Supplemental material for "*A bootstrapped test of covariance stationarity based on orthonormal transformations*"[*]

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A. Introduction

We list the assumptions for reference in Appendix B. Appendix C presents a method for automatic selection of $\mathcal{H}_T$ and $\mathcal{K}_T$. Appendix D contains omitted proofs. We provide an empirical study in Appendix E, and Appendix F contains all simulation results.

We use the following notation. $[z]$ rounds z to the nearest integer.. L_2 is the space of square integrable random variables; and $L_2[a,b)$ is the class of square integrable functions on $[a,b)$. $||\cdot||_p$ and $||\cdot||$ are the L_p and l_2 norms respectively, $p \geq 1$. Let $\mathbb{Z} \equiv \{\ldots -2, -1, 0, 1, 2, \ldots\}$, and $\mathbb{N} \equiv \{0, 1, 1, 2, \ldots\}$. $K > 0$ is a finite constant whose value may be different in different places. $awp1$ denotes "*asymptotically with probability approaching one*". Write $\max_{\mathcal{H}_T} = \max_{0 \leq h \leq \mathcal{H}_T}$. $\max_{\mathcal{K}_T} = \max_{1 \leq k \leq \mathcal{K}_T}$ and $\max_{\mathcal{H}_T, \mathcal{K}_T} = \max_{0 \leq h \leq \mathcal{H}_T, 1 \leq k \leq \mathcal{K}_T}$. Similarly, $\max_{\mathcal{H}_T} a(h, \tilde{h}) = \max_{0 \leq h, \tilde{h} \leq \mathcal{H}_T} a(h, \tilde{h})$, etc.

B. Assumptions

Write

$$z_t(h,k) \equiv \{X_t X_{t+h} b_k(t) - E\left[X_t X_{t+h}\right] b_k(t)\}$$

$$\mathcal{Z}_T(h,k) \equiv \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} z_t(h,k).$$

Define σ-fields

$$\mathcal{F}_{T,t}^{\infty} \equiv \sigma\left(\{X_t X_{t+h} : 0 \leq h \leq \mathcal{H}_T\}_{\tau \geq t}\right) \text{ and } \mathcal{F}_{T,-\infty}^{t} \equiv \sigma\left(\{X_t X_{t+h} : 0 \leq h \leq \mathcal{H}_T\}_{\tau \leq t}\right),$$

and α-mixing coefficients $\alpha_l \equiv \limsup_{T\to\infty} \sup_{t\in\mathbb{Z}} \sup_{\mathcal{A}\subset\mathcal{F}_{T,-\infty}^{t}, \mathcal{B}\subset\mathcal{F}_{T,t+l}^{\infty}} |\mathcal{P}(\mathcal{A}\cap\mathcal{B}) - \mathcal{P}(\mathcal{A})\mathcal{P}(\mathcal{B})|$, for $l > 0$.

Assumption 1.

a. (geometric mixing)*:* $\{X_t\}$ *is* α*-mixing with coefficients* $\alpha_l = O(\exp\{-l^{\phi}\})$ *for some* $\phi > 0$.

b. (subexponential tails)*:* $\max_{1\leq t\leq T} P(|X_t| > c) \leq \varpi \exp\{-c^{\vartheta_1} \mathcal{E}_T^{-\vartheta_2}\}$ *for some* $\varpi \geq 1$, $\vartheta_1 \geq 2\vartheta_2$ *and* $\vartheta_2 \geq 1$, *and some sequence of constants* $\{\mathcal{E}_T\}$, $\liminf_{T\to\infty} \mathcal{E}_T \geq 1$.

c. (nondegeneracy)*:* $\liminf_{T\to\infty} E[\mathcal{Z}_T^2(h,k)] > 0 \; \forall (h,k)$.

d. (orthonormal basis)*:* $\{\mathcal{B}_k(x) : 0 \leq k \leq \mathcal{K}\}$ *forms a complete orthonormal basis on* $\mathcal{L}[0,1)$*;* $\mathcal{B}_k(x) \in \{-1, 1\}$ *on* $[0,1)$*; and* $|\sum_{t=1}^{T} B_k(t)| = O(\eta(k))$ *for some positive strictly monotonic function* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$, $\eta(k) \nearrow \infty$ *as* $k \to \infty$

Set a block size b_T such that $1 \leq b_T < T$, $b_T/T^{\iota} \to \infty$ and $b_T/T^{1-\iota} \to 0$ for some tiny $\iota > 0$ that may be different in different places.

Assumption 2.

a. (i) $\liminf_{T\to\infty} s_T^2(h,k;\tilde{h},\tilde{k}) > 0 \; \forall (h,\tilde{h},k,\tilde{k})$*; and* (ii) $\max_{\mathcal{H}_T,\mathcal{K}_T} |s_T^2(h,k;\tilde{h},\tilde{k}) - s^2(h,k;\tilde{h},\tilde{k})| = O(T^{-\iota})$ *for some infinitessimal* $\iota > 0$.

b. $b_T/T^{\iota} \to \infty$ *and* $b_T = o(T^{1/2-\iota})$ *for some infinitessimal* $\iota > 0$.

C. Automatic $\{\mathcal{H}_T, \mathcal{K}_T\}$ selection

We propose picking $(\mathcal{H}_T, \mathcal{K}_T)$ in a data dependent manner based on the lag selection method in Hill and Motegi (2020, Section 3), an extension of Escanciano and Lobato's (2009) method for lag selection. Cf. Inglot and Ledwina (2006). We consider selecting $\mathcal{H}_T$ for a given $\mathcal{K}_T$, and selecting $\mathcal{K}_T$ for a given $\mathcal{H}_T$. Finally, we discuss an empirical approach for selecting both iteratively. Assume the orthonormal basis is comprised of Walsh functions $W_k(x)$ to focus ideas, hence $\mathcal{K}_T = o(\sqrt{T})$ is required, cf. Remark 8 in the main paper.

C.1. $\mathcal{H}_T^*(\mathcal{K}_T)$

The optimal $\mathcal{H}_T^*(\mathcal{K}_T)$ for a given $\mathcal{K}_T$ is chosen from a set $\{0, ..., \bar{\mathcal{H}}_T\}$ for some pre-chosen upper-bound $\bar{\mathcal{H}}_T$, where $\bar{\mathcal{H}}_T \to \infty$ with $\bar{\mathcal{H}}_T = O(\sqrt{T})$. Similarly $\mathcal{K}_T$ is pre-chosen from $\{1, ..., \bar{\mathcal{K}}_T\}$ where $\bar{\mathcal{K}}_T \to \infty$ and $\bar{\mathcal{K}}_T = o(\sqrt{T})$. We only consider sequences $\{\mathcal{H}_T, \mathcal{K}_T\}$ that satisfy $\mathcal{H}_T/\bar{\mathcal{H}}_T \to [0, K]$ and $\mathcal{K}_T/\bar{\mathcal{K}}_T \to [0, K]$ for finite $K > 0$. Put $K = 1$ for ease of notation. Under H_0, Escanciano and Lobato's (2009) method leads to $P(\mathcal{H}_T^*(\mathcal{K}_T) = 0) \to 1$ because higher lags do not provide useful information and incur a high penalty (see below). Thus, we need to allow for sequences $\{\mathcal{H}_T, \mathcal{K}_T\}$ that converge or diverge, e.g. $\mathcal{H}_T \to [0, ..., \infty]$.

Similar to Hill and Motegi (2020), cf. Escanciano and Lobato (2009), define a *penalized max-correlation difference*:

$$\mathcal{M}_T^{\mathcal{P}}(\mathcal{H}, \mathcal{K}) \equiv \mathcal{M}_T(\mathcal{H}, \mathcal{K}) - \mathcal{P}_T(\mathcal{H}, \mathcal{K}) \text{ where } \mathcal{M}_T(\mathcal{H}, \mathcal{K}) \equiv \max_{0 \le h \le \mathcal{H}, 1 \le k \le \mathcal{K}} \left| \sqrt{T} \left(\hat{\rho}_h^{(k)} - \hat{\rho}_h \right) \right|,$$

with penalty function $\mathcal{P}_T(\cdot)$:

$$\mathcal{P}_T(\mathcal{H}, \mathcal{K}) = \begin{cases} \sqrt{(\mathcal{H}+1)\mathcal{K} \ln T} & \text{if } \mathcal{M}_T(\mathcal{H}, \mathcal{K}) \le \sqrt{q \ln T} \\ \sqrt{2(\mathcal{H}+1)\mathcal{K}} & \text{if } \mathcal{M}_T(\mathcal{H}, \mathcal{K}) > \sqrt{q \ln T} \end{cases} \tag{C.1}$$

where q is a fixed positive constant. We need $\mathcal{H} + 1$ since $\mathcal{H} = 0$ is possible, covering a test exclusively for variance constancy. Notice $(\mathcal{H} + 1)\mathcal{K}$ is the total number of objects being searched over. A small value of q leads to the AIC penalty $\sqrt{2(\mathcal{H}+1)\mathcal{K}}$ being chosen with high probability, while a large q promotes selection of the BIC penalty $\sqrt{(\mathcal{H}+1)\mathcal{K} \ln T}$. A low q is therefore akin to the de facto AIC penalty used in Jin, Wang and Wang (2015) applied to h. In their setting, however, they are not choosing $\{\mathcal{H}, \mathcal{K}\}$; rather, they penalize the use of higher lags and systematic samples in a maximized high dimensional Wald statistic. Escanciano and Lobato (2009) find $q = 2.4$ is suitable for their penalized Q-statistic based on empirical size and power, and based on results in Inglot and Ledwina (2006). Hill and Motegi (2020) find that $q = 3$ works well for their max-correlation difference statistic. In experiments not reported here we find $q \in [1/4, 3/4]$ lead to competitive empirical size for the max-correlation difference.

Following Escanciano and Lobato (2009) and Hill and Motegi (2020), the chosen optimal lag is for each T:

$$\mathcal{H}_T^*(\mathcal{K}_T) = \min \left\{ \mathcal{H} : 0 \le \mathcal{H} \le \bar{\mathcal{H}}_T : \mathcal{M}_T^{\mathcal{P}}(\mathcal{H}, \mathcal{K}_T) \ge \mathcal{M}_T^{\mathcal{P}}(h, \mathcal{K}_T) \text{ for } h = 0, ..., \bar{\mathcal{H}}_T \right\}. \tag{C.2}$$

In order to characterize $\mathcal{H}_T^*(\mathcal{K}_T)$ we need

$$\Delta r_T(h, k) \equiv \frac{1}{\frac{1}{T}\sum_{t=1}^T E\left[X_t^2\right]} \frac{1}{T} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t).$$

Under Assumption 1 it is straightforward to prove (see the proof of Theorem C.1):

$$\max_{h,k}\left|\hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h,k)\right| \xrightarrow{p} 0.$$

Under H_0 and Lemma 3.a in Jin, Wang and Wang (2015),

$$|\Delta r_T(h,k)| = \left|\frac{E\left[X_t X_{t+h}\right]}{E\left[X_t^2\right]}\frac{1}{T}\sum_{t=1}^{T-h} B_k(t)\right| \leq |\rho_h|\left|\frac{k+1}{T}\right| = O(1/T).$$

Consider the global alternative $E[X_t X_{t+h}] = \gamma_h + c_h(t/T)$, cf, (16) in the main paper, where $c_h : [0,1] \to \mathbb{R}$ are integrable functions on $[0,1]$ uniformly over h. Then $\lim_{T\to\infty} T^{-1}\sum_{t=1}^T E[X_t^2] = \gamma_0 + \int_0^1 c_0(u)\, du > 0$, and

$$\lim_{R\to\infty} \Delta r_T(h,k) = \Delta r(h,k) \equiv \frac{\int_0^1 c_h(u)\mathcal{B}_k(u)du}{\gamma_0 + \int_0^1 c_0(u)\, du}.$$

Now define the smallest lag h at which the largest (in magnitude) asymptotic correlation difference occurs over systematic samples $\{1,...,\mathring{\mathcal{K}}\}$:

$$h^*(\mathring{\mathcal{K}}) \equiv \min\left\{h : h = \text{argmax}_{h\in\mathbb{N}}\left\{\max_{k\in\{1,...,\mathring{\mathcal{K}}\}} |\Delta r(h,k)|\right\}\right\} \text{ where } \mathring{\mathcal{K}} \equiv \lim_{T\to\infty}\mathcal{K}_T \in [1,\infty],$$

Theorem C.1. *Let Assumptions 1 and 2 hold. Let $\{\mathcal{K}_T\}$ satisfy $\mathcal{K}_T \in [1,...,\bar{\mathcal{K}}_T]$ where $\mathcal{K}_T \to [1,\infty]$ and $\bar{\mathcal{K}}_T = o(\sqrt{T})$; let $\bar{\mathcal{H}}_T = O(\sqrt{T})$; and let $\bar{\mathcal{H}}_T\bar{\mathcal{K}}_T = o(T/\ln(T))$.*

a. Under H_0, $P(\mathcal{H}_T^*(\mathcal{K}_T) = 0) \to 1$ for any $\{\mathcal{K}_T\}$. Further $\max_{1\leq k\leq\mathcal{K}}\mathcal{H}_T^*(k) \xrightarrow{p} 0$ for any fixed finite $1 \leq \mathcal{K} \leq \mathring{\mathcal{K}} \equiv \lim_{T\to\infty}\mathcal{K}_T$.

b. Under H_1, $\mathcal{H}_T^*(\mathcal{K}_T) \xrightarrow{p} h^*(\mathring{\mathcal{K}})$. Further $\max_{1\leq k\leq\mathcal{K}}|\mathcal{H}_T^*(k) - h^*(k)| \to 1$.

Remark 1. Under either hypothesis $\mathcal{H}_T^*(\mathcal{K}_T)$ converges to a plausibly *most informative and efficient* value. We say both *informative* and *efficient* because we use maximal information under H_0 asymptotically with probability approaching one (no data points are trimmed with $\mathcal{H}_T^*(\mathcal{K}_T) = 0$).

Under H_1 we use the least lag (hence the least amount of trimmed sample points) at which $\mathcal{K}_T$ optimizes the correlation difference. In our simulation study, for example, $h^*(\mathring{\mathcal{K}}) = 0$ or 1 for most models under H_1 because the non-covariance stationary processes used have zero autocovariances at lag $h \geq 2$. In alternative models 2 and 9 we have $h^*(\mathring{\mathcal{K}}) = 6$ and 25 because only at those lags are there non-zero autocovariances.

Remark 2. We require a slightly more restrictive bound $\bar{\mathcal{H}}_T\bar{\mathcal{K}}_T = o(T/\ln(T))$ compared to Theorem 4.1 where implicitly $\bar{\mathcal{H}}_T\bar{\mathcal{K}}_T = o(T)$. This ultimately arises from the logarithmic component in the penalty selection threshold $\sqrt{q\ln T}$.

C.2. $\mathcal{K}_T^*(\mathcal{H}_T)$

An identical procedure applies for selecting $\mathcal{K}_T$ for a given maximum lag $\mathcal{H}_T$. The chosen optimal $\mathcal{K}_T^*(\mathcal{H}_T)$ is:

$$\mathcal{K}_T^*(\mathcal{H}_T) = \min\left\{\mathcal{K} : 1 \le \mathcal{K} \le \bar{\mathcal{K}}_T : \mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T, \mathcal{K}) \ge \mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T, k) \text{ for } k = 1, ..., \bar{\mathcal{K}}_T\right\}. \tag{C.3}$$

Define the smallest systematic sample counter k at which the largest (in magnitude) asymptotic correlation difference occurs:

$$k^*(\mathring{\mathcal{H}}) \equiv \min\left\{k : k = \text{argmax}_{k\in\mathbb{N}}\left\{\max_{h\in\{0,...,\mathring{\mathcal{H}}\}} |\Delta r(h,k)|\right\}\right\} \text{ where } \mathring{\mathcal{H}} \equiv \lim_{T\to\infty} \mathcal{H}_T.$$

The proof of the next theorem is identical to the proof of Theorem C.1 and therefore omitted.

Theorem C.2. *Let Assumption 1 and 2 hold. Let $\{\mathcal{H}_T\}$ be an arbitrary sequence of maximum lags, $\mathcal{H}_T \in [1, ..., \bar{\mathcal{H}}_T]$, $\mathcal{H}_T \to [1,\infty]$ and $\mathcal{H}_T = O(\sqrt{T})$; let $\bar{\mathcal{K}}_T = o(\sqrt{T})$; and let $\bar{\mathcal{H}}_T\bar{\mathcal{K}}_T = o(T/\ln(T))$.*

a. Under H_0, $P(\mathcal{K}_T^*(\mathcal{H}_T) = 1) \to 1$ for any $\{\mathcal{H}_T\}$. Further $\sup_{0\le h\le\mathcal{H}} \mathcal{K}_T^*(h) \xrightarrow{p} 1$ for any fixed finite $0 \le \mathcal{H} \le \mathring{\mathcal{H}} \equiv \lim_{T\to\infty} \mathcal{H}_T$.

b. Under H_1, $\mathcal{K}_T^*(\mathcal{H}_T) \xrightarrow{p} k^*(\mathring{\mathcal{H}})$. Further $\sup_{0\le h\le\mathcal{H}} |\mathcal{K}_T^*(h) - k^*(h)| \to 1$ for any fixed finite $1 \le \mathcal{H} \le \mathring{\mathcal{H}}$.

Remark 3. Under H_0 the variance and autocovariances are constant over time. Since no systematic sample provides information for detecting a break in (co)variance, the least of the set $\{1, ..., \bar{\mathcal{K}}_T\}$ is the optimal choice.

C.3. Iterative $\{\mathcal{H}_T^*, \mathcal{K}_T^*\}$ Selection and Theorem C.1

We now discuss identifying $(\mathcal{H}_T^*, \mathcal{K}_T^*)$ iteratively for a given T, and present the main result. Write $\tilde{\mathcal{H}}_T^*(\mathcal{H}) \equiv \mathcal{H}_T^*(\mathcal{K}_T^*(\mathcal{H}))$ and $\tilde{\mathcal{K}}_T^*(\mathcal{K}) \equiv \mathcal{K}_T^*(\mathcal{H}_T^*(\mathcal{K}))$, hence $\tilde{\mathcal{H}}_T^* : \mathbb{N} \to \mathbb{N}$ and $\tilde{\mathcal{K}}_T^* : \mathbb{N} \to \mathbb{N}$. Identification of unique $\{\mathcal{H}_T^*, \mathcal{K}_T^*\}$ requires $\tilde{\mathcal{H}}_T^*$ and $\tilde{\mathcal{K}}_T^*$ to be contraction mappings (see below). If we begin with an arbitrary start h_0, set $h_1 = \tilde{\mathcal{H}}_T^*(h_0)$ and iterate $h_{m+1} = \tilde{\mathcal{H}}_T^*(h_m)$, then by the Banach fixed point theorem $h_m \xrightarrow{p} \mathcal{H}_T^*$ as $m \to \infty$.

