413
alt-4	.081, .272, .351	.062, .168, .344	.190, .413, .557	.097, .356, .486	.672, .750, .838	.674, .743, .833	.772, .917, .949	.113, .386, .509
alt-5	.102, .228, .342	.086, .149, .363	.081, .140, .191	.260, .415, .504	.494, .713, .888	.428, .739, .932	.160, .336, .471	.061, .171, .296
alt-6	.024, .081, .159	.020, .061, .162	.036, .084, .141	.204, .394, .508	.118, .244, .377	.121, .256, .352	.042, .131, .241	.070, .264, .415
alt-7	.031, .099, .135	.021, .088, .132	.054, .101, .143	.228, .399, .491	.073, .127, .239	.061, .102, .268	.058, .140, .215	.096, .219, .307
alt-8	.001, .121, .147	.003, .085, .123	.081, .140, .191	.260, .415, .504	.043, .131, .408	.027, .150, .437	.069, .110, .147	.115, .255, .352
alt-9	.050, .074, .103	.030, .047, .114	.016, .046, .099	.303, .562, .699	.067, .081, .180	.049, .098, .214	.002, .025, .063	.108, .364, .586
		= u	256			= <i>u</i>	512	
alt-1	.794, .931, .987	.754, .914, .978	.977, .997, 1.00	.045, .263, .432	1,00, 1.00, 1.00	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.317, .589, .688
alt-2	.108, .401, .488	.102, .419, .560	.004, .035, .087	.056, .283, .524	.214, .457, .512	.212, .451, .517	.008, .040, .081	.039, .230, .448
alt-3	.720, .817, .954	.710, .804, .947	.948, .986, .993	.154, .405, .510	1.00, 1.00, 1.00	.968, .999, 1.00	.988, .999, 1.00	.059, .284, .436
alt-4	.855, .985, 1.00	.876, .987, 1.00	.941, .983, .994	.094, .356, .463	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.999, 1.00, 1.00	.138, .402, .501
alt-5	.864, .988, .998	.723, .993, .999	.915, .977, .987	.060, .314, .433	1.00, 1.00, 1.00	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.697, .755, .777
alt-6	.259, .320, .533	.263, .338, .556	.069, .207, .296	.044, .213, .360	.681, .810, .904	.605, .829, .914	.153, .357, .469	.037, .234, .393
alt-7	.112, .341, .444	.115, .362, .468	.164, .375, .530	.032, .100, .166	.421, .695, .818	.449, .638, .834	.715, .878, .939	.084, .182, .291
alt-8	.162, .217, .467	.152, .240, .503	.044, .097, .148	.076, .163, .262	.585, .918, .996	.210, .935, .997	.070, .189, .318	.053, .100, .128
alt-9	.162, .218, .360	.141, .230, .368	.009, .032, .088	.065, .276, .515	.285, .466, .620	.188, .468, .606	.009, .050, .103	.041, .218, .461
				1		- annu -: vaô		and an international states of the second states of
M_T and $\sum_{n=1}^{n} \hat{O}d$	M_T^{T} are the prower w	posed max-tests wi	ith and without a $p\epsilon$	enalty, based on a t	ootstrapped p-valu	e. \mathcal{D}_T^{v} is JWW s t	est based on simul	ated critical values,
T_{T} nile	uses vousuappe	u p-values.						

alt-9	alt-8	alt-7	alt-6	alt-5	alt-4	alt-3	alt-2	alt-1		110	alt-0	alt-8	alt-7	alt-6	alt-5	alt-4	alt-3	alt-1 alt-2			
.071, .192, .232	.093, .190, .368	.092, .217, .372	.119, .245, .304	.858, .878, .944	.690, .803, .910	.612, .725, .847	.024, .058, .168	624 763 846			002 025 084	.021082125	.009, .046, .096	.020, .087, .119	.110, .270, .294	.005, .248, .386	.034, .066, .141	.064, .267, .339 .007, .041, .157	\tilde{M}_{T}	>	
.041, .124, .235	.064, .147, .388	.084, .207, .297	.118, .252, .314	.802, .868, .947	.697, .801, .904	.805, .796, .823	.033, .062, .171	689 734 821	<i>n</i> =		002 024 094	.021069114	.005, .041, .071	.014, .066, .120	.101, .183, .239	.042, .244, .279	.031, .065, .124	.072, .286, .433 .008, .047, .165	$\widetilde{M}_T^{(p)}$	^ (n)	<i>n</i> =
.006, .025, .073	.012, .056, .100	.182, .388, .515	.069, .199, .294	.888, .961, .983	.920, .974, .992	.935, .981, .991	.004, .026, .069	.978 .996 .998	256	.001, .002, .001	007 032 067	.056096138	.020, .053, .094	.021, .050, .091	.056, .096, .138	.171, .386, .535	.173, .386, .513	.160, .360, .503 .002, .024, .061	$\hat{\mathcal{D}}_{T}^{cv}$		= 64
.033, .090, .127	.012, .042, .051	.007, .038, .099	.011, .044, .080	.018, .127, .288	.029, .165, .361	.035, .239, .468	.030, .093, .144	.011116373			203 361 426	.180269313	.129, .221, .267	.147, .247, .297	.175, .270, .308	.050, .199, .288	.051, .206, .276	.021, .127, .223 .206, .336, .397	$\hat{\mathcal{D}}_T^{dw}$	× 1	
.181, .370, .428	.404, .654, .884	.481, .505, .768	.332, .418, .592	.934, .977, .987	.989, .997, 1.00	.890, .984, 1.00	.050, .161, .307	.905 .950 .976			030 077 178	.046102311	.084, .126, .249	.094, .162, .231	.295, .464, .684	.566, .698, .824	.124, .297, .365	.342, .351, .449 .024, .079, .214	\tilde{M}_T	>	
.141, .269, .433	.418, .670, .920	.498, .425, .690	.341, .440, .605	.938, .977, .987	.928, .974, .985	.828, .928, .998	.049, .166, .306	920 945 971	<i>n</i> =	.010, .007, .201	013 087 201	.008111320	.072, .102, .211	.082, .151, .234	.214, .494, .697	.467, .581, .706	.117, .274, .340	.331, .377, .385 .047, .083, .225	$\widetilde{M}_T^{(p)}$	^ (n)	<i>n</i> =
.004, .040, .084	.016, .091, .187	.697, .882, .935	.164, .350, .468	1.00, 1.00, 1.00	.997, 1.00, 1.00	.984, .999, .999	.005, .029, .061	1.00. 1.00. 1.00	512		001 025 070	.029068107	.059, .142, .231	.031, .099, .182	.130, .297, .411	.737, .899, .947	.413, .643, .753	.799, .921, .951 .007, .031, .060	$\hat{\mathcal{D}}_T^{cv}$		128
.010, .038, .059	.003, .004, .006	.029, .129, .266	.006, .034, .073	.381, .740, .816	.050, .303, .518	.013, .158, .419	.010, .039, .063	237 721 877			071 171 247	.053124163	.047, .098, .134	.025, .103, .157	.025, .084, .139	.050, .190, .358	.015, .114, .207	.021, .130, .303 .062, .194, .268	\hat{D}_T^{dw}	> 1	

$ \begin{split} \begin{array}{ c c c c c c c c c c c c c c c c c c c$	$ \begin{split} \vec{M}_T = \vec{M}_T $			= u	Case 1: $\mathcal{H}_T =$	$[\log_2(n)^{.99} - 3.$ $\epsilon_t \sim \text{GAR}$	5] and $\mathcal{K}_T = [n^{1}, \mathbf{K}_T]$	/3 + .01] n =	: 128	
alt-1 122. 247. 357 154. 317. 456 177. 376, 533 037. 206, 369 312. 435, 507 409. 412. 573 845. 946. 972 048. 243. 408 233. 555 312. 437 317. 450 173. 113. 261 005, 033, 085 097. 353, 553 575 307. 206 503. 205 533. 555 097. 353, 553 575. 307. 305 779. 911. 350 207. 244. 381 033. 282, 483 113. 306 214. 370 200. 255. 113. 309. 244. 581 072. 144 530. 505 303. 214. 370 200. 557. 333 033. 282, 483 303. 214. 370 206. 505 303. 244. 243. 483 303. 244. 243. 472 049. 155. 240. 381 039. 214. 370 307. 117. 257 049. 157. 245 049. 157. 240. 381 077. 173. 231. 351 072. 240. 381 072. 240. 381 072. 240. 381 055. 120. 230. 255. 555 047. 112. 276 033. 118. 216 077. 113. 231. 351 072. 240. 381 072. 240. 381 072. 240. 381 072. 240. 381 072. 240. 381 072. 241. 391 072. 241. 392 066. 157. 132. 390. 255. 457 044. 016. 220 007. 107. 107. 231. 307. 356 351. 356 351. 356 351. 356 351. 356 351. 356 351. 356 351. 