MATH HW 15

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1.
$\frac{\partial r}{\partial u} X \frac{\partial r}{\partial v}(u_0, v_0) \neq 0$
$(\frac{\partial r_2}{\partial u}\frac{\partial r_3}{\partial v} - \frac{\partial r_3}{\partial u}\frac{\partial r_2}{\partial v}, -\frac{\partial r_1}{\partial u}\frac{\partial r_3}{\partial v} + \frac{\partial r_3}{\partial u}\frac{\partial r_1}{\partial v}, \frac{\partial r_1}{\partial u}\frac{\partial r_2}{\partial v} - \frac{\partial r_2}{\partial u}\frac{\partial r_1}{\partial v}) \neq 0$, (all at u_0, v_0)
One of these has to be non-zero, WLOG let it be the last one
$\frac{\partial r_1}{\partial u}\frac{\partial r_2}{\partial v} - \frac{\partial r_2}{\partial u}\frac{\partial r_1}{\partial v} \neq 0$ (invertible)

If we ignore the last output variable, we have the inverse function thm for variables 1 and 2 (differentiable continuous, and invertible derivative)

Call this reduced function $\bar{r}$

By inverse func thm, $\exists O$ contains (u_0, v_0), $\exists A$ contains $\bar{r}(u_0, v_0)$, s.t. $\bar{r}$ is a bijection

Make a function $s = r(u, v)_3$, i.e. picking the 3rd output variable

$\forall (r_1, r_2) \in A, s(\bar{r}^{-1}(r_1, r_2)) = r_3$, so r_3 is the graph of (r_1, r_2) under $s \circ \bar{r}^{-1}$, which is C_1 as both composed functions are C_1.

So, in the O neighborhood, $r(u, v)$ is a C_1 graph of 2 real variables.

2.

consider the function $k = f - g \Rightarrow k(a) = 0$. Since $\frac{\partial k}{\partial x_2} \neq 0$, the implicit func thm applies

$\exists U$, $\{(a_1, a_3) : (a_1, a_3) \in U\}$, and $g(a_1, a_3) = a_2$ and g is continuously differentiable. Since U is open and contains (a_1, a_3), there is a neighborhood around a, s.t. k is a graph.

k is the intersection between S_1 and S_2, so since k is C_1, the curve is C_1 near a.

Using the equation to get the tangent plane from the gradient of $(0, 3, 0)$ we get $(x_2 - a_2) = 0, \forall x \in \mathbb{R}^3$

3.

If f fufills cauchy-riemann equations, we have the jacobian is

$[a, b] = Df(x, y)$

$[-b, a]$,

$det(Df(x, y)) = Jf(x, y) = a^2 + b^2$, so if $Jf(x, y) = 0$, a and b must be 0, so $Df(x, y) = 0$

Converse:

If $Df(x, y) = 0$, the jacobian determinant is clearly 0.

The inverse function is given by the inverse function thm as

$Df(x, y)^{-1} = Df^{-1}(f(x, y))$

I claim that $Df(x, y)^{-1}$ fufills the cauchy-riemann equations through gauss-jordian elimination

So the derivative of the inverse function fufills cauchy-riemann equations.

3.b.

Consider $f(x, y) = (x^2 + y^2, x^2 + y^2)$ $Df(1, 1) =$

$[2, 2]$

$[2, 2]$,

$\neq 0$, but this is singular and is clearly not invertible

4.

The entire space of $\mathbb{R}^n$ is open and closed.

Because f is a diffeomorphism, it preserves open and closed.

So, $f(\mathbb{R}^n)$ is open and closed.

This is possible only if $f(\mathbb{R}^n) = \mathbb{R}^n$ or $f(\mathbb{R}^n) = \emptyset$.

The latter is not possible, so the domain must be $\mathbb{R}^n$, so f is onto.