The algorithm requires going back and forth between $\mathcal{H}_T^*(\mathcal{K}_T^*(\mathcal{H}))$ and $\mathcal{K}_T^*(\mathcal{H}_T^*(\mathcal{K}))$ due to the cross-embedded arguments. Indeed, the iteration on $\tilde{\mathcal{H}}_T^*(h_m)$ implicitly simultaneously iterates on $\tilde{\mathcal{K}}_T^*(\cdot)$. The steps are as follows:

(i) Pick h_0 and compute $k_0 \equiv \mathcal{K}_T^*(h_0)$ in (C.3);
(ii) Compute $h_1 \equiv \mathcal{H}_T^*(k_0)$ and $k_1 \equiv \mathcal{K}_T^*(h_1)$ using (C.2) and (C.3);
(iii) Iterate

$$h_{m+1} \equiv \mathcal{H}_T^*(k_m) \text{ and } k_{m+1} \equiv \mathcal{K}_T^*(h_{m+1}). \tag{C.4}$$

(iv) Cease iterations when $m \ge \mathcal{M}$ for some preset maximum iteration $\mathcal{M} \in \mathbb{N}$, or

$$|h_{m+1} - h_m| \le \tau_h \text{ and } |k_{m+1} - k_m| \le \tau_k$$

where $(\tau_h, \tau_k) > 0$ are pre-chosen tolerances. Clearly $|h_{m+1} - h_m|$ and $|k_{m+1} - k_m|$ are integer valued, so $(\tau_h, \tau_k) \in \{0, 1, ...\}$. In experiments not reported here $\tau_h, \tau_k = 0$ lead to $m \leq 25$ in all simulated samples, hence convergence was easily satisfied.

We implicitly have both iterations $h_{m+1} = \tilde{\mathcal{H}}_T^*(h_m) = \mathcal{H}_T^*(\mathcal{K}_T^*(h_m))$ and $k_{m+1} \equiv \tilde{\mathcal{K}}_T^*(k_m) = \mathcal{K}_T^*(\mathcal{H}_T^*(k_m))$. Therefore $h_m \xrightarrow{p} \mathcal{H}_T^*$ and $k_m \xrightarrow{p} \mathcal{K}_T^*$ as $m \to \infty$ provided the fixed point theorem applies. We therefore need $\tilde{\mathcal{H}}_T^*(\mathcal{H})$ and $\tilde{\mathcal{K}}_T^*(\mathcal{K})$ to be contractions mappings $awp1$. Consider $\tilde{\mathcal{H}}_T^*$, and note that we require for any pair $\{h_0, h_1\}$ and some finite $\delta_\mathcal{H} > 0$:

$$\left|\tilde{\mathcal{H}}_T^*(h_1) - \tilde{\mathcal{H}}_T^*(h_0)\right| = \left|\mathcal{H}_T^*(\mathcal{K}_T^*(h_1)) - \mathcal{H}_T^*(\mathcal{K}_T^*(h_0))\right| \leq \delta_\mathcal{H} |h_1 - h_0| \quad awp1.$$

This is trivial under H_0 since by Theorem C.1.a. $\sup_{1 \leq k \leq \mathcal{K}} \mathcal{H}_T^*(k) \xrightarrow{p} 0$ for all finite $1 \leq \mathcal{K} \leq \mathring{\mathcal{K}} \equiv \lim_{T\to\infty} \mathcal{K}_T$.

Conversely, under H_1 Theorem C.2.b yields $\sup_{0 \leq h \leq \mathcal{H}} |\mathcal{K}_T^*(h) - k^*(h)| \to 1$ for any finite $1 \leq \mathcal{H} \leq \mathring{\mathcal{H}}$, where

$$k^*(h_i) \equiv \min\left\{k : k = \text{argmax}_{k\in\mathbb{N}} \left\{\max_{h\in\{0,...,h_i\}} |\Delta r(h,k)|\right\}\right\}.$$

Now invoke Theorem C.1.b to deduce:

$$\begin{aligned}
\left|\tilde{\mathcal{H}}_T^*(h_1) - \tilde{\mathcal{H}}_T^*(h_0)\right| &= \left|\mathcal{H}_T^*(\mathcal{K}_T^*(h_1)) - \mathcal{H}_T^*(\mathcal{K}_T^*(h_0))\right| \\
&= \Bigg|\min\left\{h : h = \text{argmax}_{h\in\mathbb{N}} \left\{\max_{k\in\{1,...,k^*(h_i)\}} \left|\int_0^1 c_h(u)\,\mathcal{B}_k(u)du\right|\right\}\right\} \\
&\quad - \min\left\{h : h = \text{argmax}_{h\in\mathbb{N}} \left\{\max_{k\in\{1,...,k^*(h_0)\}} \left|\int_0^1 c_h(u)\,\mathcal{B}_k(u)du\right|\right\}\right\}\Bigg| + o_p(1) \\
&\equiv \Delta h(h_1, h_0) + o_p(1),
\end{aligned}$$

where $\Delta h(\cdot)$ is implicitly defined. We therefore need

$$\Delta h(h_1, h_0) \leq \delta_\mathcal{H} |h_1 - h_0| \;\; \forall (h_0, h_1). \tag{C.5}$$

Property (C.5) effectively restricts directions of deviation from H_0.

Assumption 3. *Let $\Delta h(h_1, h_0) \leq \delta_\mathcal{H} |h_1 - h_0| \;\; \forall (h_0, h_1)$ and $\Delta k(k_1, k_0) \leq \delta_\mathcal{K} |k_1 - k_0| \;\; \forall (k_0, k_1)$, where $\delta_\mathcal{H}, \delta_\mathcal{K} > 0$ are fixed constants.*

The discussion leading to (C.5) proves the following claim.

Theorem C.3. *Let Assumptions 1-3 hold. Then $h_{m+1} \equiv \mathcal{H}_T^*(k_m)$ and $k_{m+1} \equiv \mathcal{K}_T^*(h_{m+1})$ defined in (C.4) satisfy $h_m \xrightarrow{p} \mathcal{H}_T^*$ and $k_m \xrightarrow{p} \mathcal{K}_T^*$ as $m \to \infty$.*

Theorems C.1 and C.2 now yield the following.

Theorem C.4. *Let Assumptions 1-3 hold. As $T \to \infty$, $(h_m, k_m) \xrightarrow{p} (0, 1)$ under H_0, and under H_1 we have $(h_m, k_m) \xrightarrow{p} (h^*(\mathring{\mathcal{K}}), k^*(\mathring{\mathcal{H}}))$.*

C.4. Proof of Theorem C.1

Let q be any fixed positive constant. Recall the penalty function is $\mathcal{P}_T(\mathcal{H},\mathcal{K}) = \sqrt{(\mathcal{H}+1)\mathcal{K}\ln T}$ if $\mathcal{M}_T(\mathcal{H},\mathcal{K}) \leq \sqrt{q \ln T}$, else $\mathcal{P}_T(\mathcal{H},\mathcal{K}) = \sqrt{2(\mathcal{H}+1)\mathcal{K}}$.

In order to reduce notation we drop the argument $\mathcal{K}_T$ and write, e.g., $\mathcal{M}_T(\mathcal{H}_T) = \mathcal{M}_T(\mathcal{H}_T,\mathcal{K}_T)$, $\mathcal{P}_T(\mathcal{H}_T) = \mathcal{P}_T(\mathcal{H}_T,\mathcal{K}_T)$, $\mathcal{H}_T^* = \mathcal{H}_T^*(\mathcal{K}_T)$.

Claim (a). Let H_0 be true. We will only prove the first claim $P(\mathcal{H}_T^*(\mathcal{K}_T) = 0) \rightarrow 1$ for any $\{\mathcal{K}_T\}$. It then follows that $\mathcal{H}_T^*(\mathcal{K}) \xrightarrow{p} 0$ for any fixed finite $1 \leq \mathcal{K} \leq \mathring{\mathcal{K}}$. The second claim $\max_{1\leq k\leq \mathcal{K}} \mathcal{H}_T^*(k)$ $\xrightarrow{p} 0$ requires pointwise convergence and equicontinuity. Pointwise convergence follows from the first claim, and equicontinuity is trivial for integer-valued functions.

It suffices to prove the following. *First*, for any $\mathcal{H}_T$, $\mathcal{H}_T \rightarrow [0,\infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \rightarrow [0,1]$, the penalty term satisfies:

$$P\left(\mathcal{P}_T(\mathcal{H}_T) = \sqrt{(\mathcal{H}_T+1)\mathcal{K}_T \ln T}\right) \rightarrow 1. \tag{C.6}$$

Hence $\mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T) \equiv \mathcal{M}_T(\mathcal{H}_T) - \sqrt{(\mathcal{H}_T+1)\mathcal{K}_T \ln T}$ asymptotically with probability approaching one.

Second, for such $\{\mathcal{H}_T\}$ the following holds:

$$\begin{aligned} &P\left(\mathcal{M}_T(\mathcal{H}_T) - \mathcal{M}_T(h) \geq \left(\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right)\sqrt{\ln(T)}\right) \\ &\quad \rightarrow \begin{cases} 1 \text{ if } h \geq \mathcal{H}_T \\ 0 \text{ for fixed } h = 0,\ldots,\mathcal{H}_T - 1 \end{cases} \end{aligned} \tag{C.7}$$

Together (C.6) and (C.7) prove the claim $P(\mathcal{H}_T^* = 0) \rightarrow 1$ since the following holds *for every* $h = 0,\ldots,\bar{\mathcal{H}}_T$ *if and only if* $\mathcal{H}_T \rightarrow 0$:

$$\begin{aligned} &\lim_{T\rightarrow\infty} P\left(\mathcal{M}_T^{(\mathcal{P})}(\mathcal{H}_T) \geq \mathcal{M}_T^{(\mathcal{P})}(h)\right) \\ &= \lim_{T\rightarrow\infty} P\left(\mathcal{M}_T(\mathcal{H}_T) - \mathcal{M}_T(h) \geq \left(\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right)\sqrt{\ln(T)}\right) = 1, \end{aligned} \tag{C.8}$$

while $\mathcal{H}_T^*$ is the least of sequences that satisfy (C.8) *for every* $h = 0,\ldots,\bar{\mathcal{H}}_T$.

(C.6). By construction of $\mathcal{P}_T(\mathcal{H}_T)$ it suffices to prove $P(\mathcal{M}_T(\mathcal{H}_T) > \sqrt{q\ln T}) \rightarrow 0$. Under H_0, $\sqrt{T}(\hat{\rho}_h^{(k)} - \hat{\rho}_h) = O_p(1)$ by Theorem 3.4, hence $\sqrt{T}(\hat{\rho}_h^{(k)} - \hat{\rho}_h)/\sqrt{q\ln T} \xrightarrow{p} 0$ for any fixed $q \in (0,\infty)$. Therefore, by Theorem 3.4 for some non-unique $\{\bar{\mathcal{H}}_T, \bar{\mathcal{K}}_T\}$, $\bar{\mathcal{H}}_T, \bar{\mathcal{K}}_T \rightarrow \infty$, $\bar{\mathcal{H}}_T = o(T)$:

$$\frac{\mathcal{M}_T(\bar{\mathcal{H}}_T)}{\sqrt{q\ln T}} = \frac{\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}(\hat{\rho}_h^{(k)} - \hat{\rho}_h)\right|}{\sqrt{q\ln T}} \xrightarrow{p} 0. \tag{C.9}$$

Moreover $\bar{\mathcal{H}}_T\bar{\mathcal{K}}_T = o(T/\ln(T))$ by assumption. By monotonicity of $\mathcal{M}_T(\cdot) \geq 0$, (C.9) holds for any $\{\mathcal{H}_T,\mathcal{K}_T\}$ where $\mathcal{H}_T \rightarrow [0,\infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \rightarrow [0,1]$, $\mathcal{K}_T \rightarrow [1,\infty]$ and $\mathcal{K}_T/\bar{\mathcal{K}}_T \rightarrow [0,1]$. Thus $\mathcal{M}_T(\mathcal{H}_T)/\sqrt{q\ln T} \xrightarrow{p} 0$ for all such $\{\mathcal{H}_T\}$.

(C.7). Suppose $h > \mathcal{H}_T$. By (C.9), $\mathcal{M}_T(\bar{\mathcal{H}}_T)/\sqrt{\ln T} = o_p(1)$ and therefore $\mathcal{M}_T(\mathcal{H}_T) - \mathcal{M}_T(h) = o_p(\sqrt{\ln(T)})$ for any $\{\mathcal{H}_T\}$ where $\mathcal{H}_T \rightarrow [0,\infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \rightarrow [0,1]$, and any $0 \leq h \leq \bar{\mathcal{H}}_T$. Now use

(C.6), monotonicity of $\mathcal{M}_T(\cdot)$, and $\inf_{T\geq 1}\{\sqrt{(h+1)\mathcal{K}_T} - \sqrt{(\mathcal{H}_T+1)\mathcal{K}_T}\} > 0$, to yield that as $T \to \infty$:

$$P\left(\mathcal{M}_T(\mathcal{H}_T) - \mathcal{M}_T(h) \geq \left(\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right)\sqrt{\ln(T)}\right)$$
$$= P\left(\frac{\mathcal{M}_T(\mathcal{H}_T) - \mathcal{M}_T(h)}{\sqrt{\ln(T)}} \geq \sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right)$$
$$= P\left(\sqrt{(h+1)\mathcal{K}_T} - \sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} \geq \frac{\mathcal{M}_T(h) - \mathcal{M}_T(\mathcal{H}_T)}{\sqrt{\ln(T)}}\right) \to 1.$$

Similarly, if $h = \mathcal{H}_T$ then $\sqrt{(h+1)\mathcal{K}_T} - \sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} = 0$ and $\mathcal{M}_T(h) - \mathcal{M}_T(\mathcal{H}_T) = 0$ hence the above limit holds.

Conversely, suppose $h \in \{0, ..., \mathcal{H}_T - 1\}$ and $\mathcal{H}_T > 1$. Then from $\mathcal{M}_T(\mathcal{H}_T) = o_p(\sqrt{q \ln T})$ and $1 - \sqrt{(h+1)/(\mathcal{H}_T+1)} > 0$ it follows:

$$P\left(\mathcal{M}_T(\mathcal{H}_T) - \mathcal{M}_T(h) \geq \left(\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right)\sqrt{\ln(T)}\right)$$
$$= P\left(\frac{\mathcal{M}_T(\mathcal{H}_T) - \mathcal{M}_T(h)}{\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T}\sqrt{\ln(T)}} \geq \left(1 - \sqrt{\frac{h+1}{\mathcal{H}_T+1}}\right)\right) \to 0.$$

(C.7) follows directly.

Claim (b). Let H_1 hold. Similar to (a), we need only prove $\mathcal{H}_T^*(\mathcal{K}_T) \xrightarrow{p} h^*(\mathring{\mathcal{K}})$. Recall under H_1, cf. (16) and (17), both $\gamma_0 + \int_0^1 c_0(u)\,du > 0$ and:

$$\Delta r_T(h,k) = \frac{1}{\frac{1}{T}\sum_{t=1}^T E\left[X_t^2\right]} \frac{1}{T} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t)$$

satisfies $|\hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h,k)| \xrightarrow{p} 0$ and

$$\lim_{T\to\infty} \Delta r_T(h,k) = \Delta r(h,k) \equiv \frac{\int_0^1 c_h(u)\mathcal{B}_k(u)du}{\gamma_0 + \int_0^1 c_0(u)\,du} \neq 0 \text{ for some } h \geq 0,\ k \geq 1.$$

Recall $\mathring{\mathcal{K}} \equiv \lim_{T\to\infty}\mathcal{K}_T$, and

$$h^* \equiv h^*(\mathring{\mathcal{K}}) \equiv \min\left\{h : h = \operatorname{argmax}_{h\in\mathbb{N}}\left\{\max_{k\in\{1,...,\mathring{\mathcal{K}}\}} |\Delta r(h,k)|\right\}\right\},$$

and define the finite sample version:

$$h_T^* \equiv \min\left\{h_T : h_T = \operatorname{argmax}_{0\leq h\leq \mathcal{H}_T} \max_{1\leq k\leq \mathcal{K}_T}\left|\hat{\rho}_h^{(k)} - \hat{\rho}_h\right|\right\},$$

the smallest lag at which the largest sample correlation difference in magnitude occurs.

Define

$$\mathbb{N}_1(h) \equiv \left\{ h \in \mathbb{N} : \max_{k \in \mathbb{N}} |\Delta r(h,k)| \neq 0 \right\},$$

and $\underline{\mathbb{N}}_1 \equiv \min_{h \in \mathbb{N}} \mathbb{N}_1(h)$, the smallest lag at which the largest k^{th} Walsh coefficient $\int_0^1 c_h(u)\mathcal{B}_k(u)du \neq 0$.

We prove in Step 1 that for any integer sequence $\{\mathcal{H}_T\}$ such that $\mathcal{H}_T \rightarrow [\underline{\mathbb{N}}_1, \infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \rightarrow [0,1]$, the penalty satisfies for any sequence of maximum number of systematic samples $\{\mathcal{K}_T\}$:

$$\mathcal{P}\left(\mathcal{P}_T(\mathcal{H}_T) = \sqrt{2(\mathcal{H}_T+1)\mathcal{K}_T}\right) \rightarrow \infty. \tag{C.10}$$

We then prove in Step 2 that *if and only if* $\mathcal{H}_T/h_T^* \xrightarrow{p} [1,\infty]$, for each $0 \leq h \leq \bar{\mathcal{H}}_T$ we have

$$P\left(\mathcal{M}_T(\mathcal{H}_T) \geq \mathcal{M}_T(h) + 2\left(\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right)\right) \rightarrow 1. \tag{C.11}$$

Moreover, $\mathcal{H}_T, \mathcal{K}_T \rightarrow \infty$, $|\hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h,k)| \xrightarrow{p} 0$ (see (C.13) below) and $\Delta r_T(h,k) \rightarrow \Delta r(h,k)$ yields:

$$h_T^* \xrightarrow{p} h^* \equiv \min\left\{ h : h = \text{argmax}_{h \in \mathbb{N}} \left\{ \max_{1 \leq k \leq \lim_{T\to\infty} \mathcal{K}_T} |\Delta r(h,k)| \right\} \right\}.$$

Notice $h^* \in [\underline{\mathbb{N}}_1, \infty)$ by construction of $\underline{\mathbb{N}}_1$.