356 <th< th=""><th>alt-1 [122 247, 357 154, 317, 459 177, 376, 533 037, 206, 369 312, 435, 507 409, 412, 573 845, 946, 972 048, 243, 408 alt-3 001, 095, 150 071, 094, 159 017, 055, 153 097, 355, 575 alt-37 020, 126, 333 188, 424, 584 089, 341, 468 239, 348 313 227, 307, 505 779 911, 356 500 505 alt-8 039, 144 087, 123 009, 137, 235 069, 124, 159 266, 396, 345 464 033, 282, 380, 351 32, 227, 307, 505 779 911, 356 50 505 alt-8 039, 147 215, 309 337, 112, 205 001, 137, 235 069, 124, 159 256, 390, 450 061, 157, 245 001, 137, 235 069, 124, 159 266, 390, 450 061, 157, 245 001, 137, 245 001, 137, 245 009, 124, 159 266, 390, 450 047, 112, 276 028, 116, 290 050, 086, 156 109, 230, 351 alt-8 013, 047, 113 261 043, 111, 207 00, 101, 118, 216 055, 120, 201, 201 077, 173, 261 alt-8 013, 047, 113 6 012, 045, 085 140 077, 113, 261 044, 087, 113, 261 044, 098, 147 215, 390, 450 047, 102, 000, 118, 248 315 069, 124, 159 266, 390, 451 01, 202 010, 106, 220 007, 038, 085 106, 350, 256 300, 230, 351 alt-2 022, 096, 136 013, 047, 103 069, 124, 159 266, 390, 451 11, 207 00, 101, 006, 220 007, 038, 085 106, 350, 250 350, 351 alt-2 022, 096, 134 013, 047, 103 092, 130 250, 351 alt-2 021, 016, 100, 200 057, 038, 085 100, 100, 004, 258, 657 alt-2 312, 347 alt-2 328, 561 41, 774 588 991 100, 100, 004, 258, 395 alt-2 328, 561 673 33, 393, 398, 991 135, 388, 901 719, 887, 901 997, 100, 100 044, 243, 472 alt-2 528, 614, 774 580 391, 201 997, 100, 100 100, 004, 258, 355 alt-2 328, 564 490 305, 214, 302 303, 224, 302 306, 912, 310, 328 alt-2 328, 564 437 331, 428 abs, 921 100, 100, 100 047, 243, 473 alt-2 328, 561 673 330, 355, 561 448 310, 308, 932 107, 306, 430 377, 338, 391 710, 100, 100 00, 115, 343, 452 alt-2 328, 564 690 377, 388, 391 719, 388, 391 710, 100, 100 00, 337 712, 745 alt-2 328, 564 437 319, 328 alt-2 328, 564 430 305, 912, 301, 301, 302 303, 306, 914 328 310, 422 555, 551 441 366, 492 393, 392 551 491 300, 991 710, 100 100, 100, 100, 100, 337 712, 745 alt-2 388, 391 714, 586 393, 392 300, 392 300, 391 292 357, 375 alt-2 328, 393 309, 512, 924</th><th></th><th>$\hat{\mathcal{M}}_T$</th><th>$\hat{\mathcal{M}}_{T}^{(p)}$</th><th>$\hat{\mathcal{D}}_{T}^{cv}$</th><th>$\hat{\mathcal{D}}_{T}^{d_{W}}$</th><th>$\hat{\mathcal{M}}_T$</th><th>$\hat{\mathcal{M}}_{T}^{(p)}$</th><th>$\hat{\mathcal{D}}_{T}^{cv}$</th><th>$\hat{\mathcal{D}}_{T}^{d_{W}}$</th></th<>	alt-1 [122 247, 357 154, 317, 459 177, 376, 533 037, 206, 369 312, 435, 507 409, 412, 573 845, 946, 972 048, 243, 408 alt-3 001, 095, 150 071, 094, 159 017, 055, 153 097, 355, 575 alt-37 020, 126, 333 188, 424, 584 089, 341, 468 239, 348 313 227, 307, 505 779 911, 356 500 505 alt-8 039, 144 087, 123 009, 137, 235 069, 124, 159 266, 396, 345 464 033, 282, 380, 351 32, 227, 307, 505 779 911, 356 50 505 alt-8 039, 147 215, 309 337, 112, 205 001, 137, 235 069, 124, 159 256, 390, 450 061, 157, 245 001, 137, 235 069, 124, 159 266, 390, 450 061, 157, 245 001, 137, 245 001, 137, 245 009, 124, 159 266, 390, 450 047, 112, 276 028, 116, 290 050, 086, 156 109, 230, 351 alt-8 013, 047, 113 261 043, 111, 207 00, 101, 118, 216 055, 120, 201, 201 077, 173, 261 alt-8 013, 047, 113 6 012, 045, 085 140 077, 113, 261 044, 087, 113, 261 044, 098, 147 215, 390, 450 047, 102, 000, 118, 248 315 069, 124, 159 266, 390, 451 01, 202 010, 106, 220 007, 038, 085 106, 350, 256 300, 230, 351 alt-2 022, 096, 136 013, 047, 103 069, 124, 159 266, 390, 451 11, 207 00, 101, 006, 220 007, 038, 085 106, 350, 250 350, 351 alt-2 022, 096, 134 013, 047, 103 092, 130 250, 351 alt-2 021, 016, 100, 200 057, 038, 085 100, 100, 004, 258, 657 alt-2 312, 347 alt-2 328, 561 41, 774 588 991 100, 100, 004, 258, 395 alt-2 328, 561 673 33, 393, 398, 991 135, 388, 901 719, 887, 901 997, 100, 100 044, 243, 472 alt-2 528, 614, 774 580 391, 201 997, 100, 100 100, 004, 258, 355 alt-2 328, 564 490 305, 214, 302 303, 224, 302 306, 912, 310, 328 alt-2 328, 564 437 331, 428 abs, 921 100, 100, 100 047, 243, 473 alt-2 328, 561 673 330, 355, 561 448 310, 308, 932 107, 306, 430 377, 338, 391 710, 100, 100 00, 115, 343, 452 alt-2 328, 564 690 377, 388, 391 719, 388, 391 710, 100, 100 00, 337 712, 745 alt-2 328, 564 437 319, 328 alt-2 328, 564 430 305, 912, 301, 301, 302 303, 306, 914 328 310, 422 555, 551 441 366, 492 393, 392 551 491 300, 991 710, 100 100, 100, 100, 100, 337 712, 745 alt-2 388, 391 714, 586 393, 392 300, 392 300, 391 292 357, 375 alt-2 328, 393 309, 512, 924		$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_{T}^{(p)}$	$\hat{\mathcal{D}}_{T}^{cv}$	$\hat{\mathcal{D}}_{T}^{d_{W}}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_{T}^{(p)}$	$\hat{\mathcal{D}}_{T}^{cv}$	$\hat{\mathcal{D}}_{T}^{d_{W}}$
atr-2 $071, 095, 150$ $071, 094, 159$ $017, 051, 109$ $307, 536, 660$ $014, 100, 249$ $017, 113, 261$ $005, 033, 085$ $097, 335, 575$ atr-3 $020, 057, 121$ $020, 057, 121$ $020, 057, 121$ $020, 057, 112$ $021, 157, 235$ $060, 137, 235$ $069, 134, 281$ $070, 339, 148$ $230, 364, 496$ $033, 214, 331$ $249, 165, 330$ $111, 307$ $031, 137, 232$ $031, 137, 235$ $044, 108, 137, 235$ $069, 124, 159$ $226, 330, 450$ $044, 101, 087, 112$ $041, 075, 119, 231$ $072, 240, 331$ $124, 243, 331$ $127, 232$ $039, 138$ $069, 124, 159$ $266, 330, 450$ $044, 101, 207$ $031, 137, 232$ $001, 177, 173, 261$ $007, 173, 231$ $007, 173, 231$ $007, 230, 231, 331$ $310, 230, 351, 350, 556$ atr-8 $014, 088, 144$ $014, 039, 138$ $069, 124, 159$ $256, 330, 450$ $047, 111, 207$ $033, 118, 216$ $007, 173, 231$ $007, 173, 231$ $007, 230, 230, 351$ $007, 173, 283$ $100, 100, 100, 100, 230, 230, 351$ $037, 116, 230, 230, 236, 433$ $111, 207$ $028, 116, 230, 231, 105, 231$ $077, 173, 281$ $097, 100, 100, 100, 100, 230,$	alt-2 071, 095, 150 071, 094, 159 017, 051, 099 307, 536, 660 014, 100, 249 017, 113, 261 005, 033, 085 097, 353, 575 alt-370 alt-8 033, 227, 337 020, 126, 335 113, 300, 505 311, 360, 505 alt-370 alt-6 033, 011, 139 024, 146, 139 261, 388, 444 581 333, 054, 444 038, 513 227, 307, 505 779, 911, 365 260, 303, 037, 113, 261 049, 154, 281 alt-6 036, 137, 123 069, 124, 159 261, 390, 450 043, 111, 207 033, 118, 216 055, 120, 201, 072, 240, 351 alt-8 014, 087, 112 277 041, 076, 130 047, 103, 165 216, 375, 476 043, 111, 207 033, 118, 216 055, 120, 201 077, 173, 261 049, 136 014, 088, 144 014, 039, 138 069, 124, 159 266, 390, 450 047, 101, 202 010, 106, 220 007, 038, 086, 156 07, 230, 351 alt-8 014, 088, 144 014, 039, 138 069, 124, 159 266, 390, 450 077, 112, 276 028, 116, 290 050, 086, 156 072, 330, 556 alt-9 025, 095, 138 069, 124, 139 266, 437 045, 101, 202 010, 106, 220 007, 038, 086, 156 072, 330, 556 alt-9 027, 038, 118, 216 055, 120, 201 077, 173, 261 077, 175, 261 077, 175, 261 077, 175 261 077, 175 261 077, 175 261 077, 175 261 077, 175 261 077, 175 261 077, 175 255, 575 010, 106, 120, 100, 120 077, 175, 250, 556 575 017, 038, 052, 053, 556 575 017, 038, 052, 053, 556 575 017, 038, 058, 106 0720, 545, 657 120 001, 100, 100 077, 075, 256, 557 050 076, 330, 556 575 010, 100, 100, 100 077, 055, 556 575 017, 038, 052, 058 106, 350, 556 575 017, 038, 052 006, 350, 350, 556 575 017, 038, 052 006, 350, 350, 556 575 017, 038, 052 006, 350, 350, 350, 350, 556 575 017, 038, 052 008, 157 275 000, 333, 106, 377 375 010 071, 100, 100 100, 017, 044, 243, 472 051, 126, 260, 333, 033, 096, 495 093, 306, 495 033, 306, 456 033, 426, 557 032, 109, 100, 100 100 015, 343, 452 alt-6 087, 188, 286 092 007, 330, 226, 331, 322, 348 011, 719, 837, 901 091, 100, 100 010, 004, 258, 395 alt-7 126, 357, 325 alt-1 005, 047, 345 333, 102, 201, 201, 100, 100 100, 100, 100,	alt-1 .1	22, .247, .357	.154, .317, .459	.177, .376, .533	.037, .206, .369	.312, .435, .507	.409, .412, .573	.845, .946, .972	.048, .243, .408
