THE UNIVERSITY *of* NORTH CAROLINA *at* CHAPEL HILL

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Quantitative differentiation

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Abstract. Quantitative differentiation deals with the behavior of function or geometric object at all locations and on all scales. We will give a relatively short elementary proof of the main theorem in the simplest case, $f : [0, 1] \to \mathbb{R}$, with say $\int_0^1 |f'|^2 \leq 1$. By a location and scale, we will mean a dyadic subinterval. Since for 0 each $n = 1, 2, \ldots$ there are $2n$ such subintervals, each of length 2^{-n}, it follows that the sum of the lengths of all dyadic intervals is infinite. Quantitative differentiation states that for all $\epsilon > 0$, the sum of the lengths of those intervals on which f fails to be ϵ-linear is $\leq 2\epsilon^{-2}$. Here, f is ϵ-linear on an interval of length r if it differs from the best linear approximation by at most ϵr. Instances of the basic quantitative differentiation idea have appeared in several relatively advanced contexts going back to work of Dorronsoro in 1985. However, the above simplest case seems not to be widely known. We will also describe a 2-dimensional example from riemannian geometry which typifies applications to geometric analysis over the last 10 years. These concern partial regularity theory for various nonlinear elliptic and parabolic geometric pde, as well as a bi-Lipschitz nonembedding theorem for the Heisenberg group in the target L^1. They are joint with (subsets of) Toby Colding, Bruce Kleiner, Assaf Naor, Aaron Naber, Daniele Valtorta, Bob Haslhofer, and Wenshuai Jiang.