The proof of the claim then proceeds as follows. Take any integer sequence $\{\mathcal{H}_T\}$, $\mathcal{H}_T/h_T^* \xrightarrow{p} [1,\infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \rightarrow [0,1]$. Then (C.10) holds because $h^* \in [\underline{\mathbb{N}}_1, \infty)$, hence $\mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T) \equiv \mathcal{M}_T(\mathcal{H}_T) - \sqrt{2(\mathcal{H}_T+1)\mathcal{K}_T}$ $awp1$. Since such a sequence implies (C.11), we have $\mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T) \geq \mathcal{M}_T^{\mathcal{P}}(h)$ $awp1$ for each $h = 0, ..., \bar{\mathcal{H}}_T$. Conversely, if (C.11) holds then $\mathcal{H}_T/h_T^* \xrightarrow{p} [1,\infty]$. This yields (C.10) because $h^* \in [\underline{\mathbb{N}}_1, \infty)$. Therefore $\mathcal{M}_T^{\mathcal{P}}(\mathcal{H}_T) \geq \mathcal{M}_T^{\mathcal{P}}(h)$ $awp1$ for each $h = 0, ..., \bar{\mathcal{H}}_T$ *if and only if* $\mathcal{H}_T/h_T^* \xrightarrow{p} [1,\infty]$. Since the optimal $\{\mathcal{H}_T^*\}$ is the least of such sequences, the selection $\mathcal{H}_T^*$ satisfies $\mathcal{H}_T/h_T^* \xrightarrow{p} 1$. Together $\mathcal{H}_T/h_T^* \xrightarrow{p} 1$ and $h_T^* \xrightarrow{p} h^*$ prove the claim.

Step 1: Consider (C.10). By the triangle inequality

$$\begin{aligned}
&\left| \max_{\mathcal{H}_T,\mathcal{K}_T} \left|\hat{\rho}_h^{(k)} - \hat{\rho}_h\right| - \max_{\mathcal{H}_T,\mathcal{K}_T} |\Delta r(h,k)| \right| \\
&\quad \leq \max_{\mathcal{H}_T,\mathcal{K}_T} \left|\hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r(h,k)\right| \\
&\quad \leq \max_{\mathcal{H}_T,\mathcal{K}_T} \left|\hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h,k)\right| + \max_{\mathcal{H}_T,\mathcal{K}_T} |\Delta r_T(h,k) - \Delta r(h,k)|.
\end{aligned} \tag{C.12}$$

Note under H_1:

$$\frac{1}{T}\sum_{t=1}^{T} E[X_t^2] \rightarrow g_0 \equiv \gamma_0 + \int_0^1 c_0(u)du \in (0,\infty)$$

$$\frac{1}{T}\sum_{t=1}^{T-h} E[X_t X_{t+h}]B_k(t) \to w_{h,k} \equiv \int_0^1 c_h(u)\mathcal{B}_k(u)du.$$

Hence, Lemma 3.1 and variance bound (A.5) yield for some integer sequences $\{\bar{\mathcal{H}}_T, \mathcal{K}_T\}$, $\bar{\mathcal{H}}_T, \mathcal{K}_T \to \infty$:

$$\begin{aligned}
&\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h,k)\right| \\
&= \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{\hat{\gamma}_0}\frac{1}{T}\sum_{t=1}^{T-h} X_t X_{t+h}B_k(t)\right. \\
&\qquad \left. - \frac{1}{\frac{1}{T}\sum_{t=1}^{T} E\left[X_t^2\right]}\frac{1}{T}\sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right]B_k(t)\right| \\
&\leq \frac{1}{\hat{\gamma}_0}\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{t=1}^{T-h}\left\{X_t X_{t+h}B_k(t) - E\left[X_t X_{t+h}\right]B_k(t)\right\}\right| \\
&\qquad + \left\{\left|\frac{1}{T}\sum_{t=1}^{T}\left\{X_t^2 - E\left[X_t^2\right]\right\}\right| \times \frac{1}{\frac{1}{T}\sum_{t=1}^{T} E\left[X_t^2\right]\hat{\gamma}_0}\right. \\
&\qquad\qquad \left. \times \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right]B_k(t)\right|\right\} \\
&= O_p(1/\sqrt{T}) + O_p(1/\sqrt{T}) \times \frac{1}{g_0^2} \times \max_{\mathcal{H}_T,\mathcal{K}_T}\left|w_{h,k}\right|.
\end{aligned}$$

By Assumption 1.b,d $\max_{\mathcal{H}_T,\mathcal{K}_T}|w_{h,k}| < \infty$. Hence

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\hat{\rho}_h^{(k)} - \hat{\rho}_h - \Delta r_T(h,k)\right| = O_p(1/\sqrt{T}). \tag{C.13}$$

Further:

$$\begin{aligned}
&\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\Delta r_T(h,k) - \Delta r(h,k)\right| \\
&\leq \frac{1}{\gamma_0 + \int_0^1 c_0(u)\,du}\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right]B_k(t) - \int_0^1 c_h(u)\,\mathcal{B}_k(u)du\right| \\
&\qquad + \left\{\left|\frac{1}{T}\sum_{t=1}^{T} E\left[X_t^2\right] - \left(\gamma_0 + \int_0^1 c_0(u)\,du\right)\right| \times \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\int_0^1 c_h(u)\,\mathcal{B}_k(u)du\right|\right. \\
&\qquad\qquad \left. \times \frac{1}{\frac{1}{T}\sum_{t=1}^{T} E\left[X_t^2\right]\left(\gamma_0 + \int_0^1 c_0(u)\,du\right)}\right\} \\
&= \mathcal{A}_T + \mathcal{B}_{1,T}\mathcal{B}_{2,T}\mathcal{B}_{3,T},
\end{aligned}$$

say where $(\mathcal{A}_T, \mathcal{B}_{i,T})$ are implicitly defined. By construction and (3.10)-(3.11) in the main paper, $\mathcal{A}_T \to 0$, and:

$$\mathcal{B}_{1,T} = \left| \frac{1}{T} \sum_{t=1}^{T} (\gamma_0 + c_0(t/T)) - \left(\gamma_0 + \int_0^1 c_0(u)\, du \right) \right| \to 0$$

$$\mathcal{B}_{2,T} = \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \int_0^1 c_h(u)\, \mathcal{B}_k(u) du \right| = O(1)$$

$$\mathcal{B}_{3,T} = \frac{1}{\frac{1}{T}\sum_{t=1}^{T} E\left[X_t^2\right] \left(\gamma_0 + \int_0^1 c_0(u)\, du \right)} \to \frac{1}{\left(\gamma_0 + \int_0^1 c_0(u)\, du \right)^2} \in (0, \infty).$$

Hence $\max_{\mathcal{H}_T, \mathcal{K}_T} |\Delta r_T(h,k) - \Delta r(h,k)| \to 0$. Coupled with (C.12) and (C.13), this yields:

$$\left| \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h \right| - \max_{\mathcal{H}_T, \mathcal{K}_T} |\Delta r(h,k)| \right| \xrightarrow{p} 0, \tag{C.14}$$

where

$$\lim_{T\to\infty} \max_{\mathcal{H}_T, \mathcal{K}_T} |\Delta r(h,k)| = \max_{h,k\in\mathbb{N}} \left| \frac{\int_0^1 c_h(u)\, \mathcal{B}_k(u) du}{\gamma_0 + \int_0^1 c_0(u)\, du} \right| \in (0, \infty). \tag{C.15}$$

Therefore for any $\{\mathcal{H}_T\}$, $\mathcal{H}_T \to [\underline{\mathrm{N}}_1, \infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \to [0,1]$:

$$\frac{\mathcal{M}_T(\mathcal{H}_T)}{\sqrt{q \ln T}} = \frac{\sqrt{T} \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h \right|}{\sqrt{q \ln T}}$$

$$= \frac{\sqrt{T} \left(\max_{\mathcal{H}_T, \mathcal{K}_T} |\Delta r(h,k)| + o_p(1) \right)}{\sqrt{q \ln T}} \xrightarrow{p} \infty.$$

This proves (C.10) by construction of the penalty term $\mathcal{P}_T(\mathcal{H}_T)$, cf. (C.1).

Step 2: Next, (C.11). First, by (C.14) and (C.15) $\mathcal{M}_T(\mathcal{H}_T)/\sqrt{T} \xrightarrow{p} (0,\infty)$ for any $\{\mathcal{H}_T\}$, $\mathcal{H}_T \to [\underline{\mathrm{N}}_1, \infty]$ and $\mathcal{H}_T/\bar{\mathcal{H}}_T \to [0,1]$. Hence $\mathcal{M}_T(\mathcal{H}_T)/\sqrt{T/\ln(T)} \xrightarrow{p} \infty$ for any $\mathcal{H}_T \to [\underline{\mathrm{N}}_1, \infty]$. Monotonicity ensures $\mathcal{M}_T(\mathcal{H}_T) \geq \mathcal{M}_T(h)$ for each $0 \leq h \leq \mathcal{H}_T$, hence for such h:

$$\frac{\mathcal{M}_T(h)}{\mathcal{M}_T(\mathcal{H}_T)} = \frac{\mathcal{M}_T(h)/\sqrt{T}}{\mathcal{M}_T(\mathcal{H}_T)/\sqrt{T}} \xrightarrow{p} [0,1].$$

Indeed, if both

$$(h, \mathcal{H}_T) \geq h_T^* \equiv \min\left\{ h_T : h_T = \text{argmax}_{0\leq h\leq \mathcal{H}_T} \max_{1\leq k\leq \mathcal{K}_T} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h \right| \right\}$$

then by construction $\mathcal{M}_T(h)/\mathcal{M}_T(\mathcal{H}_T) = 1$.

Now suppose $0 \leq h$ and $h/\mathcal{H}_T \to [0,1)$, and $\mathcal{H}_T/h_T^* \xrightarrow{p} [0,1)$, hence $0 \leq h < \mathcal{H}_T < h_T^*$ as $T \to \infty$ $awp1$. Then $\mathcal{M}_T(h)/\mathcal{M}_T(\mathcal{H}_T) \xrightarrow{p} [0,1)$ by monotonicity and the construction of h_T^*. Use $\mathcal{H}_T \leq \bar{\mathcal{H}}_T$, $\mathcal{K}_T\mathcal{H}_T = o(T/\ln(T))$, and $\mathcal{M}_T(\mathcal{H}_T)/\sqrt{T/\ln(T)} \xrightarrow{p} \infty$, to yield:

$$P\left(\mathcal{M}_T(\mathcal{H}_T) \geq \mathcal{M}_T(h) + 2\left(\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right)\right) \tag{C.16}$$
$$= P\left(\mathcal{M}_T(\mathcal{H}_T)\left(1 - \frac{\mathcal{M}_T(h)}{\mathcal{M}_T(\mathcal{H}_T)}\right) \geq 2\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T}\left(1 - \sqrt{\frac{h+1}{\mathcal{H}_T+1}}\right)\right)$$
$$\geq P\left(\frac{\mathcal{M}_T(\mathcal{H}_T)}{\sqrt{T/\ln(T)}}\left(1 - \frac{\mathcal{M}_T(h)}{\mathcal{M}_T(\mathcal{H}_T)}\right) \geq 2\sqrt{\frac{(\mathcal{H}_T+1)\mathcal{K}_T}{T/\ln(T)}}\right) \to 1.$$

Next, consider $0 \leq h$ and $h/h_T^* \xrightarrow{p} [0,1)$, and $\mathcal{H}_T/h_T^* \xrightarrow{p} [1,\infty]$, hence $0 \leq h \leq h_T^* - 1$ $awp1$ and $\mathcal{H}_T \geq h_T^*$ $awp1$. Then $P(\mathcal{M}_T(h) = \mathcal{M}_T(\mathcal{H}_T) \to 0$ since by construction h_T^* is the smallest lag at which the maximum correlation difference occurs. Monotonicity therefore yields $\mathcal{M}_T(h)/\mathcal{M}_T(\mathcal{H}_T)$ $\xrightarrow{p} [0,1)$, and again we deduce (C.16).

Now let $(h, \mathcal{H}_T) \geq h_T^*$ $awp1$. Then by construction $\mathcal{M}_T(\mathcal{H}_T) = \mathcal{M}_T(h)$ $awp1$. Trivially if $h < \mathcal{H}_T$ ($h \geq \mathcal{H}_T$) then $\sqrt{\mathcal{H}_T} - \sqrt{h} > 0$ ($\sqrt{\mathcal{H}_T} - \sqrt{h} \leq 0$). Hence

$$P\left(\mathcal{M}_T(\mathcal{H}_T) \geq \mathcal{M}_T(h) + 2\left[\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right]\right) \to 1 \textit{ if and if } h \geq \mathcal{H}_T.$$

Next, let $\mathcal{H}_T < h_T^* \leq h$ $awp1$ such that $\mathcal{M}_T(h) = \mathcal{M}_T(h_T^*)$ $awp1$. Use $\mathcal{H}_T/h \to [0,1)$, $h = o(T/\ln(T))$, $\mathcal{M}_T(h_T^*)/\sqrt{T/\ln(T)} \xrightarrow{p} \infty$, and $\mathcal{M}_T(\mathcal{H}_T)/\mathcal{M}_T(h_T^*) \xrightarrow{p} [0,1)$ to yield:

$$P\left(\mathcal{M}_T(\mathcal{H}_T) \geq \mathcal{M}_T(h) + 2\left[\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right]\right)$$
$$= P\left(2\left(1 - \sqrt{\frac{\mathcal{H}_T+1}{h+1}}\right)\sqrt{\frac{h}{T/\ln(T)}} \geq \frac{\mathcal{M}_T(h_T^*)}{\sqrt{T/\ln(T)}}\left(1 - \frac{\mathcal{M}_T(\mathcal{H}_T)}{\mathcal{M}_T(h_T^*)}\right)\right) \to 0.$$

Finally, generally $\mathcal{M}_T(h) = \mathcal{M}_T(\mathcal{H}_T)$ $a.s.$ for some $\{h, \mathcal{H}_T\}$ and all but a finite number of T is possible. For example, when $h = \mathcal{H}_T$. In this case, *if and only if* $h \geq \mathcal{H}_T$:

$$P\left(\mathcal{M}_T(\mathcal{H}_T) \geq \mathcal{M}_T(h) + 2\left[\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right]\right)$$
$$= P\left(0 \geq 2\left[\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right]\right) \to 1$$

Combining the above results, we deduce *for every* $0 \leq h \leq \bar{\mathcal{H}}_T$ *if and only if* $\mathcal{H}_T \geq h_T^*$:

$$P\left(\mathcal{M}_T(\mathcal{H}_T) \geq \mathcal{M}_T(h) + 2\left[\sqrt{(\mathcal{H}_T+1)\mathcal{K}_T} - \sqrt{(h+1)\mathcal{K}_T}\right]\right) \to 1.$$

This proves (C.11). $\mathcal{QED}$.

D. Omitted proofs

D.1. Lemma A.2

Lemma A.2. *Let* $\max_{1\leq t\leq T} P(|X_t| > c) \leq \varpi \exp\{-c^{\vartheta_1}\mathcal{E}_T^{-\vartheta_2}\}$ *for some* $\varpi > 0$*, any* $\vartheta_1 \geq 2\vartheta_2$ *and* $\vartheta_2 \geq 1$*, and some sequence of constants* $\{\mathcal{E}_T\}$*,* $\liminf_{T\to\infty} \mathcal{E}_T \geq 1$*. It holds that*

$$\max_{1\leq t_1,\ldots,t_r\leq T} P\left(\left|\prod_{i=1}^{r} X_{t_i}\right| > c\right) \leq r\varpi \exp\left\{-c^{\vartheta_2}\mathcal{E}_T^{-\vartheta_2}\right\}. \tag{D.1}$$

Proof. We prove (D.1) by induction. If $r = 1$ then $\max_{1\leq t\leq T} P(|\ X_t| > c) \leq \varpi \exp\{-c^{\vartheta_1}\mathcal{E}_T^{-\vartheta_2}\}$ $\leq \varpi \exp\{-c^{\vartheta_2}\mathcal{E}_T^{-\vartheta_2}\}$ by assumption, given $\vartheta_1 > \vartheta_2 \geq 1$. Now let (D.1) hold for some $r \geq 1$: $\max_{1\leq t_1,\ldots,t_r\leq T} P(|\sqcap_{i=1}^{r} X_{t_i}| > c) \leq r\varpi \exp\{-c^{\vartheta_2}\mathcal{E}_T^{-\vartheta_2}\}$. Young and Bonferroni inequalities yield for any $\vartheta_1 \geq 2\vartheta_2$

$$\begin{aligned}
\max_{1\leq t_1,\ldots,t_{r+1}\leq T} P\left(\left|\prod_{i=1}^{r+1} X_{t_i}\right| > c\right) &\leq \max_{1\leq t_1,\ldots,t_{r+1}\leq T} P\left(\frac{1}{2}\left(\prod_{i=1}^{r} X_{t_i}\right)^2 + \frac{1}{2}X_{t_{r+1}}^2 > c\right) \\
&\leq \max_{1\leq t_1,\ldots,t_r\leq T} P\left(\left|\prod_{i=1}^{r} X_{t_i}\right| > c^{\frac{1}{2}}\right) + \max_{1\leq t\leq T} P\left(|X_t| > c^{\frac{1}{2}}\right) \\
&\leq r\varpi \exp\left\{-c^{\vartheta_2}\mathcal{E}_T^{-\vartheta_2}\right\} + \varpi \exp\left\{-c^{\vartheta_1/2}\mathcal{E}_T^{-\vartheta_2}\right\} \\
&\leq (r+1)\,\varpi \exp\left\{-c^{\vartheta_2}\mathcal{E}_T^{-\vartheta_2}\right\}.
\end{aligned}$$

Hence (D.1) holds for $r + 1$. The proof is complete because r is arbitrary. $\mathcal{QED}$.