alt-3 $(020, 057, 121, 020, 055, 113, 190, 444, 581, 079, 339, 458 (63, 202, 359, 158, 293, 448, 419, 682, 803, 030, 214, 370 alt-3 (033, 227, 337, 502, 126, 335, 118, 244, 581, 033, 227, 307, 505, 779, 911, 955, 111, 360, 505 (79, 911, 952, 111, 360, 381 alt-7 (041, 038, 138, 044, 066, 129, 044, 103, 136, 375, 476 (082, 157, 245, 033, 118, 216 (055, 120, 201, 077, 173, 261 alt-8 (015, 018, 136, 013, 047, 103, 138, 064, 129, 134, 136 (012, 047, 103, 165, 350, 356, 136, 136, 136, 136, 136, 136, 136, 350, 136 (017, 173, 261 alt-8 (014, 039, 138, 047, 103, 138, 046, 101, 202 (000, 106, 220 (007, 038, 086, 156 (007, 038, 086, 096) (007, 038, 086, 096 (007, 038, 086, 096) (007, 038, 086, 096 (007, 038, 086, 096) (007, 038, 038, 090) (017, 000, 001, 000, 004, 045 (007, 044, 045 (007, 044, 045 (007, 044, 045 (007, 044, 045 (067, 038, 010, 010) (000, 115, 047, 010) (016, 010) (000, 004, 045 (008, 016) (047, 046, 024) 045, 016, 004, 045 (008, 016) (047, 006, 006, 006, 006, 006, 006, 006, 00$	alt-3 020, 057, 121 020, 055, 113 190, 444, 581 079, 339, 458 163, 202, 359 158, 293, 448 419, 682, 803 030, 214, 370 alt-4 033, 227, 337 020, 137, 235 069, 124, 159 261, 335, 446 033, 287, 307, 505 779, 911, 955 111, 360, 505 330, 071, 113, 201 037, 119, 231 072, 240, 381 alt-7 044, 076, 139 097, 124, 159 261, 335, 446 033, 034, 464 033, 287, 391, 955 111, 360, 535, 120, 201 077, 173, 261 alt-8 030, 071, 113 072, 1240 047, 103, 165 250 047, 111, 207 033, 118, 216 055, 120, 201 077, 173, 261 alt-8 012, 046, 129 047, 101, 202 047, 101, 202 010, 106, 220 077, 033, 035 150, 230, 351 alt-9 022, 006, 124, 159 265, 390, 450 047, 112, 276 028, 116, 290 050, 086, 156 109, 230, 351 alt-9 022, 006, 124, 136 012, 045, 085 266, 437 046, 101, 202 010, 106, 220 007, 038, 085 106, 230, 351, 555 alt-1 557, 641, 744 546, 616, 701 975, 998 999 038, 266, 437 739, 843, 894 716, 827, 878 100, 100, 100 0270, 545, 657 alt-2 222, 175, 882, 201 106, 220 007, 038, 085 100, 247 243, 472 alt-1 557, 641, 744 546, 610, 701 099, 105 033, 214, 300 997, 100, 100 0270, 545, 657 alt-2 222, 175, 882, 901 100, 100 0271, 306, 350, 556 alt-2 328, 295 691, 772, 882, 901 100, 1100 044, 243, 475 691 719, 885, 901 100, 100 0047, 100 044, 243, 475 alt-6 691, 776, 882 991, 100, 100 100, 100 053, 712, 745 alt-6 691, 776, 881, 991 100, 100 100, 100, 100, 100 053, 712, 745 alt-6 691, 776, 881, 901 100, 1100 115, 343, 452 alt-6 691, 716, 877, 901 997, 100, 100 100, 100 053, 712, 745 alt-6 691, 716, 350, 556 046, 224, 337, 342, 355 644 391, 306, 224, 337, 224, 330, 515 699, 737, 892 691, 777, 882, 901 100, 100 100, 100 053, 712, 745 alt-6 601, 189, 238, 901, 955 644 337, 901 997, 100, 1100, 1100 135, 374, 555 alt-6 691, 776, 882 901, 100, 100, 100 100, 100, 100, 100 053, 712, 735 alt-7 126, 360, 386, 922 006, 933 095, 146 337, 322, 324 33, 428, 654 337, 443, 667 773, 904, 945 093, 718, 256 alt-6 608, 330, 966, 445 333, 428, 654 337, 443, 667 773, 904, 945 093, 173, 246 144, 245, 737 904, 946 093, 108, 104, 241 200, 106, 106, 016, 044, 245, 231, 742 and 712, 8	alt-2 .0	71, .095, .150	.071, .094, .159	.017, .051, .099	.307, .536, .660	.014, .100, .249	.017, .113, .261	.005, .033, .085	.097, .353, .575
alt-4 $[033, 227, 337, 020, 126, 335, 188, 424, 564, 089, 341, 468 \\ alt-5 \\ (091, 142, 270 \\ 001, 137, 235 \\ 0091, 137, 235 \\ 0091, 137, 235 \\ 0091, 137, 235 \\ 0091, 137, 235 \\ 0091, 137, 235 \\ 0091, 137, 235 \\ 0091, 137, 235 \\ 0091, 137, 235 \\ 0091, 137 \\ 0141, 005, 139 \\ 0142, 013 \\ 0141, 005, 130 \\ 0141, 015, 130 \\ 0141, 015, 130 \\ 0141, 015 \\ 013, 047, 112 \\ 0141, 016 \\ 013, 047, 112 \\ 0141, 016 \\ 013, 047, 112 \\ 0141, 016 \\ 013, 047, 113 \\ 0141, 016 \\ 013, 047, 113 \\ 0141, 016 \\ 013, 047, 112 \\ 0141, 020 \\ 013, 047, 113 \\ 0141, 015 \\ 013, 047, 103 \\ 015 \\ 0141, 015 \\ 012 \\ 0141, 015 \\ 012 \\ 0141, 015 \\ 012 \\ 011, 041 \\ 010 \\ 000 \\ 00$	alt-4 (033, 227, 337 (020, 126, 135 (06), 124, 159 (06), 138, 434, 153 (05), 111, 260, 505 (06), 137, 235 (06), 137, 235 (06), 137, 235 (06), 137, 235 (06), 137, 235 (06), 137, 235 (06), 137, 232 (04), 138 (19, 231 (077, 132, 261 (077, 173, 261 (077, 167, 273, 273, 273))))))))))))))))))))))))))))))))))))	alt-3 .0	020, .057, .121	.020, .055, .113	.190, .444, .581	.079, .339, .458	.163, .202, .359	.158, .293, .448	.419, .682, .803	.030, .214, .370
alt-5 $(091, 142, 270)$ $(061, 137, 235)$ $(069, 1124, 159)$ $(261, 386, 449)$ $(330, 364, 464)$ $(033, 282, 483)$ $(15, 324)$ $(371, 173, 231)$ $(072, 240, 381)$ alt-6 $(030, 071, 139)$ $(024, 006, 112)$ $(047, 101)$ $(071, 173)$ $(071, 172)$ $(071, 173)$ $(071, 172)$ $(071, 173)$ $(071, 172)$ $(071, 173)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 172)$ $(071, 100)$ $(071, 100)$ $(071, 172)$ $(071, $	alt-5 (091, 142, 270 (061, 137, 235 (069, 124, 159 (261, 386, 449) (330, 364, 464 (033, 282, 483 (158, 343, 472 (049, 154, 281) (077, 173, 261) (071, 173, 275) (071, 101, 202) (011, 010, 106, 220) (071, 038, 085) (106, 350, 556) (071, 176, 276) (022, 196, 101, 202) (011, 010, 100, 120) (071, 073, 273, 261) (071, 173, 273) (071, 073, 273, 271) (071, 072, 243, 372) (071, 073, 273, 251) (071, 072, 073) (071, 073, 273, 251) (071, 071, 073, 273) (071, 071, 072, 243, 372) (071, 072, 073) (074, 100, 074, 243, 472) (071, 176, 276) (071, 196, 303, 038, 056, 932) (071, 038, 038, 056, 157) (071, 072, 038, 252, 321) (010, 010, 010) (071, 074, 373, 395) (071, 176, 276) (071, 176, 276) (071, 196, 303, 096, 195) (071, 198, 338, 901) (071, 071, 108, 276) (074, 274, 374) (074, 100, 100, 100, 100) (100, 116, 274) (075, 108) (074, 138) (076, 139) (076, 130, 126) (076, 130) (076, 130) (076, 127) (076, 107, 246) (071, 196, 303) (076, 130) (076, 130) (076, 127) (076, 108) (076, 127) (076, 078) (076, 108) (0	alt-4 0.	33, .227, .337	.020, .126, .335	.188, .424, .564	.089, .341, .468	.229, .308, .513	.227, .307, .505	.779, .911, .955	.111, .360, .505
alt-6 $[030, 071, 139 \ 0.24, 0.66, 129 \ 041, 039, 147 \ 215, 391, 486 \ 0.82, 165, 250 \ 061, 157, 245 \ 0.33, 118, 216 \ 0.55, 120, 201 \ 077, 173, 261 \ alt-8 \ 014, 088, 144 \ 014, 039, 138 \ 069, 124, 159 \ 265, 390, 450 \ 044, 111, 207 \ 033, 118, 216 \ 055, 120, 201 \ 077, 173, 261 \ alt-8 \ 012, 096, 136 \ 012, 045, 085 \ 046, 101, 202 \ 010, 106, 220 \ 050, 086, 156 \ 109, 230, 351 \ alt-9 \ 022, 096, 136 \ 013, 047, 110 \ 027, 033, 018 \ 016, 220 \ 007, 038, 085 \ 166, 350, 356 \ 3$	alt-6 $(030, 071, 139, 024, 066, 129, 041, 089, 147 215, 391, 486$ $(082, 165, 250, 061, 157, 245, 037, 119, 231, 072, 240, 381 alt-8 (014, 038, 138, 069, 134, 013, 065, 356, 390, 450 (047, 111, 207) 033, 118, 216 055, 120, 201, 230, 351 31, 351 (052, 350, 556, 350, 556, 375, 116, 200, 007, 038, 085, 116, 200, 037, 173, 261, 351 (052, 200, 230, 351 (052, 202, 007, 038, 085, 106, 230, 351, 556, 350, 556, 371, 112, 276 028, 116, 220 007, 038, 085 (106, 230, 556, 556, 556, 457 (046, 101, 202) 001, 106, 220 007, 038, 085 (106, 350, 556, 556, 556, 457 (046, 101, 202) 001, 106, 220 007, 038, 085 (106, 350, 556, 556, 457 (052) (052, 052, 555, 674 (051) (052) 001, 100, 100, 100, 100, 100, 100, 100$	alt-5 0.0	191, .142, .270	.061, .137, .235	.069, .124, .159	.261, .386, .449	.330, .364, .464	.033, .282, .483	.158, .343, .472	.049, .154, .281