D.2. Lemma A.4

Let $\Rightarrow^p$ denote weak convergence in probability on l_∞ (the space of bounded functions) as defined in Giné and Zinn (1990, Section 3). Recall $\{\mathcal{E}_T\}$ is the Assumption 1 exponential moment scale, $\liminf_{T\to\infty} \mathcal{E}_T \geq 1$; the bootstrap index blocks are $\mathfrak{B}_s = \{(s-1)b_T + 1, \ldots, sb_T\}$, $s = 1, \ldots, T/b_T$, with block size b_T, $1 \leq b_T < T$, $b_T \to \infty$ and $b_T/T^{1-\iota} \to 0$ for some small $\iota > 0$; ξ_i is iid $N(0,1)$; and $\varphi_t = \xi_s$ if $t \in \mathfrak{B}_s$. Recall the number of blocks $\mathcal{N}_T = [T/b_T]$, and

$$\Delta\hat{g}_T^{(dw)}(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h} \varphi_t \left\{X_t X_{t+h} B_k(t) - \frac{1}{T}\sum_{t=1}^{T-h} X_t X_{t+h} B_k(t)\right\},$$

and define

$$\mathring{\sigma}_T^2(h,k) \equiv E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h} \varphi_t \left\{X_t X_{t+h} B_k(t) - \frac{1}{T}\sum_{s=1}^{T-h} E\left[X_s X_{s+h}\right] B_k(s)\right\}\right)^2\right].$$

Recall

$$z_t(h,k) \equiv \{X_t X_{t+h} - E\left[X_t X_{t+h}\right]\} B_k(t)$$

$$\sigma_T^2(h,k) \equiv E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h} z_t(h,k)\right)^2\right].$$

We refer to the following condition below:

$$\frac{1}{T^{1/9}}\left[\mathcal{E}_T^{2/3}\left\{\ln\left(\mathcal{H}_T\mathcal{K}_T\right)\right\}^{(1+2\phi)/(3\phi)} + \mathcal{E}_T\left(\ln \mathcal{H}_T\mathcal{K}_T\right)^{7/6}\right] \to 0. \tag{D.2}$$

Lemma A.4. *Let Assumptions 1 and 2 hold.*

a. Let $\{\mathring{\mathbf{Z}}_T(h,k) : 0 \leq h \leq \mathcal{H}_T, 1 \leq k \leq \mathcal{K}_T\}_{T\leq 1}$ *be a Gaussian process,* $\mathring{\mathbf{Z}}_T(h,k) \sim N(0, \mathring{\sigma}_T^2(h,k))$, *independent of the sample* $\{X_t\}_{t=1}^T$. *For any sequences* $\{\mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$, *where* $0 \leq \mathcal{H}_T < T-1$, $\mathcal{H}_T = o(T)$, $\eta(\mathcal{K}_T) = o(\sqrt{T})$ *and (D.2) hold:*

$$\sup_{c>0}\left|P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k)\right| \leq c|\{X_t\}_{t=1}^T\right) - P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathring{\mathbf{Z}}_T(h,k)\right| \leq c\right)\right| \xrightarrow{p} 0.$$

b. Let $\{\mathring{\mathbf{Z}}(h,k)\}$ *be an independent copy of the Lemma 3.1 Gaussian process* $\{\mathbf{Z}(k,h) : h,k \in \mathbb{N}\}$, $\mathbf{Z}(h,k) \sim N(0, \lim_{T\to\infty}\sigma_T^2(h,k))$, *independent of the asymptotic draw* $\{X_t\}_{t=1}^\infty$. *For any sequences* $\{b_T, \mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$, *such that* $0 \leq \mathcal{H}_T < T-1$, $b_T/T^\iota \to \infty$, $b_T = o(T^{1/2-\iota})$, $\mathcal{H}_T = O(T^{1-\iota}/b_T)$, $\eta(\mathcal{K}_T) = o(\sqrt{T})$, *and (D.2) hold:*

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k)\right| \Rightarrow^p \max_{h,k\in\mathbb{N}}\left|\mathring{\mathbf{Z}}(h,k)\right|.$$

The proof requires two preliminary results. We first prove the following uniform sample covariance result. Write

$$\hat{g}(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) \text{ and } g_T(h,k) \equiv E\left[\hat{g}(h,k)\right] = \frac{1}{T}\sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t)$$

Lemma D.1. *Under Assumption 1, for any* $\{\mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$ *satisfying* $0 \leq \mathcal{H}_T < T-1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^\kappa)$ *for some finite* $\kappa > 0$, $\eta(\mathcal{K}_T) = o(\sqrt{T})$ *and (D.2):*

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\hat{g}(h,k) - g_T(h,k)\right| = O_p\left(1/\sqrt{T}\right).$$

Proof. Define

$$\mathcal{G}_T(h,k) \equiv \sqrt{T}\left(\hat{g}(h,k) - g_T(h,k)\right) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\left\{X_t X_{t+h} - E\left[X_t X_{t+h}\right]\right\} B_k(t)$$

and $s_T^2(h,k) \equiv E[\mathcal{G}_T^2(h,k)]$. The argument used to prove Lemma 3.1 yields

$$\sup_{z\geq 0}\left|P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathcal{G}_T(h,k)\right| \leq z\right) - P\left(\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathbf{G}_T(h,k)\right| \leq z\right)\right| \to 0$$

for some sequence of random functions $\{\mathbf{G}_T(h,k)\}_{T\geq 1}$ with $\mathbf{G}_T(h,k) \sim N(0, s_T^2(h,k))$, for any $\{\mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$ satisfying $0 \leq \mathcal{H}_T < T-1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^\kappa)$ for some finite $\kappa > 0$, $\eta(\mathcal{K}_T) = o(\sqrt{T})$ and (D.2). The claim follows instantly. $\mathcal{QED}$.

Next, define

$$y_t(h,k) \equiv \left\{ X_t X_{t+h} B_k(t) - \frac{1}{T-h} \sum_{s=1}^{T-h} E\left[X_s X_{s+h}\right] B_k(s) \right\}.$$

We decompose the following summand into big and little blocks:

$$\Delta \ddot{g}_T^*(h,k) \equiv \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t \left\{ X_t X_{t+h} B_k(t) - \frac{1}{T-h} \sum_{s=1}^{T-h} E\left[X_s X_{s+h}\right] B_k(s) \right\} = \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t y_t(h,k).$$

Let $\tilde{b}_T$ and $\tilde{l}_T$ be block sizes, $(\tilde{b}_T, \tilde{l}_T) \to \infty$, with $1 < \tilde{b}_T < T$, $\tilde{b}_T = o(T)$, $1 \leq \tilde{l}_T < \tilde{b}_T$, and $\tilde{l}_t = o(\tilde{b}_T)$. In each index set $\{1, ..., T-h\}$ the number of blocks is $\widetilde{\mathcal{N}}_T(h) = [(T-h)/\tilde{b}_T]$. Denote the blocks by $\tilde{\mathfrak{B}}_s = \{(s-1)\tilde{b}_T + 1, \ldots, s\tilde{b}_T\}$ with $s = 1, \ldots, \widetilde{\mathcal{N}}_T(h)$, and $\tilde{\mathfrak{B}}_{\widetilde{\mathcal{N}}_T(h)+1} = \{\widetilde{\mathcal{N}}_T(h)\tilde{b}_T, ..., T+h\}$. Then

$$\Delta \ddot{g}_T^*(h,k) = \frac{1}{T} \sum_{i=1}^{\widetilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\tilde{b}_T+\tilde{l}_T+1}^{i\tilde{b}_T} \varphi_t y_t(h,k) + \frac{1}{T} \sum_{i=1}^{\widetilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\tilde{b}_T+1}^{(i-1)\tilde{b}_T+\tilde{l}_T} \varphi_t y_t(h,k)$$
$$+ \frac{1}{T} \sum_{i=\widetilde{\mathcal{N}}_T(h)\tilde{b}_T+1}^{T-h} \varphi_t y_t(h,k).$$

Lemma D.2. *Under Assumptions 1 and 2, for any* $\{\mathcal{E}_T, \mathcal{H}_T, \mathcal{K}_T\}$ *satisfying* $0 \leq \mathcal{H}_T < T-1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^\kappa)$ *for some finite* $\kappa > 0$, $\eta(\mathcal{K}_T) = o(\sqrt{T})$ *and (D.2):*

$$\left| \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \Delta \ddot{g}_T^*(h,k) \right| - \max_{\mathcal{H}_T, \mathcal{K}_T} \left| \frac{1}{T} \sum_{i=1}^{\widetilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\tilde{b}_T+\tilde{l}_T+1}^{i\tilde{b}_T} \varphi_t y_t(h,k) \right| \right| = o_p\left(1/\sqrt{T}\right).$$

Proof. The triangle inequality yields for any real-valued functions $\{a(h,k), b(h,k)\}$

$$\left| \max_{\mathcal{H}_T, \mathcal{K}_T} |a(h,k)| - \max_{\mathcal{H}_T, \mathcal{K}_T} |b(h,k)| \right| \leq \max_{\mathcal{H}_T, \mathcal{K}_T} |a(h,k) - b(h,k)|.$$

We therefore prove

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \left| \Delta \ddot{g}_T^*(h,k) \right| - \left| \frac{1}{T} \sum_{i=1}^{\widetilde{\mathcal{N}}_T(h)} \sum_{t=(i-1)\tilde{b}_T+\tilde{l}_T+1}^{i\tilde{b}_T} \varphi_t y_t(h,k) \right| \right| = o_p\left(1/\sqrt{T}\right),$$

Step 1. It suffices to replace $y_t(h,k)$ with $z_t(h,k)$ uniformly *awp1*:

$$\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t \left\{ y_t(h,k) - z_t(h,k) \right\} \right| = o_p(1). \tag{D.3}$$

This follows by noting:

$$\left| \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t y_t(h,k) - \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t z_t(h,k) \right|$$

$$\leq \left| \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t \left\{ E\left[X_t X_{t+h}\right] B_k(t) - \frac{1}{T-h} \sum_{s=1}^{T-h} E\left[X_s X_{s+h}\right] B_k(s) \right\} \right|$$

$$\leq \left| \frac{1}{T} \sum_{t=1}^{T-h} \varphi_t E\left[X_t X_{t+h}\right] B_k(t) \right| + O_p\left(1/\sqrt{T}\right)$$

in view of $\max_{\mathcal{H}_T,\mathcal{K}_T} |(T-h)^{-1} \sum_{s=1}^{T-h} E\left[X_s X_{s+h}\right] B_k(s)| \leq K$ under Assumption 1.b,d, and $\max_{\mathcal{H}_T} |1/T \sum_{t=1}^{T-h} \varphi_t| = O_p(1/\sqrt{T})$. The latter follows by construction of φ_t:

$$\frac{1}{T} \sum_{t=1}^{T-h} \varphi_t = \frac{1}{T/b_T} \sum_{i=1}^{[(T-h)/b_T]} \xi_i = \frac{1}{T/b_T} \sum_{i=1}^{[\lambda_h T/b_T]} \xi_i \text{ with } \lambda_h = 1 - \frac{h}{T} \in [0,1),$$

where iid $\xi_i \sim N(0,1)$. By Donsker's theorem extended to $\mathcal{D}[0,1]$, and the mapping theorem, $\sup_{\lambda\in[0,1)} |1/\sqrt{N} \sum_{i=1}^{[\lambda N]} \xi_i| = O_p(1)$ (cf Dudley, 1999).

Next, by construction:

$$\frac{1}{T} \sum_{t=1}^{T-h} \varphi_t E\left[X_t X_{t+h}\right] B_k(t)$$

$$= \frac{1}{T/b_T} \sum_{i=1}^{[(T-h)/b_T]} \xi_i \frac{1}{b_T} \sum_{t=(i-1)b_T+1}^{ib_T} E\left[X_t X_{t+h}\right] B_k(t) = \frac{1}{T/b_T} \sum_{i=1}^{\mathcal{N}_T(h)} \xi_i \varpi_{T,i}(h,k)$$

say, where $\varpi_{T,i}(h,k) \equiv 1/b_T \sum_{t=(i-1)b_T+1}^{ib_T} E[X_t X_{t+h}] B_k(t)$ and $\mathcal{N}_T(h) \equiv [(T-h)/b_T]$. Given ξ_i is iid $N(0,1)$, a generalization of Nemirovski's $\mathcal{L}_q$-moment bound, $q \geq 1$, for independent sequences yields (see, e.g., Bühlmann and Van De Geer, 2011, Lemma 14.24):

$$E\left[\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T/b_T} \sum_{i=1}^{\mathcal{N}_T(h)} \xi_i \varpi_{T,i}(h,k) \right|^q\right] \leq \left\{ \frac{8\ln(2\mathcal{H}_T\mathcal{K}_T) \max_{\mathcal{H}_T,\mathcal{K}_T} \max_{1\leq i\leq \mathcal{N}_T(h)} \varpi_{T,i}^2(h,k)}{T/b_T} \right\}^{q/2}. \quad \text{(D.4)}$$

Moreover:

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \max_{1\leq i\leq \mathcal{N}_T(h)} \varpi_{T,i}^2(h,k) = \max_{\mathcal{H}_T,\mathcal{K}_T} \max_{1\leq i\leq \mathcal{N}_T(h)} \left(\frac{1}{b_T} \sum_{t=(i-1)b_T+1}^{ib_T} E\left[X_t X_{t+h}\right] B_k(t) \right)^2 \quad \text{(D.5)}$$

$$\leq \left(\max_{\mathcal{H}_T} \max_{1\leq t\leq T} |E\left[X_t X_{t+h}\right]| \right)^2 \equiv \bar{\varpi}_T^2.$$

Now combine (D.4) and (D.5), choose $q = 2$, and invoke $b_T = o(T^{1/2-\iota})$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^\kappa)$ for some finite $\kappa > 0$, and $\bar{\varpi}_T = O(1)$ under Assumption 1.b and the Cauchy-Schwartz inequality, to yield:

$$E\left[\max_{\mathcal{H}_T,\mathcal{K}_T} \left(\frac{1}{T/b_T} \sum_{i=1}^{\mathcal{N}_T(h)} \xi_i \varpi_{T,i}(h,k) \right)^2\right] \leq K \frac{\ln(\mathcal{H}_T\mathcal{K}_T)}{T/b_T} \bar{\varpi}_T^2$$

$$= o\left(\frac{\ln(T)}{T^{1/2+\iota}}\right) \times \bar{\varpi}_T^2 = o\left(\frac{\ln(T)}{T^{1/2+\iota}}\right) \times o(T^{1/2}) = o(1).$$

This proves (D.3) by Chebyshev's inequality.

Step 2. Now observe that

$$\begin{aligned}&\left|\Delta\ddot{g}_T^*(h,k)-\frac{1}{T}\sum_{i=1}^{(T-h)/b_T}\sum_{t=(i-1)b_T+l_T+1}^{ib_T}\varphi_t z_t(h,k)\right|\\&\qquad\leq\left|\frac{1}{T}\sum_{i=1}^{\widetilde{N}_T(h)}\sum_{t=(i-1)\tilde{b}_T+1}^{(i-1)\tilde{b}_T+\tilde{l}_T}\varphi_t z_t(h,k)\right|+\left|\frac{1}{T}\sum_{i=\widetilde{N}_T(h)\tilde{b}_T+1}^{T-h}\varphi_t z_t(h,k)\right|.\end{aligned}$$

Lemma 3.1 and $\tilde{b}_T/\tilde{l}_T=o(1)$ yield under the assumed properties for $\{\mathcal{E}_T,\mathcal{H}_T,\mathcal{K}_T\}$:

$$\begin{aligned}\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{i=1}^{\widetilde{N}_T(h)}\sum_{t=(i-1)\tilde{b}_T+1}^{(i-1)\tilde{b}_T+\tilde{l}_T}\varphi_t z_t(h,k)\right|&=.\frac{\tilde{l}_T}{\tilde{b}_T}\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T\tilde{l}_T/\tilde{b}_T}\sum_{i=1}^{\widetilde{N}_T(h)}\sum_{t=(i-1)\tilde{b}_T+1}^{(i-1)\tilde{b}_T+\tilde{l}_T}\varphi_t z_t(h,k)\right|\\&=O_p\left(\frac{\tilde{l}_T/\tilde{b}_T}{\sqrt{T\tilde{l}_T/\tilde{b}_T}}\right)=O_p\left(1/\sqrt{T\tilde{b}_T/\tilde{l}_T}\right)=o_p\left(1/\sqrt{T}\right).\end{aligned}$$

Similarly, for any (h,k), the integer-valued discrepancy implicit in $T-h-\widetilde{N}_T(h)\tilde{b}_T=T-h-[(T-h)/\tilde{b}_T]\tilde{b}_T$ yields:

$$\begin{aligned}\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{i=\widetilde{N}_T(h)\tilde{b}_T+1}^{T-h}\varphi_t z_t(h,k)\right|&=O_p\left(\max_{\mathcal{H}_T}\frac{\sqrt{(T-h)-\widetilde{N}_T(h)\tilde{b}_T}}{T}\right)\qquad\text{(D.6)}\\&=O_p\left(\max_{\mathcal{H}_T}\frac{\sqrt{1-\left[\frac{T}{\tilde{b}_T}(1-h/T)\right]\frac{\tilde{b}_T}{T(1-h/T)}}}{\sqrt{T}}\right)=o_p\left(1/T\right).\end{aligned}$$

This completes the proof. $\mathcal{QED}$

We are now ready to prove Lemma A.4. Assume $(T-h)/b_T$ and related ratios are integers to reduce notation. The resulting error otherwise is asymptotically negligible, as in (D.6).

Proof of Lemma A.4.

Claim (a). Define the sample $\mathfrak{X}_T\equiv\{X_t\}_{t=1}^T$, and define

$$\Delta g_T^*(h,k)\equiv\frac{1}{T}\sum_{t=1}^{T-h}\varphi_t\left\{X_tX_{t+h}B_k(t)-\frac{1}{T}\sum_{s=1}^{T-h}E\left[X_sX_{s+h}\right]B_k(s)\right\}.$$

Let $\{\Delta\hat{g}_T^{(dw)}(i)\}_{i=0}^{\mathcal{H}_T\mathcal{K}_T}$, etc., denote the stacked $\{\Delta\hat{g}_T^{(dw)}(h,k)\}_{h=0,k=1}^{\mathcal{H}_T,\mathcal{K}_T}$:

$$\Delta\hat{g}_T^{(dw)}(i)=\Delta\hat{g}_T^{(dw)}(h,k)\text{ with index correspondence }i=(k-1)\mathcal{H}_T+h.\qquad\text{(D.7)}$$

and define

$$\hat{s}_T^2(i,j) = TE\left[\Delta\hat{g}_T^{(dw)}(i)\Delta\hat{g}_T^{(dw)}(j)|\mathfrak{X}_T\right] \text{ and } \mathring{s}_T^2(i,j) = TE\left[\Delta g_T^*(i)\Delta g_T^*(j)|\mathfrak{X}_T\right]$$

$$\Delta_T \equiv \max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T}\left|\hat{s}_T^2(i,j) - \mathring{s}_T^2(i,j)\right|,$$

hence $\mathring{s}_T^2(i,i) \equiv \mathring{\sigma}_T^2(h,k)$ where $i = (k-1)\mathcal{H}_T + h$.[1]

Let $\{\mathring{\mathbf{Z}}_T(i)\}_{T\leq 1}$ be sequences of normal random variables $\mathring{\mathbf{Z}}_T(i) \sim N(0, \mathring{s}_T^2(i,i))$ independent of $\mathfrak{X}_T$. Lemma 3.1 in Chernozhukov, Chetverikov and Kato (2013), cf. Chernozhukov, Chetverikov and Kato (2015, Theorem 2, Proposition 1) and Chen (2018, Lemma C.1), yields:

$$\mathcal{E}_T \equiv \sup_{c>0}\left|P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T}\left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(i)\right| \leq c|\mathfrak{X}_T\right) - P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T}\left|\mathring{\mathbf{Z}}_T(i)\right| \leq c\right)\right| \tag{D.8}$$

$$= O_p\left(\Delta_T^{1/3}\max\left\{1, \ln\left(\mathcal{H}_T\mathcal{K}_T/\Delta_T\right)\right\}^{2/3}\right).$$

We prove below $\Delta_T = O_p(1/T^\iota)$ for some $\iota > 0$. Now use $\mathcal{H}_T = o(T)$ and $\mathcal{K}_T = o(T^\kappa)$ for some finite $\kappa > 0$ to reach:

$$\mathcal{E}_T = O_p\left(\Delta_T^{1/3}\max\left\{1, \ln\left(\mathcal{H}_T\mathcal{K}_T/\Delta_T\right)\right\}^{2/3}\right)$$
$$= O_p\left(\Delta_T^{1/3}\max\left\{1, \ln\left(\sqrt{T}\mathcal{H}_T\mathcal{K}_T\right) + \ln\left(\sqrt{T}\Delta_T\right)\right\}^{2/3}\right) = O_p\left(\frac{1}{T^{\iota/6}}\left\{\ln(T)\right\}^{2/3}\right) \xrightarrow{p} 0.$$

This suffices to prove the claim in view of the correspondence $i = (k-1)\mathcal{H}_T + h$.

We now prove $\Delta_T = O_p(1/T^\iota)$. Define for any $g \in \mathbb{R}$

$$\mathfrak{E}_{l,T}(h,k;g) \equiv \sum_{t=(l-1)b_T+1}^{lb_T}\left\{X_tX_{t+h}B_k(t) - g\right\},$$

and define

$$\hat{g}(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h}X_tX_{t+h}B_k(t) \text{ and } g_T(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h}E\left[X_tX_{t+h}\right]B_k(t).$$

By construction of φ_t via iid $\{\xi_l\}_{l=1}^{(T-h)/b_T}$, $\xi_l \sim N(0,1)$:

$$\Delta\hat{g}_T^{(dw)}(h,k) = \frac{1}{T}\sum_{l=1}^{(T-h)/b_T}\xi_l\mathfrak{E}_{l,T}(h,k;\hat{g}(h,k))$$

$$\Delta g_T^*(h,k) = \frac{1}{T}\sum_{l=1}^{(T-h)/b_T}\xi_l\mathfrak{E}_{l,T}(h,k;g_T(h,k)).$$

[1]The correspondence $i = (k-1)\mathcal{H}_T + h$ is unique. In particular, it is understood that we set k and move through $h \in \{0,,,.\mathcal{H}_T\}$. Thus, (i) set $k=1$ and move through $h = 0,\ldots,\mathcal{H}_T$ for $i = 0,\ldots,\mathcal{H}_T$; (ii) set $k=2$ and then $h=0,\ldots,\mathcal{H}_T$ to yield $i = \mathcal{H}_T+1,\ldots,2\mathcal{H}_T$; and so on. Thus, if $\mathcal{H}_T = 100$ then $i = 299$ is uniquely matched with $k=3$ and $h=99$.

Serial independence, and independence of $\mathfrak{X}_T$, for ξ_t yield for some couplets (h,k) and $(\tilde{h},\tilde{k})$:

$$\hat{s}_T^2(i,j) = TE\left[\Delta\hat{g}_T^{(dw)}(i)\Delta\hat{g}_T^{(dw)}(j)|\mathfrak{X}_T\right]$$
$$= TE\left[\frac{1}{T}\sum_{l=1}^{(T-h)/b_T}\xi_l\mathfrak{E}_{l,T}(h,k;\hat{g}(h,k))\frac{1}{T}\sum_{m=1}^{(T-\tilde{h})/b_T}\xi_m\mathfrak{E}_{m,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k}))|\mathfrak{X}_T\right]$$
$$= \frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\mathfrak{E}_{l,T}(h,k;\hat{g}(h,k))\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k})).$$

Similarly:

$$\mathring{s}_T^2(i,j) = \frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\mathfrak{E}_{l,T}(h,k;g_T(h,k))\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_T(\tilde{h},\tilde{k})).$$

Now observe for any (i,j) and some associated couplets (h,k) and $(\tilde{h},\tilde{k})$:

$$\left|\hat{s}_T^2(i,j) - \mathring{s}_T^2(i,j)\right|$$
$$\leq \left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\left\{\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k})) - \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_T(\tilde{h},\tilde{k}))\right\}\right.$$
$$\left.\times\left\{\mathfrak{E}_{l,T}(h,k;\hat{g}(h,k)) - \mathfrak{E}_{l,T}(h,k;g_T(h,k))\right\}\right|$$
$$+\left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_T(\tilde{h},\tilde{k}))\left\{\mathfrak{E}_{l,T}(h,k;\hat{g}(h,k)) - \mathfrak{E}_{l,T}(h,k;g_T(h,k))\right\}\right|$$
$$+\left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\mathfrak{E}_{l,T}(h,k;g_T(h,k))\left\{\mathfrak{E}_{l,T}(\tilde{h},\tilde{k};\hat{g}(\tilde{h},\tilde{k})) - \mathfrak{E}_{l,T}(\tilde{h},\tilde{k};g_T(\tilde{h},\tilde{k}))\right\}\right|$$
$$= \mathcal{S}_{1,T}(h,k,\tilde{h},\tilde{k}) + \mathcal{S}_{2,T}(h,k,\tilde{h},\tilde{k}) + \mathcal{S}_{3,T}(h,k,\tilde{h},\tilde{k}).$$

It follows $\Delta_T = O_p(1/T^\iota)$ for some tiny $\iota > 0$ if we show each:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathcal{S}_{i,T}(h,k,\tilde{h},\tilde{k})\right| = O_p\left(1/T^\iota\right). \tag{D.9}$$

Consider $\mathcal{S}_{2,T}(\cdot)$; $\mathcal{S}_{1,T}(\cdot)$ and $\mathcal{S}_{3,T}(\cdot)$ are similar. Use

$$\{X_tX_{t+h}B_k(t) - \hat{g}(h,k)\} - \{X_tX_{t+h}B_k(t) - g_T(h,k)\} = -\{\hat{g}(h,k) - g_T(h,k)\}$$

with Lemma D.1 to yield:

$$\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\mathcal{S}_{2,T}(h,k,\tilde{h},\tilde{k})\right|$$
$$\leq \max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{t=(l-1)b_T+1}^{lb_T}\left\{X_tX_{t+h}B_{\tilde{k}}(t) - g_T(h,k)\right\}\right| \times b_T\max_{\mathcal{H}_T,\mathcal{K}_T}|\hat{g}(h,k) - g_T(h,k)|$$

$$= \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T} \sum_{t=1}^{T-h\vee\tilde{h}} \{X_t X_{t+h} - E[X_t X_{t+h}]\} B_k(t) \right| \times O_p(b_T/\sqrt{T}).$$

Moreover, by the same argument used to prove (see eq. (A.4) in the main paper):

$$\sup_{z\geq 0} \left| P\left(\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} |\mathcal{Z}_T(i)| \leq z \right) - P\left(\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} |\mathbf{Z}_T(i)| \leq z \right) \right| \to 0,$$

we have for any $\{\mathcal{H}_T\}$, $0 \leq \mathcal{H}_T < T-1$, $\mathcal{H}_T = o(T)$, $\mathcal{K}_T = o(T^\kappa)$ for some finite $\kappa > 0$ and $\eta(\mathcal{K}_T) = o(\sqrt{T})$, provided (D.2) holds:

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T} \sum_{t=1}^{T-h\vee\tilde{h}} \left\{ X_t X_{t+\tilde{h}} - E\left[X_t X_{t+\tilde{h}}\right] \right\} B_{\tilde{k}}(t) \right| = O_p(1/\sqrt{T}).$$

Therefore

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \mathcal{S}_{2,T}(h,k,\tilde{h},\tilde{k}) \right| = O_p(b_T/T) = o_p(1/T^\iota)$$

given $b_T = o(T^{1-\iota})$, proving (D.9).

Claim (b). Now let $\{\mathring{\mathbf{Z}}(h,k) : 0 \leq h \leq \mathcal{H}_T, 1 \leq k \leq \mathcal{K}_T\}$ be an independent copy of the Lemma 3.1 law $\mathbf{Z}(h,k) \sim N(0, \lim_{T\to\infty} \sigma_T^2(h,k))$, independent of the asymptotic draw $\{X_t\}_{t=1}^\infty$, where

$$\sigma_T^2(h,k) = \frac{1}{T} \sum_{s,t=1}^{T-h} E\left[z_s(h,k) z_t(h,k)\right].$$

Let $[\mathring{\mathbf{Z}}(i)]_{i=0}^{\mathcal{H}_T\mathcal{K}_T}$ be the stacked version, cf. (D.7) and footnote 1, and define

$$v^2(i,j) \equiv E[\mathring{\mathbf{Z}}(i)\mathring{\mathbf{Z}}(j)],$$

hence $v^2(i,i) \equiv \lim_{T\to\infty} \sigma_T^2(h,k)$ with $i = (k-1)\mathcal{H}_T + h$. We prove below

$$\tilde{\mathcal{E}}_T \equiv \sup_{c>0} \left| P\left(\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} \left| \mathring{\mathbf{Z}}_T(i) \right| \leq c | \mathfrak{X}_T \right) - P\left(\max_{0\leq i\leq \mathcal{H}_T\mathcal{K}_T} \left| \mathring{\mathbf{Z}}(i) \right| \leq c \right) \right| \xrightarrow{p} 0. \tag{D.10}$$

Together Claim (a) with (D.10) yield

$$\sup_{c>0} \left| P\left(\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k) \right| \leq c | \mathfrak{X}_T \right) - P\left(\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \mathring{\mathbf{Z}}(h,k) \right| \leq c \right) \right| \xrightarrow{p} 0.$$

Therefore

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k) \right| \xrightarrow{d} \max_{h,k\in\mathbb{N}} \left| \mathring{\mathbf{Z}}(h,k) \right| \; awp1 \text{ with respect to } \{X_t\}_{t=1}^\infty.$$

This yields as claimed by definition (cf. Giné and Zinn, 1990, Section 3):

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \left| \sqrt{T}\Delta\hat{g}_T^{(dw)}(h,k) \right| \Rightarrow^p \max_{h,k\in\mathbb{N}} \left| \mathring{\mathbf{Z}}(h,k) \right|.$$

We now prove (D.10). With $\mathring{s}_T^2(i,j) = TE[\Delta g_T^*(i)\Delta g_T^*(j)|\mathfrak{X}_T]$ and $v^2(i,j)$ define

$$\tilde{\Delta}_T \equiv \max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T} \left|\mathring{s}_T^2(i,j) - v^2(i,j)\right|.$$

As above $\tilde{\mathcal{E}}_T = O_p(\tilde{\Delta}_T^{1/3} \times \max\{1, \ln(\mathcal{H}_T\mathcal{K}_T/\tilde{\Delta}_T)\}^{2/3})$. The proof is complete if we show

$$\tilde{\Delta}_T = O(1/T^\iota) \text{ for some } \iota > 0, \tag{D.11}$$

since then

$$\tilde{\mathcal{E}}_T = O_p\left(\tilde{\Delta}_T^{1/3} \max\left\{1, \ln(\mathcal{H}_T\mathcal{K}_T/\tilde{\Delta}_T)\right\}^{2/3}\right) = O_p\left(T^{-\iota/3}\left\{\ln(T)\right\}^{2/3}\right) \xrightarrow{p} 0.$$

We now prove (D.11). Define

$$\Delta\ddot{g}_T^*(h,k) \equiv \frac{1}{T}\sum_{t=1}^{T-h}\varphi_t\left\{X_tX_{t+h}B_k(t) - \frac{1}{T-h}\sum_{s=1}^{T-h}E\left[X_sX_{s+h}\right]B_k(s)\right\},$$

and let $\Delta\ddot{g}_T^*(i)$ stack $\Delta\ddot{g}_T^*(h,k)$. Define

$$\mathring{s}_T^2(i,j) = TE\left[\Delta g_T^*(i)\Delta g_T^*(j)|\mathfrak{X}_T\right]$$

$$\ddot{s}_T^2(i,j) = TE\left[\Delta\ddot{g}_T^*(i)\Delta\ddot{g}_T^*(j)|\mathfrak{X}_T\right]$$

$$s_T^2(i,j) = TE\left[\Delta\ddot{g}_T^*(i)\Delta\ddot{g}_T^*(j)\right]$$

$$s^2(i,j) = \lim_{T\to\infty} TE\left[\Delta\ddot{g}_T^*(i)\Delta\ddot{g}_T^*(j)\right].$$

We prove (D.11) by showing in order:

$$\max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T}\left|\mathring{s}_T^2(i,j) - \ddot{s}_T^2(i,j)\right| = O_p\left(T^{-\iota}\right) \tag{D.12}$$

$$\max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T}\left|\ddot{s}_T^2(i,j) - s_T^2(i,j)\right| = O_p(T^{-\iota}) \tag{D.13}$$

$$\max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T}\left|s_T^2(i,j) - s^2(i,j)\right| = O(T^{-\iota}) \tag{D.14}$$

$$\max_{0\leq i,j\leq \mathcal{H}_T\mathcal{K}_T}\left|s^2(i,j) - v^2(i,j)\right| = O(T^{-\iota}). \tag{D.15}$$

Step 1 ($\mathring{s}_T^2(i,j)$, $\ddot{s}_T^2(i,j)$). Recall $g_T(h,k) \equiv 1/T\sum_{t=1}^{T-h}E[X_tX_{t+h}]B_k(t)$. After expanding, and cancelling like terms, we have for any (i,j) and some unique couplet $(h,k;\tilde{h},\tilde{k})$, where $i = (k-1)\mathcal{H}_T +$

h and $j = (k-1)\mathcal{H}_T + h$:

$$\begin{aligned}
&\left|\mathring{s}^2_T(i,j) - \ddot{s}^2_T(i,j)\right| \\
&= \left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}\left\{-g_T(h,k)X_tX_{t+\tilde{h}}B_{\tilde{k}}(t) - g_T(\tilde{h},\tilde{k})X_tX_{t+h}B_k(t)\right.\right. \\
&\quad + g_T(h,k)g_T(\tilde{h},\tilde{k}) + \frac{T}{T-h}g_T(h,k)X_tX_{t+\tilde{h}}B_{\tilde{k}}(t) \\
&\quad \left.\left.+\frac{T}{T-\tilde{h}}g_T(\tilde{h},\tilde{k})X_tX_{t+h}B_k(t) - \frac{T}{T-h}\frac{T}{T-\tilde{h}}g_T(h,k)g_T(\tilde{h},\tilde{k})\right\}\right| \\
&\le \frac{h}{T-h}\left|g_T(h,k)\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}X_tX_{t+\tilde{h}}B_{\tilde{k}}(t)\right| \\
&\quad + \frac{\tilde{h}}{T-\tilde{h}}\left|g_T(\tilde{h},\tilde{k})\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}X_tX_{t+h}B_k(t)\right| \\
&\quad + \frac{T(h+\tilde{h})+h\tilde{h}}{(T-h)\left(T-\tilde{h}\right)}\left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}g_T(h,k)g_T(\tilde{h},\tilde{k})\right| \\
&= \mathcal{D}_{T,1}(h,k;\tilde{h},\tilde{k}) + \mathcal{D}_{T,2}(h,k;\tilde{h},\tilde{k}) + \mathcal{D}_{T,3}(h,k;\tilde{h},\tilde{k})
\end{aligned}$$

Now twice wield the fact that uniform exponential tails Assumption 1.b implies uniform $\mathcal{L}_r$-boundedness for any $r \ge 1$, with uniform law Lemma D.1 and $\mathcal{H}_T = O(T^{1-\iota}/b_T)$, to yield:

$$\begin{aligned}
\max_{\mathcal{H}_T,\mathcal{K}_T}\mathcal{D}_{T,1}(h,k;\tilde{h},\tilde{k}) &\le \max_{\mathcal{H}_T,\mathcal{K}_T}\left\{\frac{h}{T-h}\left|g_T(h,k)\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}X_tX_{t+\tilde{h}}B_{\tilde{k}}(t)\right|\right\} \\
&\le K\frac{\mathcal{H}_T}{T-\mathcal{H}_T}\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}X_tX_{t+\tilde{h}}B_{\tilde{k}}(t)\right| \\
&= K\frac{\mathcal{H}_T}{T-\mathcal{H}_T}\max_{\mathcal{H}_T,\mathcal{K}_T}\left|\frac{1}{T/b_T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{t=(l-1)b_T+1}^{lb_T}X_tX_{t+\tilde{h}}B_{\tilde{k}}(t)\right| \\
&= O_p\left(\frac{\mathcal{H}_T}{T-\mathcal{H}_T}\right) = O_p\left(\frac{\mathcal{H}_T}{T}\right) = O_p\left(\frac{1}{T^\iota b_T}\right) = o_p\left(1/T^\iota\right).
\end{aligned}$$

Similarly, $\max_{\mathcal{H}_T,\mathcal{K}_T}\mathcal{D}_{T,2}(h,k;\tilde{h},\tilde{k}) = o_p\left(1/T^\iota\right)$. Furthermore, use for any $h \vee \tilde{h} \in \{1,\ldots,\mathcal{H}_T\}$:

$$\left|\frac{1}{T}\sum_{l=1}^{(T-h\vee\tilde{h})/b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T}g_T(h,k)g_T(\tilde{h},\tilde{k})\right| \le b_T\left|g_T(h,k)g_T(\tilde{h},\tilde{k})\right|$$

with $\mathcal{H}_T = O(T^{1-\iota}/b_T)$ to arrive at:

$$\max_{\mathcal{H}_T,\mathcal{K}_T} \mathcal{D}_{T,3}(h,k;\tilde{h},\tilde{k}) \leq \max_{\mathcal{H}_T,\mathcal{K}_T} \left\{ \frac{T(h+\tilde{h})+h\tilde{h}}{(T-h)(T-\tilde{h})} \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} g_T(h,k) g_T(\tilde{h},\tilde{k}) \right| \right\}$$

$$\leq b_T \frac{2T\mathcal{H}_T + \mathcal{H}_T^2}{(T-\mathcal{H}_T)^2} \leq K \frac{b_T \mathcal{H}_T}{T} (1+o(1)) = O\left(T^{-\iota}\right),$$

proving (D.12).