alt-7 .041, .087, .127 .041, .076, .130 .047, .103, .165 .216, .375, .476 .043, .111, .207 .033, .118, .216 .055, .120, .201 .077, .173, .261 alt-8 .014, .088, .144 .014, .039, .138 .069, .124, .159 .265, .390, .450 .047, .112, .276 .028, .116, .290 .050, .086, .156 .109, .230, .351 alt-9 .022, .096, .136 .013, .047, .136 .012, .045, .085 .262, .525, .675 .046, .101, .202 .010, .106, .220 .007, .038, .085 .106, .350, .356 alt-1 .557, .641, .744 .546, .616, .701 .975, .998, .999 .038, .266, .437 .739, .843, .894 .716, .827, .878 1.00, 1.00, 1.00 0.240, .543, .472 alt-1 .557, .641, .774 .546, .616, .701 .975, .998, .991 .139, .338, .984, .991 .152, .380, .254, .553 .344, .243, .472 alt-1 .557, .611 .418, .510, .693 .933, .984, .991 .152, .380, .254, .331 .001, .001, .00 .044, .243, .452 alt-2 .021, .176, .276 .023, .138, .266, .437 .739, .894, .911, .772, .882 .991, 1.00, 1.00 .004, .258, .395 .456 alt-3 .252, .551, .611, .673 .303, .966, .1437 .035, .255, .514	alt-7 $0.41, 0.87, 127$ $0.41, 0.76, 130$ $0.47, 103, 165$ $216, 375, 476$ $0.43, 111, 207$ $0.33, 118, 216$ $0.55, 120, 201$ $0.77, 173, 261$ alt-8 $0.14, 0.88, 144$ $0.14, 0.039, 138$ $0.69, 124, 159$ $265, 390, 450$ $0.47, 112, 276$ $0.28, 116, 290$ $0.50, 0.86, 156$ $109, 230, 351$ 102, 0.06, 136 $0.13, 0.47, 136$ $0.12, 0.45, 0.85$ $262, 525, 675$ $0.46, 101, 202$ $0.10, 106, 220$ $0.07, 0.38, 0.85$ $106, 350, 556112, 0.22, 0.96, 1136$ $0.13, 0.47, 136$ $0.12, 0.49, 105$ $0.38, 266, 437$ $739, 843, 894$ $716, 827, 878$ $1.00, 1.00, 1.00$ $0.270, 545, 657112, 0.21, 176, 276$ $0.27, 182, 280$ $0.011, 0.49, 105$ $0.59, 286, 490$ $0.32, 214, 302$ $0.33, 252, 321$ $0.10, 0.47, 100$ $0.44, 243, 472112, 0.21, 176, 276$ $0.27, 182, 280$ $0.011, 0.49, 105$ $0.59, 286, 490$ $0.32, 214, 302$ $0.33, 252, 321$ $0.10, 0.47, 100$ $0.44, 243, 452112, 0.21, 176, 276$ $0.27, 182, 280, 0.91, 0.49, 105$ $0.59, 286, 490$ $0.32, 214, 302$ $0.38, 252, 321$ $0.10, 0.47, 100$ $0.44, 243, 452112, 255, 611$ $1418, 510, 693$ $933, 984, 991$ $152, 380, 557, 880$ 516 $337, 244, 327, 382$ $691, 772, 882$ $991, 100, 1.00$ $100, 0.04, 283, 335$ 142 $126, 339, 255, 614, 772, 882$ $901, 1702, 100, 100, 100, 100, 100, 100, 100, 1$	alt-6 .0	30, .071, .139	.024, .066, .129	.041, .089, .147	.215, .391, .486	.082, .165, .250	.061, .157, .245	.037, .119, .231	.072, .240, .381
alt-8 [014, 088, 144 [014, 039, 138 [009, 124, 159 [265, 390, 450 [047, 112, 276 [028, 116, 290 [050, 086, 156 [100, 230, 351] [010, 202 [007, 038, 085 [106, 350, 556] [010, 202, 007, 038, 085 [106, 350, 556] [010, 202, 007, 038, 085 [100, 230, 556] [010, 202, 007, 038, 085 [100, 230, 556] [010, 201, 176, 256] [011, 049, 105 [038, 266, 437 [739, 843, 894 [716, 827, 878 [100, 100, 100 [00, 270, 545, 457 [14, 254] [148, 510, 693 [933, 984, 991 [152, 338, 251] [157, 322, 321, 302 [091, 712, 882 [991, 100, 100, 100 [064, 258, 395] [154, 275, 561] [152, 333, 984, 991 [152, 338, 251, 321, 302 [691, 772, 882 [991, 100, 100 [064, 258, 395] [157, 325, 351 [170, 533, 712, 745] [157, 343, 452 [157, 333, 452 [157, 232, 487 [150, 100, 100, 100, 100, 100, 100, 100,	alt-8 0.14, 0.08, 1.44 0.14, 0.39, 1.38 0.69, 1.24, 1.59 2.65, .390, .450 0.47, .112, .276 0.28, 1.16, .220 0.07, 0.38, 0.85 1.06, .350, .356, .356, .356 1.02, 0.02, 0.13, 0.47, .136 0.12, 0.045, 0.085 0.136, 0.12, 0.045, 0.085 0.136 0.12, 0.045, 0.085 0.156 0.136, 0.120, 0.07, 0.38, 0.85 1.00, 1.00, 1.00, 1.00, 0.04, .236, .556 att-2 0.21, 1.76, .276 0.257, .611 0.418, .510 0.933, .999 0.38, .266, .437 0.32, .214, .302 0.38, .252, .321 0.00, 1.00, 1.00, 0.00, 0.04, .234, .452 att-3 0.22, .156, .673 0.353, .085 0.011, .049, .105 0.59, .286, .490 0.32, .214, .302 0.38, .252, .321 0.00, 0.07, 1.00 0.270, .545, .657 att-3 422, .525, .611 0.418, .510, .693 0.33, .266, .437 0.320, .371, .892 0.991, 1.100, 1.00, 0.00, 1.00 0.270, .545, .657 att-3 422, .525, .611 0.418, .510, .693 0.33, .984, .991 1.157, .382 0.91, 1.100, 1.00 0.044, .233, .452 att-3 422, .525, .611 0.418, .510, .693 0.33, .986, .992 0.071, .306, .333, .252, .321 0.010, .001, 1.00 0.100 0.100 0.115, .343, .452 att-3 422, .525, .611 0.418, .510, .693 0.331, .986, .992 0.071, .306, .333, .252, .321 0.01, .001, 1.00 0.100 0.100 1.15, .343, .452 att-3 422, .525, .611 0.418, .510, .693 0.331, .906, .445 0.333, .906, .445 0.333, .906, .445 0.333, .906, .445 0.333, .901 0.997, 1.001, 1.00 0.100 0.103, .452 att-3 422 att-3 1.20, .386, .326 0.06, .310, .320, .303, .905, .445 0.333, .906, .445 0.333, .902, .244 0.333, .904, .945 0.933, .173 0.44, .245 .455 0.93, .173 0.88, .287 0.93, .173 0.88, .288 0.911, .001, 1.00, 1.00 1.00, 1.00 1.00, .100 1.15, .245 .456 0.333, .104, .245 0.333, .104, .245 0.333, .172 0.383, .251 0.333, .104, .245 0.333 0.33, .006, .145 0.333, .106, .247 0.333, .246, .242, .376 0.333, .173 0.346 0.333, .173 0.348 0.333, .173 0.344, .245 0.366 0.456 0.333, .106, .246, .244 0.306, .906, .906, .906, .906, .906, .906, .906, .906, .906, .906, .906, .906, .906, .906, .906, .906, .906, .9145 0.333, .924 0.006, .906, .906, .906, .906, .906, .905 0.333, .906, .445 0.333, .426 0.333, .246, .422, .702 0.251, .441 0.001, .001, .904, .945 0.903, .173	alt-7 .0	141, .087, .127	.041, .076, .130	.047, .103, .165	.216, .375, .476	.043, .111, .207	.033, .118, .216	.055, .120, .201	.077, .173, .261
alt-9 .022, .096, .136 .013, .047, .136 .012, .045, .085 .262, .525, .675 .046, .101, .202 .010, .106, .220 .007, .038, .085 .106, .350, .356 alt-1 .557, .641, .744 .546, .616, .701 .975, .998, .999 .038, .256, .437 .739, .843, .894 .716, .827, .878 1.00, 1.00, 1.00 0.270, .545, .657 alt-2 .021, .176, .276 .027, .182, .280 .011, .049, .105 .059, .286, .490 .032, .214, .302 .038, .252, .321 .010, .001, 1.00 0.270, .545, .657 alt-2 .021, .176, .276 .027, .182, .280 .011, .049, .105 .059, .286, .490 .032, .214, .302 .038, .252, .321 .010, .001, 1.00 .024, .253, .452 alt-3 .258, .561, .673 .553, .674 .991 .152, .382 .901 .719, .837, .901 .901, .001, 1.00 .044, .253, .452 alt-4 .528, .561, .673 .533, .954, .991 .152, .322, .321 .010, .001, 1.00 .033, .712, .745 alt-6 .087, .189, .296 .074, .305, .384, .901 .719, .833, .901 .719, .837, .901 .991, 1.00, 1.00 .033, .712, .745 alt-7 <td< td=""><td>alt-9 022, .096, .136 013, .047, .136 012, .045, .085 .262, .525, .675 .046, .101, .202 .010, .106, .220 .007, .038, .085 .106, .350, .556 alt-1 .557, .641, .744 .546, .616, .701 .975, .998, .999 .038, .266, .437 .739, .843, .894 .716, .827, .878 1.00, 1.00, 1.00 0.0270, .545, .657 alt-2 .021, .176, .276 027, .182, .280 .011, .049, .105 .059, .286, .490 .032, .214, .302 .038, .252, .321 .010, 0.01, 0.0 .044, .243, .472 alt-3 .252, .555, .611 .418, .510, .693 .933, .984, .991 .157, .382 .691, .772, .882 .991, 1.00, 1.00, 0.00 .044, .243, .472 alt-4 .528, .561, .673 .530, .555, .674 .931, .986, .992 .071, .305, .312, .322, .321 .010, 0.10, 0.100 .00, 1.00, 1.00 .00, 