Step 2 ($\ddot{s}_T^2(i,j)$, $s_T^2(i,j)$). Write

$$\ddot{g}_T(h,k) \equiv \frac{1}{T-h} \sum_{t=1}^{T-h} E\left[X_t X_{t+h}\right] B_k(t).$$

For some unique couplet $(h,k;\tilde{h},\tilde{k})$ with $i = (k-1)\mathcal{H}_T + h$ and $j = (k-1)\mathcal{H}_T + h$, expand terms in $\ddot{s}_T^2(i,j)$, and use

$$\frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} = \left(1 - h \vee \tilde{h}/T\right) b_T$$

to deduce:

$$\ddot{s}_T^2(i,j) = \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} X_s X_{s+h} X_t X_{t+\tilde{h}} B_k(s) B_k(t) \tag{D.16}$$

$$- \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s=(l-1)b_T+1}^{lb_T} X_s X_{s+h} B_k(s) \times \ddot{g}_T(\tilde{h},\tilde{k})$$

$$- \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{t=(l-1)b_T+1}^{lb_T} X_t X_{t+\tilde{h}} B_{\tilde{k}}(t) \times \ddot{g}_T(h,k)$$

$$+ (1 - \{h \vee \tilde{h}\}/T) b_T \ddot{g}_T(h,k) \ddot{g}_T(\tilde{h},\tilde{k}).$$

Now use $s_T^2(i,j) = E[\ddot{s}_T^2(i,j)]$ to obtain:

$$\max_{0 \leq i,j \leq \mathcal{H}_T \mathcal{K}_T} \left| \ddot{s}_T^2(i,j) - s_T^2(i,j) \right| \leq \mathcal{D}_{1,T} + \mathcal{D}_{2,T},$$

where

$$\mathcal{D}_{1,T} = \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} \left\{ X_s X_{s+h} X_t X_{t+\tilde{h}} - E\left[X_s X_{s+h} X_t X_{t+\tilde{h}}\right] \right\} B_k(s) B_{\tilde{k}}(t) \right|$$

$$\mathcal{D}_{2,T} = 2 \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s=(l-1)b_T+1}^{lb_T} \left\{ X_s X_{s+h} - E\left[X_s X_{s+h}\right] \right\} B_k(s) \times \ddot{g}_T(\tilde{h},\tilde{k}) \right|.$$

Consider $\mathcal{D}_{1,T}$ and write

$$\mathfrak{X}_{T,l}(h,k) \equiv \frac{1}{\sqrt{b_T}} \sum_{t=(l-1)b_T+1}^{lb_T} X_t X_{t+h} B_k(t) \text{ and } \mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k}) \equiv \mathfrak{X}_{T,l}(h,k)\mathfrak{X}_{T,l}(\tilde{h},\tilde{k}),$$

hence:

$$\mathcal{D}_{1,T} = \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \left(\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k}) - E\left[\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k})\right]\right) \right|.$$

Let $[\boldsymbol{Y}_{T,l}(i,j)]_{i,j=0}^{\mathcal{H}_T\mathcal{K}_T}$ stack $\mathfrak{Y}_{T,l}(h,k;\tilde{h},\tilde{k})$, with correspondence $i = (k-1)\mathcal{H}_T + h$ and $j = (\tilde{k}-1)\mathcal{H}_T + \tilde{h}$. Similarly $[\mathring{\boldsymbol{Y}}_{T,l}(l)]_{l=0}^{\mathcal{H}_T^2\mathcal{K}_T^2}$ stacks $[\boldsymbol{Y}_{T,l}(i,j)]_{i,j=0}^{\mathcal{H}_T\mathcal{K}_T}$ with $l = (j-1)\mathcal{H}_T\mathcal{K}_T + i$. Hence

$$\mathcal{D}_{1,T} = \max_{0\leq l\leq\mathcal{H}_T^2\mathcal{K}_T^2} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \left(\mathring{\boldsymbol{Y}}_{T,l}(l) - E\left[\mathring{\boldsymbol{Y}}_{T,l}(l)\right]\right) \right|.$$

We show below that $\mathring{\boldsymbol{Y}}_{T,l}(l)$ satisfies Conditions 1-3 in Chang, Chen and Wu (2021). Hence, similar to (A.4)-(A.6) in the main paper, $\mathcal{D}_{1,T} = O_p(b_T^{1/2}/T^{1/2}) = o_p(1)$ provided

$$\frac{1}{T^{1/9}}\left[\mathcal{E}_T^{2/3}\left\{\ln\left(\mathcal{H}_T\mathcal{K}_T\right)\right\}^{(1+2\phi)/(3\phi)} + \mathcal{E}_T\left(\ln\mathcal{H}_T\mathcal{K}_T\right)^{7/6}\right] \to 0$$
$$(\ln(\mathcal{H}_T\mathcal{K}_T))^{3-\phi} = o(T^{3\phi}).$$

The latter hold since $\mathcal{H}_T = O(T^{1-\iota}/b_T)$, $b_T/T^{\iota} \to \infty$, and $\mathcal{K}_T = o(T^{\kappa})$ for some finite $\kappa > 0$, and therefore $\mathcal{E}_T = o(T^{1/6}/\{\ln(T)\}^{(1+2\phi)/(2\phi)})$. Now $b_T = o(T^{1-\iota})$ yields $\mathcal{D}_{1,T} = o_p(1/T^{\iota})$ for some $\iota > 0$.

We now show $\mathring{\boldsymbol{Y}}_{T,l}(l)$ satisfies Chang, Chen and Wu's (2021) Conditions 1-3. For Condition 1, Bonferroni's inequality and Lemma A.2 yield

$$\max_{0\leq l\leq\mathcal{H}_T^2\mathcal{K}_T^2} P\left(\left|\mathring{\boldsymbol{Y}}_{T,l}(l)\right| > c\right) = \max_{\mathcal{H}_T,\mathcal{K}_T} P\left(\left|\frac{1}{b_T}\sum_{s,t=(l-1)b_T+1}^{lb_T} X_s X_{t+h} X_t X_{t+\tilde{h}} B_k(s) B_{\tilde{k}}(t)\right| > c\right)$$
$$\leq b_T^2 \max_{\mathcal{H}_T} \max_{1\leq t\leq T-h} P\left(\left|X_s X_{t+h} X_t X_{t+\tilde{h}}\right| > b_T c\right) \leq 2\varpi b_T^2 \exp\left\{-\frac{b_T^{\vartheta_2}}{\mathcal{E}_T^{\vartheta_2}} c^{\vartheta_2}\right\}.$$

Use $b_T/T^{\iota} \to \infty$ by supposition to deduce for any $c > 0$ $\exists\mathcal{T} \in \mathbb{N}$ such that

$$b_T^2 \exp\{-b_T^{\vartheta_2/2}\mathcal{E}_T^{-\vartheta_2}c^{\vartheta_2}\} \leq \exp\{-c^{\vartheta_2}\mathcal{E}_T^{-\vartheta_2}\}\ \forall T \geq \mathcal{T}.$$

Hence Condition 1 holds:

$$\max_{0\leq l\leq\mathcal{H}_T^2\mathcal{K}_T^2} P\left(\left|\mathring{\boldsymbol{Y}}_{T,l}(l)\right| > c\right) \leq \tilde{\varpi}\exp\{-c^{\vartheta_2}\mathcal{E}_T^{-\vartheta_2}\}\ \forall T \geq \mathcal{T} \text{ and some } \tilde{\varpi} \geq 2.$$

Condition 2 holds by Assumption 1.a and measurability. Condition 3 holds by Assumption 2.a(i).

For $\mathcal{D}_{2,T}$, use Lemma D.1, and $b_T = O(T^{1/2-\iota})$ under Assumption 2.b, to get:

$$\mathcal{D}_{2,T} \leq K \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T/b_T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s=(l-1)b_T+1}^{lb_T} \{X_s X_{s+h} - E[X_s X_{s+h}]\} B_k(s) \right|$$

$$= K b_T \max_{\mathcal{H}_T,\mathcal{K}_T} \left| \frac{1}{T} \sum_{t=1}^{T-h\vee\tilde{h}} \{X_t X_{t+h} - E[X_t X_{t+h}]\} B_k(t) \right| = O_p\left(b_T/T^{1/2}\right) = O_p\left(T^{-\iota}\right).$$

Step 3 ($s_T^2(i,j)$, $s^2(i,j)$). The property holds by Assumption 2.a(ii).

Step 4 ($s^2(i,j), v^2(i,j)$). For some $(h,k;\tilde{h},\tilde{k})$, $s^2(i,j)$ is identically

$$\lim_{T\to\infty} \frac{1}{T} \sum_{l=1}^{(T-h\vee\tilde{h})/b_T} \sum_{s,t=(l-1)b_T+1}^{lb_T} E\left[\left\{X_s X_{s+h} B_k(s) - \frac{1}{T-h}\sum_{u=1}^{T-h} E[X_u X_{u+h}] B_k(u)\right\}\right.$$
$$\left. \times \left\{X_t X_{t+\tilde{h}} B_{\tilde{k}}(t) - \frac{1}{T-\tilde{h}}\sum_{u=1}^{T-\tilde{h}} E\left[X_u X_{u+\tilde{h}}\right] B_{\tilde{k}}(u)\right\}\right]$$

and by rearranging terms

$$v^2(i,j) = \lim_{T\to\infty} \frac{1}{T} \sum_{s,t=1}^{T-h\vee\tilde{h}} E\left[\left\{X_s X_{s+h} B_k(s) - \frac{1}{T-h}\sum_{u=1}^{T-h} E[X_u X_{u+h}] B_k(u)\right\}\right.$$
$$\left. \times \left\{X_t X_{t+\tilde{h}} B_{\tilde{k}}(t) - \frac{1}{T-\tilde{h}}\sum_{u=1}^{T-\tilde{h}} E\left[X_u X_{u+\tilde{h}}\right] B_{\tilde{k}}(u)\right\}\right].$$

Further, block size $b_T \to \infty$. Hence $s^2(i,j) = v^2(i,j)\ \forall i,j$. This completes the proof. $\mathcal{QED}$.

Remark 4. We technically only need the iid random numbers $\{\xi_1,\ldots,\xi_{\mathcal{N}_T}\}$ to satisfy $E[\xi_i] = 0$, $E[\xi_i^2] = 1$, and $E[\xi_i^4] < \infty$. In this general setting $\sqrt{T}\Delta\hat{g}_T^{(dw)}(i)|\mathfrak{X}_T$ is not necessarily normally distributed, hence the Gaussian-to-Gaussian result (D.8) may not hold. We will need the added step:

$$\sup_{c>0} \left| P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T} \left|\sqrt{T}\Delta\hat{g}_T^{(dw)}(i)\right| \leq c|\mathfrak{X}_T\right) - P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T} \left|\sqrt{T}\Delta\mathring{\boldsymbol{g}}_T(i)|\mathfrak{X}_T\right| \leq c\right)\right| \xrightarrow{p} 0$$

where $\sqrt{T}\Delta\mathring{\boldsymbol{g}}_T(i)|\mathfrak{X}_T \sim N(0, TE[\Delta\hat{g}_T^{(dw)}(i)^2|\mathfrak{X}_T])$. We would then need to alter (D.8), and prove instead

$$\mathcal{E}_T \equiv \sup_{c>0} \left| P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T} \left|\sqrt{T}\Delta\mathring{\boldsymbol{g}}_T(i)\right| \leq c|\mathfrak{X}_T\right) - P\left(\max_{0\leq i\leq\mathcal{H}_T\mathcal{K}_T} \left|\mathring{\boldsymbol{Z}}_T(i)\right| \leq c\right)\right|$$
$$= O_p\left(\Delta_T^{1/3} \max\{1, \ln(\mathcal{H}_T\mathcal{K}_T/\Delta_T)\}^{2/3}\right) \xrightarrow{p} 0.$$

E. Empirical study

We now apply our test and the test in Jin, Wang and Wang (2015) to quarterly international (ex post) real interest rates. We analyze 16 countries over the period 1960.Q1 - 2019.Q4. The data were collected from the U.S. Federal Reserve Bank data archive (FRED), which itself is taken from the OECD data archives. The countries are Australia, Austria, Belgium, Canada, Denmark, France, Germany, Ireland Italy, Japan, Netherlands, Norway, Switzerland, UK and US.

Following Rapach and Weber (2004), we use the 10-year government bond yield as our measure of the nominal interest rate $r_{n,t}$, and the Consumer Price Index in order to compute inflation i_t. The (ex post) real bond rate is $r_{r,t} = r_{n,t} - i_t$. See Table A.1 for the exact date range available for each series and subsequent size. Figure 1 contains plots of each series.

Unit root tests have been proposed as a standard for testing for non-stationarity in interest rates. See, e.g., Rose (1988) and Rapach and Weber (2004) and their historical references. In that framework, it is implicitly assumed that real interest rates are unbounded (asymptotically with probability approaching one), in particular if a unit root is present. In the case of a unit root, of course, variance is unbounded asymptotically, and α-mixing fails to hold.

Testing real interest rates is complicated by the fact that nominal rates $r_{n,t}$ and inflation i_t may be nonstationary while real rates $r_{r,t} = r_{n,t} - i_t$ can yet be stationary. In a unit root test setting, it is possible that $r_{n,t} \sim I(1)$ and $i_t \sim I(1)$ yet $(r_{n,t}, i_t)$ are cointegrated with integrating vector $[-1, 1]$, hence $r_{r,t}$ are stationary. Conversely, nonstationarity necessarily exists when just $r_{r,t} \sim I(1)$ or just $i_t \sim I(1)$. Rose (1988) finds the latter for each country in our study based on quarterly post-war data and conventional unit root tests, hence Rose (1988) broadly concludes unit root nonstationarity. Rapach and Weber (2004) obtain more nuanced results. They find nonstationarity in nominal rates for all countries except Germany and Switzerland, and mixed results for inflation based on Phillips and Perron (1988) and Ng and Perron (1997, 2001) unit root tests. In order to handle the evident cases $r_{n,t} \sim I(1)$ and $i_t \sim I(1)$ they apply several cointegration tests, including tests by Ng and Perron (2001) and one eventually published in Perron and Rodriguez (2016).

A different approach for studying structural time variation in interest rates couches rates in a parametric regime switching regression model. See, e.g., Garcia and Perron (1996), Bekdache (1000), and Ang and Bekaert (2002). See also Teräsvirtra (1994) and Gray (1996).

In our setting, under either hypothesis we assume a moment generating function exists uniformly over t, and a geometric mixing condition holds. Thus, we implicitly assume a unit root does not exist. The moment conditions can be assured simply by assuming nominal interest rates and inflation are bounded. This is a fairly natural assumption empirically for interest rates which are typically managed by government market actions, and lie in the range $[-1, 1]$. In any case, in our sample range bond yields and inflation never surpass the total range $[-.02, .30]$. We therefore test for a (non-unit root based) deviation from covariance stationarity. Our setting of course is nonparametric: we do not need to specify a (switching) regression model (e.g. Augmented Dickey Fuller, or Markov Switching), and indeed our test is relevant irrespective of any underlying parametric features.

We report test results for the max-test based on a dependent wild bootstrap, and Jin, Wang and Wang's (2015) test based both on simulated critical values and dependent wild bootstrap. Both tests exploit a Walsh basis in view of simulation evidence suggesting the inferiority of the composite Haar basis. We simulate critical values for each series and each country (hence, 54 simulated sets of critical values), rather than for each sample size. We use $\mathcal{H}_T = [2T^{.49}]$ and $\mathcal{K}_T = [.5T^{.49}]$. See Table A.2 for test results. Tests are performed on nominal and real bond yields, and inflation, but we focus our discussion on real bond yields given is importance in the literature.

Consider the max-correlation difference test. In all countries except one, when the test finds evidence of non-covariance stationarity in nominal rates, the same result applies for real rates. Consider Italy:

the p-values are .024 and .032 for nominal and real rates respectively, while the p-value for inflation is .216. Thus, nominal rates are the driving force for non-stationarity. New Zealand is the sole exception: p-values for nominal and real rates and inflation are .156, .080 and .162. Thus, we reject stationarity at the 10% level for real rates, but *fail to reject* for nominal rates and inflation. It is easily verified, however, that if random variables X_t and Y_t are covariance stationary then so is any linear combination. A deeper study into this is left for future work.

The bootstrapped JWW test, on par with the Monte Carlo study, almost never leads to a rejection of the covariance stationarity null hypothesis. Tests based on simulated critical values, however, match across nominal and real bond yields, with four exceptions: Belgium, Japan, New Zealand and the UK. The JWW test generally yields strong rejections (well under the 1% level) when nonstationarity is detected, while the max-correlation test is more moderate, with rejections variously at the 1%, 5%, and 10% levels.

Finally, in five countries the max-correlation test and JWW test disagree: Australia, France, Italy, New Zealand and Switzerland (denoted by bold in Table A.2). In the first four the max-correlation difference test yielded rejections of covariance stationarity (p-values are .056, .022, .032, and .080), while the JWW test failed to reject. The JWW test with simulated critical value detected non-stationarity for Switzerland at the 1% level ($\hat{\mathcal{D}}_T = 58.1$, 1% c.v. = 7.9), but the max-correlation test did not at the 10% (p-value .144).

Table A.1. Dates and Sample Sizes

	Nominal Bond r_n		Inflation i		Real Bond r_r	
	Dates	n	Dates	n	Dates	n
Australia	1969.Q3-2021.Q4	210	1960.Q2-2021.Q4	246	1969.Q3-2021.Q4	210
Austria	1990.Q1-2021.Q4	128	1960.Q2-2021.Q4	246	1990.Q1-2021.Q4	128
Belguim	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
Canada	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	244	1960.Q2-2021.Q4	246
Denmark	1987.Q1-2021.Q4	140	1967.Q2-2021.Q4	218	1987.Q1-2021.Q4	140
France	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
Germany	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
Ireland	1971.Q1-2021.Q4	204	1976.Q2-2021.Q4	182	1976.Q2-2021.Q4	182
Italy	1991.Q2-2021.Q4	122	1960.Q2-2021.Q4	246	1991.Q2-2021.Q4	122
Japan	1989.Q1-2021.Q4	132	1960.Q2-2021.Q4	246	1989.Q1-2021.Q4	132
Netherlands	1960.Q1-2021.Q4	248	1960.Q3-2021.Q4	246	1960.Q3-2021.Q4	246
New Zealand	1970.Q1-2021.Q4	208	1960.Q2-2021.Q4	246	1970.Q1-2021.Q4	208
Norway	1985.Q1-2021.Q4	148	1960.Q2-2021.Q4	246	1985.Q1-2021.Q4	148
Switzerlnad	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
UK	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246
US	1960.Q1-2021.Q4	248	1960.Q2-2021.Q4	246	1960.Q2-2021.Q4	246

Nominal bond r_n are 10 year government bond yields; inflation i is derived from the Consumer Price Index for all goods and services; and real bond yields $r_r = r_n - i$.

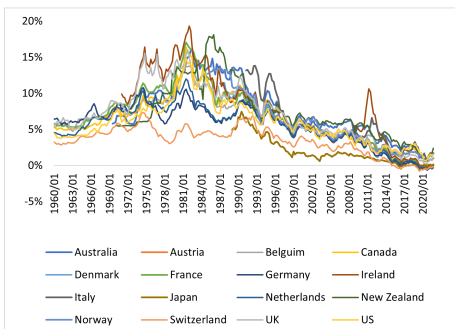
Table A.2. Empirical Study: Covariance Stationarity Tests

	Nominal Bond r_n			Inflation i			Real Bond r_r		
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
Australia	.080	65.5 (3.5, 4.8, 7.5) ***	.729	.174	5.57 (3.8, 5.1, 7.9) **	.605	**.056**	**-2.28 (3.5, 4.8, 7.5)**	**.854**
Austria	.002	12.3 (2.9, 4.9, 6.6) ***	.198	.158	62.7 (3.8, 5.1, 7.9) ***	.134	.000	352 (2.9, 4.1, 6.6) ***	.024
Belguim	.032	2.03 (3.8, 5.1, 7.9)	.876	.236	9.90 (3.8, 5.1, 8.0) ***	.537	.023	44 (3.8, 5.1, 7.9) ***	.919
Canada	.014	24.1 (3.8, 5.1, 7.9) ***	.904	.158	10.4 (3.8, 5.1, 7.9) ***	.361	.018	17.7 (3.8, 5.1, 7.9) ***	.756
Denmark	.000	241 (3.0, 4.1, 6.7) ***	.246	.066	29.1 (3.6, 4.8, 7.6) ***	.319	.000	217 (3.0, 4.1, 6.6) ***	.273
France	.020	-.543 (3.8, 5.1, 7.9)	.661	.174	2.27 (3.8, 5.1, 7.9)	.541	**.022**	**-2.00 (3.8, 5.1, 7.9)**	**.866**
Germany	.180	28.9 (3.8, 5.1, 7.9) ***	.858	.046	109 (3.8, 5.1, 7.9) ***	.170	.090	879 (3.8, 5.1, 7.9) ***	.399
Ireland	.101	6.12 (3.4, 4.7, 7.5) **	.998	.242	30.6 (3.4, 4.6, 7.2) ***	.248	.012	161 (3.4, 4.6, 7.2) ***	.563
Italy	.024	1.92 (2.9, 4.0, 6.5)	.246	.216	218 (3.8, 5.1, 7.8) ***	.076	**.032**	**1.80 (2.9, 4.0, 6.6)**	**.836**
Japan	.054	1.43 (2.9, 4.1, 6.6)	.331	.331	3.34 (3.8, 5.1, 7.9)	.473	.014	92.7 (2.9, 4.0, 6.6) ***	.581
Netherlands	.068	284 (3.8, 5.1, 7.9) ***	.585	.114	12.9 (3.8, 5.1, 7.9) ***	.251	.026	184 (3.8, 5.1, 7.9) ***	.394
New Zealand	.156	11.0 (3.5, 4.8, 7.5) ***	.820	.162	2.13 (3.8, 5.1, 7.9)	.819	**.080**	**-2.86 (3.5, 4.8, 7.5)**	**.982**
Norway	.014	63.2 (3.0, 4.2, 6.8) ***	.273	.042	-3.49 (3.8, 5.1, 7.9)	.719	.006	23.7 (3.0, 4.1, 6.8) ***	.102
Switzerlnad	.136	853 (3.8, 5.1, 7.9) ***	.345	.265	16.2 (3.8, 5.1, 7.9) ***	.371	**.144**	**58.1 (3.8, 5.1, 7.9) ***	**.334**
UK	.032	-2.02 (3.8, 5.1, 7.9)	.994	.222	54.2 (3.8, 5.1, 7.9) ***	.699	.006	21.5 (3.8, 5.1, 7.9) ***	.890
US	.036	555 (3.8, 5.1, 7.9) ***	.647	.124	9.49 (3.8, 5.1, 7.9) ***	.307	.024	652 (3.8, 5.1, 7.9) ***	.222

$\hat{\mathcal{M}}_T$ is the proposed max-test based on a bootstrapped p-value: reported values are p-values computed by dependent wild bootstrap. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, shown in parentheses: *, **, *** denote rejection at the 10%, 5% and 1% levels. $\hat{\mathcal{D}}_T^{dw}$ is JWW's test based dependent wild bootstrapped p-values.

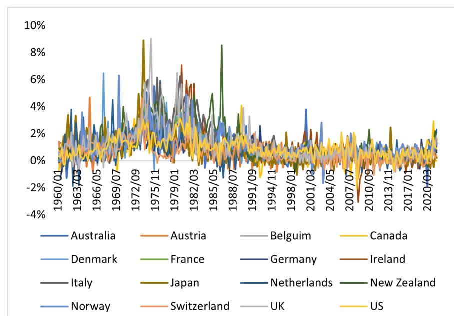
Figure 1: Quarterly Interest Rates and Inflation

(a) Quarterly Interest Rates r_n



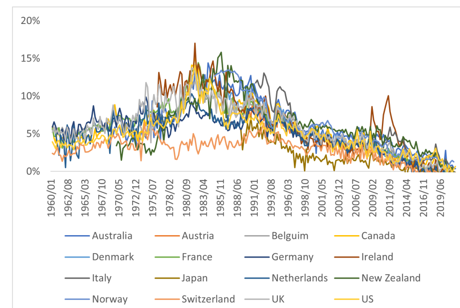
Rates are the nominal 10 year government bond yield.

(b) Quarterly CPI Inflation i



Inflation is computed as the quarterly change in the log-consumer price index.

(c) Real Interest Rates $r_r = r_n - i$



Rates are the nominal 10 year government bond yield minus the inflation rate.

F. Complete simulation results

Table A.3.: Rejection Frequencies under H_0: **Walsh Basis**
Case 1: $\mathcal{H}_T = [\log_2(n)^{.99} - 3.5]$ and $\mathcal{K}_T = [n^{1/3} + .01]$

	$\epsilon_t \overset{iid}{\sim} N(0,1)$							
	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.005, .025, .093	.002, .024, .092	.012, .041, .077	.304, .548, .675	.001, .036, .103	.003, .041, .111	.002, .027, .066	.118, .376, .596
AR(1)	.006, .043, .106	.006, .045, .105	.066, .105, .162	.263, .427, .504	.006, .052, .130	.005, .054, .132	.064, .109, .159	.141, .271, .380
SETAR	.006, .026, .053	.003, .036, .076	.052, .102, .159	.230, .424, .508	.004, .025, .064	.006, .035, .067	.050, .124, .173	.114, .276, .388
GARCH	.004, .038, .099	.000, .029, .098	.014, .045, .096	.249, .504, .635	.002, .056, .158	.004, .055, .158	.004, .030, .091	.094, .349, .547
	$n = 256$				$n = 512$			
MA(1)	.002, .024, .090	.003, .022, .096	.005, .041, .082	.051, .281, .526	.006, .038, .105	.006, .041, .106	.010, .053, .097	.045, .227, .445
AR(1)	.006, .036, .103	.005, .037, .107	.045, .093, .146	.067, .173, .279	.005, .052, .132	.004, .061, .132	.035, .076, .140	.042, .100, .190
SETAR	.004, .034, .069	.005, .033, .063	.034, .099, .178	.052, .171, .269	.004, .032, .078	.003, .036, .079	.024, .100, .177	.024, .092, .163
GARCH	.002, .031, .093	.002, .031, .093	.005, .040, .098	.074, .310, .530	.004, .046, .120	.003, .047, .119	.017, .051, .087	.045, .244, .500
	$\epsilon_t \overset{iid}{\sim} t_5$							
	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.001, .032, .085	.002, .034, .088	.003, .027, .069	.168, .311, .379	.004, .039, .095	.005, .041, .095	.002, .020, .062	.059, .167, .233
AR(1)	.009, .044, .116	.009, .048, .117	.053, .085, .138	.164, .253, .305	.007, .051, .134	.007, .054, .134	.026, .069, .116	.053, .113, .148
SETAR	.004, .019, .050	.004, .021, .053	.032, .080, .131	.143, .249, .292	.004, .013, .035	.001, .013, .039	.025, .080, .158	.042, .107, .142
	$n = 256$				$n = 512$			
MA(1)	.001, .028, .087	.001, .031, .083	.000, .025, .062	.026, .098, .155	.002, .035, .073	.001, .032, .073	.004, .034, .074	.011, .036, .053
AR(1)	.005, .033, .107	.005, .034, .113	.017, .048, .104	.024, .050, .058	.002, .038, .104	.002, .034, .107	.005, .054, .088	.003, .007, .011
SETAR	.000, .005, .032	.000, .005, .035	.038, .129, .210	.013, .041, .057	.000, .018, .047	.000, .018, .048	.065, .175, .274	.002, .005, .013
	$\epsilon_t \sim$ GARCH							
	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.001, .030, .088	.001, .034, .091	.014, .045, .096	.258, .498, .642	.002, .042, .119	.002, .047, .130	0.014, .045, .096	.258, .498, .642
AR(1)	.011, .060, .121	.011, .057, .121	.080, .120, .186	.244, .407, .488	.008, .048, .127	.008, .054, .134	.080, .120, .186	.244, .407, .488
SETAR	.005, .021, .052	.004, .022, .056	.064, .122, .174	.242, .417, .512	.003, .018, .051	.003, .017, .054	.064, .122, .174	.242, .417, .512
	$n = 256$				$n = 512$			
MA(1)	.001, .020, .097	.002, .021, .096	.005, .040, .098	.072, .306, .528	.001, .045, .114	.001, .042, .116	.017, .051, .087	.050, .247, .499
AR(1)	.002, .035, .084	.002, .035, .086	.063, .106, .162	.083, .168, .258	.003, .044, .107	.003, .046, .113	.034, .085, .129	.036, .107, .176
SETAR	.000, .013, .065	.000, .014, .068	.029, .099, .164	.049, .172, .264	.000, .020, .069	.002, .020, .074	.028, .103, .177	.018, .096, .167

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values. The GARCH error is based on an iid $N(0,1)$ innovation.

Table A.4.: Rejection Frequencies under H_0: **Walsh Basis**
Case 2: $\mathcal{H}_T = 2T^{.49}$ and $\mathcal{K}_T = .5T^{.49}$

	$\epsilon_t \overset{iid}{\sim} N(0,1)$, $n=64$: $\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$n=128$: $\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.000, .012, .054	.001. .010, .066	.018, .040, .075	.005, .085, .255	.001, .008, .046	.001, .017, .053	.008, .034, .074	.001, .031, .195
AR(1)	.001, .019, .072	.001, .024, .087	.068, .110, .161	.027, .073, .166	.001, .020, .084	.001, .030, .094	.074, .123, .172	.022, .068, .130
SETAR	.001, .017, .037	.003, .019, .041	.040, .089, .152	.014, .093, .202	.001, .027, .050	.003, .024, .046	.051, .124, .171	.015, .062, .148
GARCH	.000, .021, .095	.000, .025, .109	.014, .047, .103	.004, .077, .247	.003, .044, .113	.004, .047, .128	.008, .034, .085	.000, .023, .156
	$n=256$				$n=512$			
MA(1)	.000, .011, .049	.000, .015, .059	.008, .045, .088	.000, .019, .146	.004, .027, .061	.004, .032, .079	.013, .047, .109	.000, .025, .161
AR(1)	.002, .015, .057	.002, .019, .058	.048, .100, .154	.017, .036, .087	.002, .027, .085	.005, .028, .080	.033, .079, .149	.004, .021, .065
SETAR	.004, .033, .059	.003, .032, .058	.030, .100, .174	.003, .024, .108	.004, .031, .067	.005, .033, .069	.026, .097, .176	.000, .016, .069
GARCH	.003, .033, .095	.004, .032, .098	.007, .037, .086	.001, .031, .173	.003, .035, .108	.002, .035, .116	.017, .045, .083	.001, .030, .159

	$\epsilon_t \overset{iid}{\sim} t_5$, $n=64$: $\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$n=128$: $\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.000, .013, .050	.001, .013, .057	.005, .030, .078	.008, .072, .175	.001, .021, .059	.002, .026, .073	.006, .026, .064	.001, .022, .094
AR(1)	.002, .020, .066	.005, .030, .081	.056, .091, .150	.025, .067, .132	.000, .027, .081	.000, .033, .091	.034, .077, .124	.005, .017, .056
SETAR	.001, .018, .057	.001, .023, .057	.026, .067, .118	.012, .061, .155	.001, .014, .038	.001, .014, .045	.020, .083, .152	.002, .018, .048
	$n=256$				$n=512$			
MA(1)	.000, .017, .060	.000, .019, .067	.003, .028, .064	.001, .016, .061	.002, .021, .080	.003, .028, .086	.006, .033, .070	.000, .006, .026
AR(1)	.003, .014, .049	.003, .018, .052	.018, .048, .093	.002, .013, .023	.003, .027, .087	.003, .034, .094	.007, .049, .090	.000, .001, .005
SETAR	.001, .014, .033	.002, .012, .037	.038, .115, .191	.001, .007, .024	.003, .014, .049	.002, .015, .047	.056, .165, .263	.000, .002, .008

	$\epsilon_t \sim$ GARCH, $n=64$: $\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$n=128$: $\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
MA(1)	.000, .019, .071	.000, .020, .082	.014, .047, .103	.006, .082, .253	.002, .031, .096	.002, .035, .112	.008, .034, .085	.000, .017, .160
AR(1)	.003, .025, .077	.004, .031, .087	.082, .130, .190	.034, .086, .195	.001, .022, .085	.001, .033, .092	.073, .123, .170	.031, .057, .127
SETAR	.004, .019, .057	.004, .022, .063	.063, .122, .179	.028, .094, .226	.001, .014, .050	.001, .016, .056	.040, .106, .155	.01, .041, .108
	$n=256$				$n=512$			
MA(1)	.002, .021, .076	.002, .020, .091	.007, .037, .086	.000, .024, .161	.004, .038, .110	.003, .044, .116	.017, .045, .083	.000, .025, .158
AR(1)	.001, .019, .064	.001, .025, .077	.064, .114, .164	.016, .044, .098	.000, .034, .109	.001, .032, .108	.033, .077, .123	.006, .028, .070
SETAR	.000, .013, .042	.000, .013, .044	.023, .092, .159	.002, .017, .091	.000, .023, .071	.001, .025, .082	.027, .100, .174	.001, .009, .064

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values. The GARCH error is based on an iid $N(0,1)$ innovation.

Table A.5.: a. Rejection Frequencies under H_1: **Walsh Basis**
Case 1: $\mathcal{H}_T = [\log_2(n)^{.99} - 3.5]$ and $\mathcal{K}_T = [n^{1/3} + .01]$
$\boldsymbol{\epsilon_t \sim N(0,1)}$

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.122, ,227, .336	.165, .287, .431	.173, .401, .549	.037, .198, .380	.436, .613, .742	.374, .454, .672	.827, .936, .967	.054, .275, .440
alt-2	.031, .147, .268	.032, .151, .274	.019, .048, .093	.275, .535, .679	.088, .398, .424	.090, .409, .541	.010, .044, .089	.116, .379, .602
alt-3	.021, .053, .201	.023, .038, .165	.173, .409, .555	.095, .336, .490	.354, .475, .630	.342, .436, .599	.434, .696, .809	.030, .225, .413
alt-4	.081, .272, .351	.062, .168, .344	.190, .413, .557	.097, .356, .486	.672, .750, .838	.674, .743, .833	.772, .917, .949	.113, .386, .509
alt-5	.102, .228, .342	.086, .149, .363	.081, .140, .191	.260, .415, .504	.494, .713, .888	.428, .739, .932	.160, .336, .471	.061, .171, .296
alt-6	.024, .081, .159	.020, .061, .162	.036, .084, .141	.204, .394, .508	.118, .244, .377	.121, .256, .352	.042, .131, .241	.070, .264, .415
alt-7	.031, .099, .135	.021, .088, .132	.054, .101, .143	.228, .399, .491	.073, .127, .239	.061, .102, .268	.058, .140, .215	.096, .219, .307
alt-8	.001, .121, .147	.003, .085, .123	.081, .140, .191	.260, .415, .504	.043, .131, .408	.027, .150, .437	.069, .110, .147	.115, .255, .352
alt-9	.050, .074, .103	.030, .047, .114	.016, .046, .099	.303, .562, .699	.067, .081, .180	.049, .098, .214	.002, .025, .063	.108, .364, .586
	$n = 256$				$n = 512$			
alt-1	.794, .931, .987	.754, .914, .978	.977, .997, 1.00	.045, .263, .432	1,00, 1.00, 1.00	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.317, .589, .688
alt-2	.108, .401, .488	.102, .419, .560	.004, .035, .087	.056, .283, .524	.214, .457, .512	.212, .451, .517	.008, .040, .081	.039, .230, .448
alt-3	.720, .817, .954	.710, .804, .947	.948, .986, .993	.154, .405, .510	1.00, 1.00, 1.00	.968, .999, 1.00	.988, .999, 1.00	.059, .284, .436
alt-4	.855, .985, 1.00	.876, .987, 1.00	.941, .983, .994	.094, .356, .463	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.999, 1.00, 1.00	.138, .402, .501
alt-5	.864, .988, .998	.723, .993, .999	.915, .977, .987	.060, .314, .433	1.00, 1.00, 1.00	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.697, .755, .777
alt-6	.259, .320, .533	.263, .338, .556	.069, .207, .296	.044, .213, .360	.681, .810, .904	.605, .829, .914	.153, .357, .469	.037, .234, .393
alt-7	.112, .341, .444	.115, .362, .468	.164, .375, .530	.032, .100, .166	.421, .695, .818	.449, .638, .834	.715, .878, .939	.084, .182, .291
alt-8	.162, .217, .467	.152, .240, .503	.044, .097, .148	.076, .163, .262	.585, .918, .996	.210, .935, .997	.070, .189, .318	.053, .100, .128
alt-9	.162, .218, .360	.141, .230, .368	.009, .032, .088	.065, .276, .515	.285, .466, .620	.188, .468, .606	.009, .050, .103	.041, .218, .461

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.5.: b. Rejection Frequencies under H_1: **Walsh Basis**
Case 1: $\mathcal{H}_T = [\log_2(n)^{.99} - 3.5]$ and $\mathcal{K}_T = [n^{1/3} + .01]$
$\boldsymbol{\epsilon_t \sim t_5}$