1.00, 1.00 .053, .372, .343, .452 alt-6 .641, .705, .815 .637, .720, .831 .907, .968, .986 .076, .333, .322, .487 .159, .339, .521 .901, 1.00, 1.00, 1.00 .004, .273, .375 alt-6 .641, .705, .815 .637, .720, .831 .907, .968, .986 .076, .333, .423, .667 .377, .443, .667 .777, .945 .343, .655 .443, .667 .777, .945<</td><td>alt-8 .0</td><td>14, .088, .144</td><td>.014, .039, .138</td><td>.069, .124, .159</td><td>.265, .390, .450</td><td>.047, .112, .276</td><td>.028, .116, .290</td><td>.050, .086, .156</td><td>.109, .230, .351</td></td<>	alt-9 022, .096, .136 013, .047, .136 012, .045, .085 .262, .525, .675 .046, .101, .202 .010, .106, .220 .007, .038, .085 .106, .350, .556 alt-1 .557, .641, .744 .546, .616, .701 .975, .998, .999 .038, .266, .437 .739, .843, .894 .716, .827, .878 1.00, 1.00, 1.00 0.0270, .545, .657 alt-2 .021, .176, .276 027, .182, .280 .011, .049, .105 .059, .286, .490 .032, .214, .302 .038, .252, .321 .010, 0.01, 0.0 .044, .243, .472 alt-3 .252, .555, .611 .418, .510, .693 .933, .984, .991 .157, .382 .691, .772, .882 .991, 1.00, 1.00, 0.00 .044, .243, .472 alt-4 .528, .561, .673 .530, .555, .674 .931, .986, .992 .071, .305, .312, .322, .321 .010, 0.10, 0.100 .00, 1.00, 1.00 .00, 1.00, 1.00 .053, .372, .343, .452 alt-6 .641, .705, .815 .637, .720, .831 .907, .968, .986 .076, .333, .322, .487 .159, .339, .521 .901, 1.00, 1.00, 1.00 .004, .273, .375 alt-6 .641, .705, .815 .637, .720, .831 .907, .968, .986 .076, .333, .423, .667 .377, .443, .667 .777, .945 .343, .655 .443, .667 .777, .945<	alt-8 .0	14, .088, .144	.014, .039, .138	.069, .124, .159	.265, .390, .450	.047, .112, .276	.028, .116, .290	.050, .086, .156	.109, .230, .351
n = 556 $n = 256$ $n = 256$ $n = 512$ alt-1 $.557, 641, .744$ $.546, 616, .701$ $.975, .998, .999$ $.038, .266, .437$ $.739, .843, .894$ $.716, .827, .878$ $1.00, 1.00, 1.00$ $0.270, .545, .657$ alt-2 $.021, .176, .276$ $.027, .182, .280$ $.011, .049, .105$ $.059, .286, .490$ $.032, .214, .302$ $.038, .252, .321$ $.010, .047, .100$ $0.244, .243, .472$ alt-3 $.422, .555, .611$ $.418, .510, .693$ $.933, .984, .991$ $.152, .380, .515$ $.699, .777, .882$ $.991, 1.00, 1.00$ $0.64, .258, .395$ alt-4 $.528, .561, .673$ $.533, .555, .674$ $.931, .986, .992$ $.107, .306, .310, .425$ $.699, .777, .892$ $.691, .772, .882$ $.991, 1.00, 1.00$ $.064, .258, .355$ alt-5 $.637, .720, .831$ $.907, .968, .992$ $.107, .306, .310, .425$ $.893, .925, .961$ $.805, .912, .924$ $.100, 1.00, 1.00$ $.064, .223, .712, .745$ alt-6 $.087, .189, .296$ $.071, .196, .303$ $.092, .224, .337$ $.041, .189, .328$ $.157, .322, .487$ $.159, .339, .521$ $.100, 1.00, 1.00, 1.00$ $.046, .227, .375$ alt-7 $.126, .369, .382$ $.026, .088, .303$ $.092, .224, .337, .096, .145$ $.333, .428, .654$ $.337, .443, .667$ $.737, .904, .945$ $.093, .178, .269$ alt-8 $.026, .088, .287$ $.006, .096, .303$ $.042, .292, .527$ $.181, .245, .436$ $.164, .249, .440$ $.016, .047, .102$ $.045, .231, .473$ alt-9 $.018, .114, .271$ $.010, .115, .275$ $.011, .050, .098$ $.054, .292,$	n = 256 $n = 256$ $n = 512$ alt-1 557 , 641 , 744 546 , 616 , 701 975 , 998 , 999 038 , 266 , 437 739 , 843 , 894 716 , 827 , 878 100 , 1.00 , 1.00 0.270 , 555 , 657 alt-2 021 , 176 , 276 027 , 182 , 280 011 , 049 , 105 059 , 286 , 490 033 , 252 , 214 , 302 038 , 252 , 321 010 , 047 , 100 047 , 100 047 , 243 , 472 alt-3 021 , 176 , 276 027 , 182 , 280 011 , 049 , 105 059 , 286 , 490 033 , 205 , 018 , 291 0047 , 100 004 , 258 , 393 391 719 , 837 , 901 971 , 100 , 100 064 , 258 , 393 345 , 357 , 575 , 614 931 , 986 , 992 107 , 336 , 430 893 , 925 , 961 805 , 912 , 924 100 , 100 , 100 , 100 105 , 343 , 452 alt-6 641 , 705 , 815 637 , 770 , 831 907 , 968 , 986 076 , 3310 , 425 893 , 925 , 961 805 , 912 , 924 100 , 100 , 100 , 100 , 100 105 , 343 , 452 $alt-6$ 087 , 189 , 226 071 , 196 , 333 096 , 145 337 , 448 </td <td>alt-9 .0</td> <td>)22, .096, .136</td> <td>.013, .047, .136</td> <td>.012, .045, .085</td> <td>.262, .525, .675</td> <td>.046, .101, .202</td> <td>.010, .106, .220</td> <td>.007, .038, .085</td> <td>.106, .350, .556</td>	alt-9 .0)22, .096, .136	.013, .047, .136	.012, .045, .085	.262, .525, .675	.046, .101, .202	.010, .106, .220	.007, .038, .085	.106, .350, .556
alt-1 557, 641, 744 546, 616, 701 975, 998, 999 038, 266, 437 739, 843, 894 716, 827, 878 1.00, 1.00, 1.00 0.270, 545, 657 545, 657 alt-2 021, 176, 276 027, 182, 280 011, 049, 105 059, 286, 490 032, 214, 302 038, 252, 321 010, 047, 100 044, 243, 472 alt-3 .422, .555, 611 .418, 510, 693 .933, 984, 991 .152, .380, .515 .699, .737, 892 .691, .772, 882 .991, 100, 1.00 0.64, .258, .345 alt-4 .528, .561, 673 .553, .674 .931, .986, .992 .107, .306, .430 .033, .925, .961 .712, .882 .991, 1.00, 1.00 0.64, .258, .355 alt-5 .644, .705, .815 .637, .720, .831 .907, .966, .435 .393, .925, .961 .805, 912, .924 1.00, 1.00 .033, .712, .745 alt-6 .087, .189, .296 .071, .196, .303 .092, .224, .337 .041, .189, .325, .339, .351 .901, .100, 1.00 .001, .00, .100 .633, .712, .345 alt-6 .087, .189, .296 .071, .196, .303 .092, .304, .425 .333, .428, .654 .337, .443, .667 .137, .904, .945 .093, .178, .2	alt-1 $557, 641, 744$ $546, 616, 701$ $975, 998, 999$ $038, 266, 437$ $739, 843, 894$ $716, 827, 878$ $1.00, 100, 100$ $0.270, 555, 657$ 323 $323, 392, 393, 394, 991$ $152, 380, 515$ $699, 737, 892$ $691, 772, 882$ $991, 100, 100$ $0.047, 100$ $044, 223, 375$ 335 472 $353, 551, 673$ $553, 554$ $931, 986, 992$ $107, 306, 430$ $719, 838, 901$ $719, 837, 901$ $997, 100, 100$ $0.04, 258, 345$ 335 $452, 555, 674$ $931, 907, 986, 992$ $107, 306, 430$ $719, 838, 901$ $719, 837, 901$ $997, 100, 100$ $105, 343, 452$ $458, 353, 555, 674$ $931, 907, 968, 986$ $076, 310, 425$ $893, 925, 961$ $805, 912, 924$ $1.00, 100, 100$ $0.04, 254, 375$ $312, 641, 705, 815$ $637, 720, 831$ $907, 968, 986$ $076, 310, 425$ $157, 322, 487$ $159, 339, 521$ $190, 378, 505$ $046, 227, 375$ $31-7$			= <i>u</i>	256			= u	= 512	
alt-2 021, 176, 276 027, 182, 280 011, 049, 105 059, 286, 490 032, 214, 302 038, 252, 321 010, 047, 100 044, 243, 472 alt-3 122, 356, 611 418, 510, 693 933, 984, 991 152, 380, 515 699, 737, 892 691, 772, 882 991, 100, 100 064, 258, 395 alt-4 528, 561, 673 530, 555, 674 931, 986, 992 107, 306, 430 719, 838, 901 719, 837, 901 997, 100, 100 106, 233, 712, 745 alt-6 087, 189, 296 071, 196, 333 986 972, 310, 425 893, 925, 961 805, 912, 924 100, 100 100, 100 633, 712, 745 alt-6 087, 189, 296 071, 196, 303 092, 224, 337 041, 189, 328 101 719, 837, 901 997, 100, 100, 100 106, 233, 712, 745 alt-6 087, 189, 296 071, 196, 303 092, 224, 337 041, 189, 328 157, 322, 487 159, 339, 521 190, 378, 505 046, 227, 375 alt-7 126, 369, 333 095, 145 333, 995, 146 533, 712, 745 333, 333, 428, 654 337, 443, 667 737, 904, 945 093, 178, 269 alt-8 006, 096, 303 042, 091, 166, 024, 128, 554 337, 443, 667 737, 904, 945 093, 178, 269 alt-8 008, 018, 114, 271 001, 115, 275 011, 050, 098 054, 222, 527 1181, 245, 436 164, 249, 440 016, 047, 102 045, 231, 473	alt-2 $0.21, 176, 276$ $0.27, 182, 280$ $0.011, 0.49, 105$ $0.59, 286, 490$ $0.32, 214, 302$ $0.38, 252, 321$ $0.01, 0.47, 100$ $0.44, 243, 472$ alt-3 $142, 528, 561, 673$ $530, 555, 674$ $931, 986, 992$ $107, 306, 430$ $599, 737, 892$ $691, 772, 882$ $991, 1.00, 1.00$ $0.64, 258, 395$ alt-4 $528, 561, 673$ $530, 555, 674$ $931, 986, 992$ $107, 306, 430$ $719, 838, 901$ $719, 837, 901$ $997, 1.00, 1.00$ $106, 238, 452$ alt-5 $641, 705, 815$ $637, 720, 831$ $907, 968, 986$ $076, 310, 425$ $893, 925, 961$ $719, 837, 901$ $997, 1.00, 1.00$ $100, 1.00$ $15, 343, 452$ alt-6 $087, 189, 296$ $071, 196, 303$ $092, 224, 337$ $041, 189, 328$ $157, 322, 487$ $159, 339, 521$ $190, 378, 505$ $046, 227, 375$ alt-7 $126, 369, 382$ $121, 305, 303, 096, 145$ $333, 428, 654$ $337, 443, 667$ $737, 904, 945$ $093, 178, 269$ alt-8 $026, 088, 287$ $006, 096, 303$ $042, 091, 164$ $059, 160, 247$ $246, 422, 702$ $251, 451, 719$ $069, 198, 318$ $050, 085, 110$ alt-9 $018, 114, 271$ $010, 115, 275$ $011, 050, 098$ $054, 292, 527$ $181, 245, 436$ $164, 249, 440$ $016, 047, 102$ $045, 231, 473$ 17 and $\hat{M}_{e}^{(P)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_{e}^{e}$ is JWW's test based on simulated critical value.	alt-1 .5	57, .641, .744	.546, .616, .701	.975, .998, .999	.038, .266, .437	.739, .843, .894	.716, .827, .878	1.00, 1.00, 1.00	0.270, .545, .65
alt-3 422, 525, 611 418, 510, 693 933, 984, 991 152, 380, 515 699, 777, 892 691, 772, 882 991, 100, 100 064, 258, 395 435 alt-4 528, 561, 673 553, 674 931, 986, 992 107, 306, 430 719, 838, 901 719, 837, 901 997, 100, 100 115, 343, 452 alt-5 641, 705, 815 637, 720, 831 907, 968, 986 076, 310, 425 893, 925, 961 805, 912, 924 1.00, 100, 100 105, 343, 712, 745 alt-7 126, 382 011, 196, 333 092, 224, 337 041, 189, 328 157, 322, 487 159, 339, 521 190, 378, 505 046, 227, 375 alt-7 126, 389, 382 121, 305, 373 092, 104, 189, 328 157, 322, 487 159, 339, 521 190, 378, 505 046, 227, 375 alt-7 126, 389, 382 121, 305, 303 092, 224, 337 041, 189, 328 157, 322, 487 159, 339, 521 190, 378, 505 046, 227, 375 alt-8 0.056, 088, 287 006, 096, 303 042, 091, 164 059, 160, 247 246, 422, 702 251, 451, 779 069, 198, 318 050, 085, 110 alt-8 0.08, 114, 271 010, 115, 275 011, 050, 098 054, 292, 527 181, 245, 436 164, 249, 440 016, 047, 102 045, 231, 473	alt-3 at 22, 525, 611 at 8, 510, 693 933, 984, 991 152, 380, 515 (699, 777, 892 691, 772, 882 991, 100, 100 064, 258, 395 at 452 (611, 705, 815 637, 7720, 831 907, 968, 992 107, 306, 430 (719, 838, 901 719, 837, 901 997, 100, 100 105, 343, 452 at 564 (717, 705, 815 637, 7720, 831 907, 968, 986 (719, 303, 925, 961 805, 912, 924 100, 100, 100, 100 (715, 343, 742 715 (719) (710, 130, 373, 371, 371 371 371 371 305, 333, 328 (715, 333, 325, 961 805, 912, 924 100, 100, 100, 100, 100, 135, 343, 742 at 71 126, 369, 332 (712, 303, 096, 145 333, 033, 096, 145 (715, 333, 428, 654 (337, 443, 667 737, 904, 945 093, 178, 269 at -800, 010, 115, 275 (711, 006, 096, 303 045, 2224, 337, 422, 702 (251, 451, 719) (719, 378, 505 046, 227, 375 at -901, 114, 271 (710, 115, 275 011, 050, 098 (554, 292, 527 181, 245, 436 (164, 249, 440 016, 047, 102 (045, 231, 440) 106, 104, 102 (713, 441) 106, 104, 102 (713, 404) 106, 104, 104) 106, 104, 104, 104, 104 180 (713, 104) 106, 104, 104, 104, 106, 104, 106, 104, 102 (104, 102) 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 104) 106, 104, 102 (104, 102) 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 102 (104, 104) 106, 104, 103 (104, 104) 106, 104, 103 (104, 104) 106	alt-2 0.	021, .176, .276	.027, .182, .280	.011, .049, .105	.059, .286, .490	.032, .214, .302	.038, .252, .321	.010, .047, .100	.044, .243, .472
alt-4 528, 561, 673 530, 555, 674 931, 986, 992 107, 306, 430 719, 838, 901 719, 837, 901 997, 100, 1.00 115, 343, 452 alt-5 641, 705, 815 637, 720, 831 907, 968, 986 076, 310, 425 893, 925, 961 805, 912, 924 1.00, 1.00, 1.00 633, 712, 745 alt-6 0.87, 189, 296 071, 196, 303 092, 224, 337 041, 189, 328 1.57, 322, 487 1.59, 339, 521 190, 378, 505 046, 227, 375 alt-7 1.26, 369, 382 1.21, 305, 372 200, 408, 533 093, 106, 145 333, 428, 654 337, 443, 667 7.37, 904, 945 093, 178, 269 alt-8 0.06, 096, 303 042, 091, 164 059, 160, 247 246, 422, 702 251, 451, 719 0.69, 198, 318 050, 085, 110 alt-8 0.08, 114, 271 0.00, 115, 275 0.11, 050, 098 054, 292, 527 1.81, 245, 436 1.64, 249, 440 0.16, 047, 102 045, 231, 473	alt-4 $\begin{bmatrix} .528, .561, .673 \\ .641, .705, .815 \\ .637, .720, .831 \\ .907, .966, .992 \\ .907, .968, .986 \\ .076, .310, .425 \\ .308, .925, .961 \\ .803, .925, .961 \\ .805, 912, .924 \\ .100, 1.00, 1.00 \\ .100, 1.00, 1.00 \\ .033, .712, .745 \\ .372, .487 \\ .157, .322, .487 \\ .157, .322, .487 \\ .157, .322, .487 \\ .157, .322, .487 \\ .157, .322, .487 \\ .157, .322, .487 \\ .157, .322, .487 \\ .157, .322, .487 \\ .157, .322, .487 \\ .157, .322, .487 \\ .159, .339, .521 \\ .190, .378, .505 \\ .046, .227, .375 \\ .011, .050, .098 \\ .054, .292, .527 \\ .181, .245, .436 \\ .164, .249, .440 \\ .016, .047, .102 \\ .016, .047, .102 \\ .015, .045, .231, .473 \\ .050, .085, .110 \\ .016, .047, .102 \\ .045, .231, .471 \\ .010, .115, .275 \\ .011, .050, .098 \\ .054, .292, .527 \\ .181, .245, .436 \\ .164, .249, .440 \\ .016, .047, .102 \\ .045, .231, .473 \\ .045, .231, .473 \\ .050, .085, .110 \\ .045, .245, .231, .471 \\ .010, .115, .275 \\ .011, .050, .098 \\ .054, .292, .527 \\ .181, .245, .436 \\ .164, .249, .440 \\ .016, .047, .102 \\ .045, .245, .231, .473 \\ .101 \\ .101 \\ .114, .271 \\ .010, .115, .275 \\ .011, .050, .098 \\ .054, .292, .527 \\ .181, .245, .436 \\ .164, .249, .440 \\ .016, .047, .102 \\ .045, .245, .231, .473 \\ .101 \\ .101 \\ .101 \\ .114, .271 \\ .010, .115, .275 \\ .011, .050, .098 \\ .054, .292, .527 \\ .181, .245, .436 \\ .164, .249, .440 \\ .016, .047, .102 \\ .045, .245, .231, .473 \\ .101 \\ .101 \\ .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .016, .047, .102 \\ .045, .245, .231 \\ .410 \\ .0$	alt-3 .4	122, .525, .611	.418, .510, .693	.933, .984, .991	.152, .380, .515	.699, .737, .892	.691, .772, .882	.991, 1.00, 1.00	.064, .258, .395
alt-5 [641, 705, 815 637, 720, 831 907, 968, 986 076, 310, 425 893, 925, 961 805, 912, 924 1.00, 1.00, 1.00 633, 712, 745 alt-6 087, 189, 296 071, 196, 303 092, 224, 337 041, 189, 328 1.57, 322, 487 1.59, 339, 521 190, 378, 505 046, 227, 375 alt-7 1.26, 369, 382 1.21, 305, 372 2.00, 408, 533 0.96, 145 3.33, 428, 654 3.37, 443, 667 7.37, 904, 945 093, 178, 269 alt-8 0.06, 096, 303 042, 091, 164 0.59, 160, 247 2.45, 422, 702 2.51, 451, 719 0.69, 198, 318 0.50, 085, 110 alt-1, 271 0.00, 115, 275 0.11, 050, 098 0.54, 292, 527 1.81, 245, 436 1.64, 249, 440 0.16, 047, 102 0.45, 231, 473	alt-5 [641, 705, 815 637, 720, 831 907, 968, 986 076, .310, 425 [893, 925, 961 .805, 912, .924 1.00, 1.00, 1.00 633, .712, .745 alt-6 .087, 189, .296 071, .196, .303 .092, .224, .337 041, .189, .328 1.57, .322, .487 1.59, .339, .521 1.90, .378, .505 0.46, .227, .375 alt-7 1.26, .369, .382 1.21, .305, .372 .200, 408, .533 0.33, .096, .145 3.33, .428, .654 3.37, .443, .667 7.37, .904, .945 0.93, .178, .269 alt-8 0.056, .088, .287 0.06, .096, .303 0.926, .098 0.54, .292, .527 1.81, .245, .436 1.64, .249, .440 0.16, .047, .102 0.455, .231, .473 (7737) and $\hat{M}_{0}^{(P)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_{0}^{C}$ is JWW's test based on simulated critical value.	alt-4 .5	528, .561, .673	.530, .555, .674	.931, .986, .992	.107, .306, .430	.719, .838, .901	.719, .837, .901	.997, 1.00, 1.00	.115, .343, .452