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.064, .267, .339	.072, .286, .433	.160, .360, .503	.021, .127, .223	.342, .351, .449	.331, .377, .385	.799, .921, .951	.021, .130, .303
alt-2	.007, .041, .157	.008, .047, .165	.002, .024, .061	.206, .336, .397	.024, .079, .214	.047, .083, .225	.007, .031, .060	.062, .194, .268
alt-3	.034, .066, .141	.031, .065, .124	.173, .386, .513	.051, .206, .276	.124, .297, .365	.117, .274, .340	.413, .643, .753	.015, .114, .207
alt-4	.005, .248, .386	.042, .244, .279	.171, .386, .535	.050, .199, .288	.566, .698, .824	.467, .581, .706	.737, .899, .947	.050, .190, .358
alt-5	.110, .270, .294	.101, .183, .239	.056, .096, .138	.175, .270, .308	.295, .464, .684	.214, .494, .697	.130, .297, .411	.025, .084, .139
alt-6	.020, .087, .119	.014, .066, .120	.021, .050, .091	.147, .247, .297	.094, .162, .231	.082, .151, .234	.031, .099, .182	.025, .103, .157
alt-7	.009, .046, .096	.005, .041, .071	.020, .053, .094	.129, .221, .267	.084, .126, .249	.072, .102, .211	.059, .142, .231	.047, .098, .134
alt-8	.021, .082, .125	.021, .069, .114	.056, .096, .138	.180, .269, .313	.046, .102, .311	.008, .111, .320	.029, .068, .107	.053, .124, .163
alt-9	.002, .025, .084	.002, .024, .094	.007, .032, .067	.203, .361, .426	.039, .077, .178	.013, .087, .201	.001, .025, .070	.071, .171, .247
	$n = 256$				$n = 512$			
alt-1	.624, .763, .846	.689, .734, .821	.978, .996, .998	.011, .116, .373	.905, .950, .976	.920, .945, .971	1.00, 1.00, 1.00	.237, .721, .877
alt-2	.024, .058, .168	.033, .062, .171	.004, .026, .069	.030, .093, .144	.050, .161, .307	.049, .166, .306	.005, .029, .061	.010, .039, .063
alt-3	.612, .725, .847	.805, .796, .823	.935, .981, .991	.035, .239, .468	.890, .984, 1.00	.828, .928, .998	.984, .999, .999	.013, .158, .419
alt-4	.690, .803, .910	.697, .801, .904	.920, .974, .992	.029, .165, .361	.989, .997, 1.00	.928, .974, .985	.997, 1.00, 1.00	.050, .303, .518
alt-5	.858, .878, .944	.802, .868, .947	.888, .961, .983	.018, .127, .288	.934, .977, .987	.938, .977, .987	1.00, 1.00, 1.00	.381, .740, .816
alt-6	.119, .245, .304	.118, .252, .314	.069, .199, .294	.011, .044, .080	.332, .418, .592	.341, .440, .605	.164, .350, .468	.006, .034, .073
alt-7	.092, .217, .372	.084, .207, .297	.182, .388, .515	.007, .038, .099	.481, .505, .768	.498, .425, .690	.697, .882, .935	.029, .129, .266
alt-8	.093, .190, .368	.064, .147, .388	.012, .056, .100	.012, .042, .051	.404, .654, .884	.418, .670, .920	.016, .091, .187	.003, .004, .006
alt-9	.071, .192, .232	.041, .124, .235	.006, .025, .073	.033, .090, .127	.181, .370, .428	.141, .269, .433	.004, .040, .084	.010, .038, .059

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.5.: c. Rejection Frequencies under H_1: **Walsh Basis**
Case 1: $\mathcal{H}_T = [\log_2(n)^{.99} - 3.5]$ and $\mathcal{K}_T = [n^{1/3} + .01]$
$\boldsymbol{\epsilon_t}$ **~ GARCH**

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.122, .247, .357	.154, .317, .459	.177, .376, .533	.037, .206, .369	.312, .435, .507	.409, .412, .573	.845, .946, .972	.048, .243, .408
alt-2	.071, .095, .150	.071, .094, .159	.017, .051, .099	.307, .536, .660	.014, .100, .249	.017, .113, .261	.005, .033, .085	.097, .353, .575
alt-3	.020, .057, .121	.020, .055, .113	.190, .444, .581	.079, .339, .458	.163, .202, .359	.158, .293, .448	.419, .682, .803	.030, .214, .370
alt-4	.033, .227, .337	.020, .126, .335	.188, .424, .564	.089, .341, .468	.229, .308, .513	.227, .307, .505	.779, .911, .955	.111, .360, .505
alt-5	.091, .142, .270	.061, .137, .235	.069, .124, .159	.261, .386, .449	.330, .364, .464	.033, .282, .483	.158, .343, .472	.049, .154, .281
alt-6	.030, .071, .139	.024, .066, .129	.041, .089, .147	.215, .391, .486	.082, .165, .250	.061, .157, .245	.037, .119, .231	.072, .240, .381
alt-7	.041, .087, .127	.041, .076, .130	.047, .103, .165	.216, .375, .476	.043, .111, .207	.033, .118, .216	.055, .120, .201	.077, .173, .261
alt-8	.014, .088, .144	.014, .039, .138	.069, .124, .159	.265, .390, .450	.047, .112, .276	.028, .116, .290	.050, .086, .156	.109, .230, .351
alt-9	.022, .096, .136	.013, .047, .136	.012, .045, .085	.262, .525, .675	.046, .101, .202	.010, .106, .220	.007, .038, .085	.106, .350, .556
	$n = 256$				$n = 512$			
alt-1	.557, .641, .744	.546, .616, .701	.975, .998, .999	.038, .266, .437	.739, .843, .894	.716, .827, .878	1.00, 1.00, 1.00	0.270, .545, .657
alt-2	.021, .176, .276	.027, .182, .280	.011, .049, .105	.059, .286, .490	.032, .214, .302	.038, .252, .321	.010, .047, .100	.044, .243, .472
alt-3	.422, .525, .611	.418, .510, .693	.933, .984, .991	.152, .380, .515	.699, .737, .892	.691, .772, .882	.991, 1.00, 1.00	.064, .258, .395
alt-4	.528, .561, .673	.530, .555, .674	.931, .986, .992	.107, .306, .430	.719, .838, .901	.719, .837, .901	.997, 1.00, 1.00	.115, .343, .452
alt-5	.641, .705, .815	.637, .720, .831	.907, .968, .986	.076, .310, .425	.893, .925, .961	.805, 912, .924	1.00, 1.00, 1.00	.633, .712, .745
alt-6	.087, .189, .296	.071, .196, .303	.092, .224, .337	.041, .189, .328	.157, .322, .487	.159, .339, .521	.190, .378, .505	.046, .227, .375
alt-7	.126, .369, .382	.121, .305, .372	.200, .408, .533	.033, .096, .145	.333, .428, .654	.337, .443, .667	.737, .904, .945	.093, .178, .269
alt-8	.026, .088, .287	.006, .096, .303	.042, .091, .164	.059, .160, .247	.246, .422, .702	.251, .451, .719	.069, .198, .318	.050, .085, .110
alt-9	.018, .114, .271	.010, .115, .275	.011, .050, .098	.054, .292, .527	.181, .245, .436	.164, .249, .440	.016, .047, .102	.045, .231, .473

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.6.: a. Rejection Frequencies under H_1: **Walsh Basis**
Case 2: $\mathcal{H}_T = 2T^{.49}$ and $\mathcal{K}_T = .5T^{.49}$
$\boldsymbol{\epsilon_t \sim N(0,1)}$

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.121, .234, .326	.142, .276, .431	.146, .395, .552	.003, .031, .129	.583, .685, .702	.563, .628, .667	.803, .934, .967	.001, .065, .211
alt-2	.020, .059, .190	.022, .067, .202	.025, .058, .089	.010, .066, .233	.033, .197, .410	.034, .213, .427	.009, .043, .086	.002, .025, .177
alt-3	.200, .313, .417	.200, .303, .422	.150, .406, .553	.006, .060, .195	.332, .579, .687	.333, .518, .661	.382, .674, .790	.000, .022, .127
alt-4	.073, .292, .399	.065, .284, .397	.160, .402, .548	.011, .050, .190	.661, .678, .828	.668, .681, .837	.739, .906, .942	.001, .097, .292
alt-5	.121, .207, .335	.101, .145, .383	.080, .129, .178	.033, .085, .188	.403, .688, .869	.455, .731, .895	.143, .323, .471	.012, .040, .106
alt-6	.041, .123 .166	.031, .113, .179	.064, .121, .190	.017, .062, .156	.072, .134, .278	.076, .155, .308	.053, .137, .252	.010, .024, .089
alt-7	.021, .073, .137	.017, .063, .137	.050, .096, .145	.022, .072, .180	.058, .115, .211	.045, .103, .219	.066, .135, .220	.018, .052, .104
alt-8	.020, .115, .164	.021, .082, .133	.080, .129, .178	.033, .085, .188	.052, .106, .365	.032, .122, .414	.075, .115, .157	.030, .067, .119
alt-9	.011, .039, .102	.012, .027, .088	.014, .051, .099	.010, .095, .287	.043, .038, .123	.025, .060, .168	.003, .024, .070	.000, .036, .159
	$n = 256$				$n = 512$			
alt-1	.815, .932, .987	.889, .908, .987	.972, .998, 1.00	.001, .059, .220	1.00, 1.00, 1.00	.991, 1.00, 1.00	1.00, 1.00, 1.00	.079, .401, .585
alt-2	.483, .946, .994	.518, .942, .989	.012, .033, .074	.000, .016, .146	1.00, 1.00, 1.00	.999, 1.00, 1.00	.010, .037, .085	.000, .020, .127
alt-3	.883, .891, .939	.878, .879, .937	.931, .982, .990	.003, .132, .325	1.00 1.00, 1.00	.965, 1.00, 1.00	.984, .999, 1.00	.003, .086, .249
alt-4	.893, .977, .996	.809, .979, .995	.929, .981, .992	.005, .096, .273	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.997, 1.00, 1.00	.013, .209, .384
alt-5	.886, .983, .999	.867, .989, .999	.908, .977, .991	.003, .058, .223	1.00, 1.00, 1.00	1.00, 1.00, 1.00	1.00, 1.00, 1.00	.363, .720, .753
alt-6	.131, .250, .428	.141, .276, .478	.067, .189, .293	.002, .012, .091	.403, .762, .877	.449, .805, .907	.131, .306, .449	.000, .024, .136
alt-7	.121, .247, .331	.119, .224, .314	.151, .349, .503	.007, .022, .058	.514, .666, .709	.557, .636, .768	.675, .850, .920	.035, .087, .171
alt-8	.083, .252, .358	.042, .174, .410	.049, .106, .158	.018, .035, .091	.537, .851, .975	.516, .870, .983	.060, .173, .285	.001, .020, .067
alt-9	.702, .973, .998	.673, .951, .989	.004, .034, .081	1.00, .026, .145	.994, 1.00, 1.00	.994, 1.00, 1.00	.009, .048, .101	1.00, .029, .147

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.6.: b. Rejection Frequencies under H_1: **Walsh Basis**
Case 2: $\mathcal{H}_T = 2T^{.49}$ and $\mathcal{K}_T = .5T^{.49}$
$\boldsymbol{\epsilon_t \sim t_5}$

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.102, .214, .333	.143, .314, .433	.134, .357, .512	.000, .021, .091	.407, .432, .538	.428, .490, .597	.777, .919, .948	.001, .029, .131
alt-2	.013, .044, .146	.013, .049, .161	.010, .033, .070	.007, .071, .175	.035, .112, .264	.037, .131, .265	.004, .036, .065	.000, .025, .122
alt-3	.213, .317, .446	.200, .315, .453	.126, .379, .505	.001, .035, .114	.316, .468, .567	.317, .453, .542	.375, .628, .738	.000, .019, .077
alt-4	.082, .264, .317	.052, .157, .298	.142, .385, .524	.003, .042, .124	.464, .493, .696	.468, .500, .698	.717, .887, .933	.002, .049, .164
alt-5	.045, .192, .264	.023, .117, .295	.056, .099, .137	.021, .068, .143	.282, .430, .669	.214, .465, .673	.114, .281, .407	.005, .015, .043
alt-6	.031, .056, .138	.028, .041, .146	.050, .095, .139	.008, .038, .111	.055, .094, .187	.047, .084, .206	.036, .108, .194	.004, .016, .047
alt-7	.012, .055, .086	.010, .042, .100	.022, .063, .094	.005, .041, .108	.092, .121, .213	.082, .115, .187	.060, .140, .216	.008, .022, .057
alt-8	.021, .081, .118	.012, .043, .115	.056, .099, .137	.023, .066, .150	.036, .074, .266	.007, .087, .291	.040, .071, .114	.011, .025, .057
alt-9	.020, .038, .072	.014, .024, .081	.011, .036, .069	.006, .087, .204	.023, .056, .135	.007, .070, .173	.004, .024, .074	.000, .023, .097
	$n = 256$				$n = 512$			
alt-1	.727, .792, .894	.773, .795, .874	.976, .995, .998	.000, .022, .111	.963, .974, .986	.949, .977, .983	1.00, 1.00, 1.00	0.035, .407, .722
alt-2	.441, .678, .833	.455, .658, .826	.004, .027, .059	.000, .019, .061	.841, .958, .975	.821, .954, .971	.005, .035, .065	.001, .010, .026
alt-3	.783, .803, .824	.781, .784, .842	.923, .976, .987	.002, .036, .190	.929, .969, .998	.725, .909, .958	.979, .998, .999	.000, .026, .153
alt-4	.840, .896, .906	.753, .798, .906	.909, .972, .988	.000, .034, .150	.914, .979, .989	.915, .979, .989	.997, 1.00, 1.00	.001, .108, .326
alt-5	.753, .816, .912	.707, .833, .917	.867, .956, .976	.000, .023, .104	.927, .972, .985	.936, .975, .987	1.00, 1.00, 1.00	.102, .527, .740
alt-6	.107, .154, .316	.103, .124, .306	.072, .186, .278	.000, .006, .022	.214, .387, .561	.230, .409, .579	.146, .307, .430	.000, .003, .026
alt-7	.132, .261, .294	.106, .284, .247	.165, .361, .480	.000, .005, .025	.472, .521, .659	.426, .510, .623	.647, .850, .914	.012, .032, .120
alt-8	.082, .275, .367	.080, .187, .308	.020, .062, .098	.003, .008, .023	.472, .597, .869	.486, .637, .887	.019, .081, .159	.001, .003, .004
alt-9	.405, .795, .902	.354, .736, .866	.007, .023, .060	.000, .008, .057	.828, .954, .973	.771, .950, .967	.003, .039, .084	.000, .007, .027

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

Table A.6.: c. Rejection Frequencies under H_1: **Walsh Basis**
Case 2: $\mathcal{H}_T = 2T^{.49}$ and $\mathcal{K}_T = .5T^{.49}$
$\boldsymbol{\epsilon_t}$ **~ GARCH**

	$n = 64$				$n = 128$			
	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$	$\hat{\mathcal{M}}_T$	$\hat{\mathcal{M}}_T^{(p)}$	$\hat{\mathcal{D}}_T^{cv}$	$\hat{\mathcal{D}}_T^{dw}$
alt-1	.065, .211, .360	.085, .277, .367	.139, .378, .546	.002, .030, .128	.308, .422, .591	.305, .407, .560	.820, .939, .970	.000, .057, .183
alt-2	.021, .054, .162	.021, .062, .173	.022, .051, .106	.008, .079, .257	.044, .099, .266	.043, .104, .277	.005, .033, .088	.001, .032, .168
alt-3	.085, .217, .371	.078, .216, .379	.156, .418, .588	.007, .046, .170	.218, .311, .467	.209, .300, .453	.371, .657, .778	.001, .017, .114
alt-4	.120, .339, .389	.087, .242, .291	.152, .408, .554	.007, .046, .171	.421, .496, .590	.325, .400, .593	.758, .909, .953	.001, .086, .281
alt-5	.056, .156, .223	.043, .127, .216	.073, .120, .163	.027, .071, .150	.157, .368, .522	.136, .394, .506	.148, .324, .455	.015, .028, .078
alt-6	.041, .116, .195	.041, .101, .181	.066, .133, .203	.019, .040, .141	.082, .172, .278	.081, .164, .270	.046, .149, .245	.004, .017, .081
alt-7	.052, .123, .199	.030, .105, .135	.056, .117, .159	.018, .061, .163	.092, .117, .208	.094, .106, .196	.062, .129, .210	.018, .035, .092
alt-8	.046, .117, .159	.035, .103, .147	.073, .120, .163	.028, .076, .158	.043, .095, .248	.034, .089, .269	.055, .090, .148	.020, .051, .115
alt-9	.020, .038, .098	.016, .029, .096	.012, .049, .083	.004, .067, .271	.057, .087, .162	.049, .080, .189	.009, .038, .081	.000, .031, .168
	$n = 256$				$n = 512$			
alt-1	.447, .589, .677	.442, .567, .647	.974, .998, .999	.005, .051, .228	.557, .665, .807	.539, .657, .803	1.00, 1.00, 1.00	.076, .375, .540
alt-2	.282, .370, .576	.287, .353, .551	.009, .042, .102	.000, .029, .142	.517, .689, .803	.590, .667, .798	.009, .051, .099	.000, .031, .155
alt-3	.421, .498, .600	.418, .489, .688	.927, .982, .992	.003, .117, .311	.635, .767, .865	.636, .766, .868	.986, .999, 1.00	.004, .080, .247
alt-4	.518, .580, .675	.527, .583, .675	.916, .979, .989	.003, .087, .241	.746, .794, .888	.755, .808, .889	.995, 1.00, 1.00	.015, .181, .339
alt-5	.424, .596, .721	.449, .518, .709	.892, .962, .983	.002, .056, .229	.829, .916, .963	.587, .828, .909	1.00, 1.00, 1.00	.347, .674, .719
alt-6	.107, .174, .296	.108, .179, .295	.085, .214, .325	.000, .017, .085	.135, .384, .428	.143, .302, .434	.169, .350, .491	1.00, .030, .137
alt-7	.102, .247, .357	.102, .227, .377	.171, .378, .507	.006, .027, .055	.331, .468, .660	.321, .428, .622	.686, .880, .928	.031, .093, .178
alt-8	.103, .275, .350	.104, .240, .321	.049, .100, .166	.010, .029, .082	.335, .415, .638	.252, .392, .649	.062, .181, .284	.006, .028, .063
alt-9	.197, .577, .754	.168, .499, .711	.012, .046, .105	.001, .025, .160	.503, .775, .884	.424, .740, .859	.009, .045, .090	.000, .028, .150

$\hat{\mathcal{M}}_T$ and $\hat{\mathcal{M}}_T^{(p)}$ are the proposed max-tests with and without a penalty, based on a bootstrapped p-value. $\hat{\mathcal{D}}_T^{cv}$ is JWW's test based on simulated critical values, and $\hat{\mathcal{D}}_T^{dw}$ uses bootstrapped p-values.

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