alt-6 0.87, 189, 296 0.01, 196, 303 0.92, 224, 337 0.41, 189, 328 1.57, 322, 487 1.59, 339, 521 1.90, 378, 505 0.46, 227, 375 alt-7 1.26, 369, 382 1.21, 305, 372 200, 408, 533 0.33, 0.96, 145 3.33, 428, 654 3.37, 443, 667 7.37, 904, 945 0.93, 178, 269 alt-8 0.26, 0.88, 287 0.06, 0.96, 303 0.42, 0.91, 164 0.59, 160, 247 2.46, 422, 702 2.51, 451, 719 0.69, 198, 318 0.50, 0.85, 110 alt-9 0.08, 114, 271 0.00, 0.15, 275 0.11, 0.50, 0.98 0.54, 292, 527 1.81, 245, 436 1.64, 249, 440 0.16, 0.47, 102 0.45, 231, 473	alt-6 $\begin{bmatrix} .087, .189, .296 & .071, .196, .303 & .092, .224, .337 & .041, .189, .328 \\ .126, .369, .382 & .121, .305, .372 & .200, .408, .533 & .033, .096, .145 \\ .333, .428, .654 & .337, .443, .667 & .737, .904, .945 & .093, .178, .269 \\ .126, .369, .387 & .006, .096, .303 & .042, .091, .164 & .059, .166, .247 \\ .246, .422, .702 & .251, .451, .719 & .069, .198, .318 & .050, .085, .110 \\ .126, .088, .287 & .006, .096, .303 & .042, .091, .164 & .059, .160, .247 \\ .246, .422, .702 & .251, .451, .719 & .069, .198, .318 & .050, .085, .110 \\ .126, .016, .016, .047, .102 & .045, .231, .473 \\ .018, .114, .271 & .010, .115, .275 & .011, .050, .098 & .054, .292, .527 \\ .181, .245, .436 & .164, .249, .440 & .016, .047, .102 & .045, .231, .473 \\ .77 \text{ and } \widehat{\mathcal{M}_{*}^{(P)}} \text{ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. } \widehat{\mathcal{D}_{*}^{C}} \text{ is JWW's test based on simulated critical value} value. }$	alt-5 6	541, .705, .815	.637, .720, .831	.907, .968, .986	.076, .310, .425	.893, .925, .961	.805, 912, .924	1.00, 1.00, 1.00	.633, .712, .745
alt-7 1.26, 369, 382 121, 305, 372 200, 408, 533 033, 096, 145 333, 428, 654 337, 443, 667 737, 904, 945 093, 178, 269 alt-8 0.26, 088, 287 006, 096, 303 042, 091, 164 0.59, 160, 247 246, 422, 702 251, 451, 719 069, 198, 318 050, 085, 110 alt-9 0.08, 114, 271 0.00, 015, 275 0.011, 050, 098 054, 292, 527 181, 245, 436 164, 249, 440 0.016, 047, 102 045, 231, 473	alt-7 1.26, 369, 382 1.21, 305, 372 200, 408, 533 033, 096, 145 333, 428, 654 337, 443, 667 737, 904, 945 093, 178, 269 alt-8 0.26, 0.88, 287 006, 0.96, 303 0.408, 164 0.59, 160, 247 246, 422, 702 251, 451, 719 0.69, 198, 318 050, 0.85, 110 alt-9 0.08, 114, 271 0.10, 115, 275 0.11, 050, 0.98 0.54, 292, 527 3.181, 245, 436 1.64, 249, 440 0.16, 047, 102 0.45, 231, 473 $(r - 1)^{12}$ and $\widehat{\mathcal{M}}_{r}^{(P)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\widehat{\mathcal{D}}_{r}^{c}$ is JWW's test based on simulated critical value.	alt-6 .0	187, .189, .296	.071, .196, .303	.092, .224, .337	.041, .189, .328	.157, .322, .487	.159, .339, .521	.190, .378, .505	.046, .227, .375
alt-8 0.026, 088, 287 0.006, 096, 303 0.42, 091, 164 0.59, 160, 247 246, 422, 702 251, 451, 719 0.69, 198, 318 0.50, 085, 110 alt-9 0.08, 114, 271 0.010, 115, 275 0.11, 050, 098 0.54, 292, 527 381, 245, 436 164, 249, 440 0.16, 047, 102 0.45, 231, 473	alt-8 0.026, 088, 287 006, 096, 303 0.42, 091, 164 0.59, 160, 247 246, 422, 702 251, 451, 719 0.69, 198, .318 0.50, 085, 110 alt-9 0.08, 114, .271 0.00, .015, .275 0.11, .050, 098 0.54, .292, .527 341, .245, .436 1.64, .249, .440 0.016, .047, 102 0.45, .231, .473 $(r - n)^{10}$ and $\hat{\mathcal{M}}_{r}^{(P)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_{r}^{cv}$ is JWW's test based on simulated critical value	alt-7 .1	26, .369, .382	.121, .305, .372	.200, .408, .533	.033, .096, .145	.333, .428, .654	.337, .443, .667	.737, .904, .945	.093, .178, .269
alt-9 018, 114, 271 010, 115, 275 011, 050, 098 054, 292, 527 181, 245, 436 164, 249, 440 016, 047, 102 045, 231, 473	alt-9 0.018, .114, .271 0.00, .115, .275 0.011, .050, .098 0.54, .292, .527 .181, .245, .436 1.64, .249, .440 0.16, .047, .102 0.45, .231, .473 $(T_{r} \text{ and } \hat{\mathcal{M}}_{r}^{(P)})$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_{r}^{CV}$ is JWW's test based on simulated critical value	alt-8 .0)26, .088, .287	.006, .096, .303	.042, .091, .164	.059, .160, .247	.246, .422, .702	.251, .451, .719	.069, .198, .318	.050, .085, .110
	$(\tau \text{ and } \hat{\mathcal{M}}_{\tau}^{(p)})$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_{\tau}^{c,v}$ is JWW's test based on simulated critical value	alt-9 .0	118, .114, .271	.010, .115, .275	.011, .050, .098	.054, .292, .527	.181, .245, .436	.164, .249, .440	.016, .047, .102	.045, .231, .473

Supplemental Material for "A bootstrapped test of covariance stationarity"

•	alt-9	alt-8	alt-7	alt-6	alt-5	alt-4	alt-3	alt-2	alt-1		all-9	olt 0	alt-8	alt-7	alt-6	alt-5	alt-4	alt-3	alt-1 alt-2			
	.702, .973, .998	.083, .252, .358	.121, .247, .331	.131, .250, .428	.886, .983, .999	.893, .977, .996	.883, .891, .939	.483, .946, .994	.815, .932, .987		.011, .029, .102	011 020, 100 100	020 115 164	.021, .073, .137	.041, .123 .166	.121, .207, .335	.073, .292, .399	.200, .313, .417	.121, .234, .326 .020, .059, .190	$\hat{\mathcal{M}}_T$		
	.673, .951, .989	.042, .174, .410	.119, .224, .314	.141, .276, .478	.867, .989, .999	.809, .979, .995	.878, .879, .937	.518, .942, .989	.889, .908, .987	<i>n</i> =	.012, .027, .000	010 007 000	021 082 133	.017, .063, .137	.031, .113, .179	.101, .145, .383	.065, .284, .397	.200, .303, .422	.142, .276, .431 .022, .067, .202	$\hat{\mathcal{M}}_T^{(p)}$	<i>n</i> :	
	.004, .034, .081	.049, .106, .158	.151, .349, .503	.067, .189, .293	.908, .977, .991	.929, .981, .992	.931, .982, .990	.012, .033, .074	.972, .998, 1.00	: 256	.014, .001, .099	014 051 000	080 129 178	.050, .096, .145	.064, .121, .190	.080, .129, .178	.160, .402, .548	.150, .406, .553	.146, .395, .552 .025, .058, .089	$\hat{\mathcal{D}}_T^{cv}$	=64	
	1.00, .026, .145	.018, .035, .091	.007, .022, .058	.002, .012, .091	.003, .058, .223	.005, .096, .273	.003, .132, .325	.000, .016, .146	.001, .059, .220		.010, .023, .287	010 005 707	033 085 188	.022, .072, .180	.017, .062, .156	.033, .085, .188	.011, .050, .190	.006, .060, .195	.003, .031, .129 .010, .066, .233	\hat{D}_T^{dw}		$\epsilon_t \sim N(0)$
	.994, 1.00, 1.00	.537, .851, .975	.514, .666, .709	.403, .762, .877	1.00, 1.00, 1.00	1.00, 1.00, 1.00	1.00 1.00, 1.00	1.00, 1.00, 1.00	1.00, 1.00, 1.00		.040, .000, .120	042, 100, 102	052 106 365	.058, .115, .211	.072, .134, .278	.403, .688, .869	.661, .678, .828	.332, .579, .687	.583, .685, .702 .033, .197, .410	\hat{M}_T		, 1)
	.994, 1.00, 1.00	.516, .870, .983	.557, .636, .768	.449, .805, .907	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.965, 1.00, 1.00	.999, 1.00, 1.00	.991, 1.00, 1.00	n	.020, .000, .100	005 060 160	032 122 414	.045, .103, .219	.076, .155, .308	.455, .731, .895	.668, .681, .837	.333, .518, .661	.563, .628, .667 .034, .213, .427	$\hat{\mathcal{M}}_T^{(p)}$	<i>n</i> :	
	.009, .048, .101	.060, .173, .285	.675, .850, .920	.131, .306, .449	1.00, 1.00, 1.00	.997, 1.00, 1.00	.984, .999, 1.00	.010, .037, .085	1.00, 1.00, 1.00	= 512	.003, .024, .070	002 021 021 070	075 115 157	.066, .135, .220	.053, .137, .252	.143, .323, .471	.739, .906, .942	.382, .674, .790	.803, .934, .967 .009, .043, .086	$\hat{\mathcal{D}}_T^{cv}$	= 128	
	1.00, .029, .14;	.001, .020, .06	.035, .087, .171	.000, .024, .13	.363, .720, .75.	.013, .209, .38-	.003, .086, .24	.000, .020, .12	.079, .401, .58;		.000, .000, .13	000 026 15	030 067 119	$.018, .052, .10^{2}$.010, .024, .089	.012, .040, .100	.001, .097, .292	.000, .022, .12	.001, .065, .21i .002, .025, .17;	$\hat{D}_T^{d_W}$		

			Table A.6.: b. Re Case	ection Frequenc $2: \mathcal{H}_T = 2T \cdot 4^9 a$ $\epsilon_t \sim t_5$	sies under H_1 : Wind $\mathcal{K}_T = .5T^{.49}$	alsh Basis		
		= u	= 64			= <i>u</i>	128	
	\hat{M}_T	$\hat{\mathcal{M}}_{T}^{(p)}$	$\hat{\mathcal{D}}_{T}^{cv}$	$\hat{\mathcal{D}}_{T}^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_{T}^{(p)}$	$\hat{\mathcal{D}}_{T}^{cv}$	$\hat{\mathcal{D}}_{T}^{dw}$
alt-1	.102, .214, .333	.143, .314, .433	.134, .357, .512	.000, .021, .091	.407, .432, .538	.428, .490, .597	.777, .919, .948	.001, .029, .131
alt-2	.013, .044, .146	.013, .049, .161	.010, .033, .070	.007, .071, .175	.035, .112, .264	.037, .131, .265	.004, .036, .065	.000, .025, .122
alt-3	.213, .317, .446	.200, .315, .453	.126, .379, .505	.001, .035, .114	.316, .468, .567	.317, .453, .542	.375, .628, .738	.000, .019, .077
alt-4	.082, .264, .317	.052, .157, .298	.142, .385, .524	.003, .042, .124	.464, .493, .696	.468, .500, .698	.717, .887, .933	.002, .049, .164
alt-5	.045, .192, .264	.023, .117, .295	.056, .099, .137	.021, .068, .143	.282, .430, .669	.214, .465, .673	.114, .281, .407	.005, .015, .043
alt-6	.031, .056, .138	.028, .041, .146	.050, .095, .139	.008, .038, .111	.055, .094, .187	.047, .084, .206	.036, .108, .194	.004, .016, .047
alt-7	.012, .055, .086	.010, .042, .100	.022, .063, .094	.005, .041, .108	.092, .121, .213	.082, .115, .187	.060, .140, .216	.008, .022, .057
alt-8	.021, .081, .118	.012, .043, .115	.056, .099, .137	.023, .066, .150	.036, .074, .266	.007, .087, .291	.040, .071, .114	.011, .025, .057
alt-9	.020, .038, .072	.014, .024, .081	.011, .036, .069	.006, .087, .204	.023, .056, .135	.007, .070, .173	.004, .024, .074	.000, .023, .097
		= <i>u</i>	256			= <i>u</i>	512	
alt-1	.727, .792, .894	.773, .795, .874	.976, .995, .998	.000, .022, .111	.963, .974, .986	.949, .977, .983	1.00, 1.00, 1.00	0.035, .407, .722
alt-2	.441, .678, .833	.455, .658, .826	.004, .027, .059	.000, .019, .061	.841, .958, .975	.821, .954, .971	.005, .035, .065	.001, .010, .026
alt-3	.783, .803, .824	.781, .784, .842	.923, .976, .987	.002, .036, .190	.929, .969, .998	.725, .909, .958	.979, .998, .999	.000, .026, .153
alt-4	.840, .896, .906	.753, .798, .906	.909, .972, .988	.000, .034, .150	.914, .979, .989	.915, .979, .989	.997, 1.00, 1.00	.001, .108, .326
alt-5	.753, .816, .912	.707, .833, .917	.867, .956, .976	.000, .023, .104	.927, .972, .985	.936, .975, .987	1.00, 1.00, 1.00	.102, .527, .740
alt-6	.107, .154, .316	.103, .124, .306	.072, .186, .278	.000, .006, .022	.214, .387, .561	.230, .409, .579	.146, .307, .430	.000, .003, .026
alt-7	.132, .261, .294	.106, .284, .247	.165, .361, .480	.000, .005, .025	.472, .521, .659	.426, .510, .623	.647, .850, .914	.012, .032, .120
alt-8	.082, .275, .367	.080, .187, .308	.020, .062, .098	.003, .008, .023	.472, .597, .869	.486, .637, .887	.019, .081, .159	.001, .003, .004
alt-9	.405, .795, .902	.354, .736, .866	.007, .023, .060	.000, .008, .057	.828, .954, .973	.771, .950, .967	.003, .039, .084	.000, .007, .027
$\hat{\mathcal{M}}_{T}$ and	$\hat{\mathcal{M}}_{(P)}^{(P)}$ are the pro-	mosed max-tests wi	ith and without a pe	analty, based on a l	hootstrapped p-valu	re. Â ^{cv} is JWW's	test based on simu	lated critical values.
and $\hat{\mathcal{D}}_{\pi}^{d}$	W uses bootstrappe	d p-values.						
-	t t	4						

Supplemental Material for "A bootstrapped test of covariance stationarity"

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T$	alt-3 alt-4 alt-5 alt-6 alt-7 alt-8 alt-8	alt-1 alt-2	alt-1 alt-2 alt-2 alt-3 alt-4 alt-5 alt-6 alt-7 alt-7
$\hat{M}^{(p)}$ are the pro-	.5421, 4958, 0000 .518, 5880, 675 .424, 596, 721 .107, 174, 296 .102, 247, 357 .103, 275, 350 .197, .577, .754	.447, .589, .677 .282, .370, .576	$\begin{split} \hat{\mathcal{M}}_T \\ \hat{\mathcal{M}}_T \\ .065, .211, .360 \\ .021, .054, .162 \\ .085, .217, .371 \\ .120, .339, .389 \\ .056, .156, .223 \\ .041, .116, .195 \\ .052, .123, .199 \\ .052, .123, .199 \\ .046, .117, .159 \\ .020, .038, .098 \end{split}$
	.418, .489, .088 .527, .583, .675 .449, .518, .709 .108, .179, .295 .102, .227, .377 .104, .240, .321 .168, .499, .711	<i>n</i> = .442, .567, .647 .287, .353, .551	$n: \\ \mathcal{M}_T^{(p)} \\ \mathcal{M}_T^{(p)} \\ .085, .277, .367 \\ .021, .062, .173 \\ .078, .216, .379 \\ .087, .242, .291 \\ .043, .127, .216 \\ .041, .101, .181 \\ .030, .105, .135 \\ .035, .103, .147 \\ .016, .029, .096 \\ .041 \\ .016, .029, .096 \\ .016 \\ .029, .096 \\ .016 \\ .016 \\ .029, .096 \\ .016 \\ .016 \\ .029 \\ .016 \\ .029 \\ .016 \\ .029 \\ .016 \\ .029 \\ .016 \\ .029 \\ .016 \\ .029 \\ .016 \\ .$
	.927, .982, .992 .916, .979, .989 .892, .962, .983 .085, .214, .325 .171, .378, .507 .049, .100, .166 .012, .046, .105	- 256 .974, .998, .999 .009, .042, .102	Cas = 64 \hat{D}_{T}^{cv} .139, .378, .546 .022, .051, .106 .156, .418, .588 .152, .408, .554 .073, .120, .163 .056, .117, .159 .073, .120, .163 .012, .049, .083
	.003, .087, .241 .003, .087, .241 .002, .056, .229 .000, .017, .085 .006, .027, .055 .010, .029, .082 .001, .025, .160	.005, .051, .228	$\epsilon_{f} \sim GA$ $\epsilon_{f} \sim GA$ δ_{T}^{dw} $002, 030, 128$ $007, 046, 171$ $027, 071, 150$ $019, 040, 141$ $018, 061, 163$ $028, 076, 158$
	.746, .794, .888 .829, .916, .963 .135, .384, .428 .331, .468, .660 .335, .415, .638 .503, .775, .884	.557, .665, .807	MT = .5T.49 RCH MT .308, .422, .591 .044, .099, .266 .218, .311, .467 .421, .496, .590 .157, .368, .522 .082, .172, .278 .092, .117, .208 .043, .095, .248
	.055, .700, .700, .800 .755, .808, .889 .587, .828, .909 .143, .302, .434 .321, .428, .622 .252, .392, .649 .424, .740, .859	<i>n</i> : .539, .657, .803 .590, .667, .798	n: \$\$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$
	.995, 1.00, 1.00 .995, 1.00, 1.00 1.00, 1.00, 1.00 .169, .350, .491 .686, .880, .928 .062, .181, .284 .009, .045, .090	= 512 1.00, 1.00, 1.00 .009, .051, .099	= 128 $\hat{\mathcal{D}}_{T}^{cv}$.8.20, .939, .970 .005, .033, .088 .371, .657, .778 .758, .909, .953 .148, .324, .455 .046, .149, .245 .062, .129, .210 .055, .090, .148 .009, .038, .081
	.004, .080, .24 .015, .181, .339 .347, .674, .719 1.00, .030, .13 .031, .093, .178 .006, .028, .06 .000, .028, .15	.076, .375, .54	$\frac{\hat{D}_T^{d_W}}{\hat{D}_T}$.000, .057, .18: .001, .032, .16: .001, .017, .114 .001, .086, .28 .015, .028, .07 .004, .017, .08 .018, .035, .09 .020, .051, .115 .000, .031, .16

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