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On the stability of discrete-time homogeneous polynomial dynamical systems

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Abstract

Every homogeneous polynomial dynamical system (HPDS) can be uniquely represented by a tensor. In our recent article (Chen, IEEE Trans Autom Control), we established necessary and sufficient stability criteria for certain continuous-time HPDSs by exploiting tensor spectral theory. In this article, we extend these results to discrete-time HPDSs. In particular, if the state transition tensor of a discrete-time HPDS is orthogonally decomposable (odeco), we can derive its explicit solution. We refer to such HPDSs as odeco HPDSs. Building upon the form of the explicit solution, we demonstrate that the Z-eigenvalues of the state transition tensor offer necessary and sufficient stability conditions, analogous to the continuous-time case. The region of attraction can also be obtained for the odeco HPDS. Additionally, by employing the upper bounds of Z-spectral radii, we can efficiently determine the asymptotic stability of odeco HPDSs. Finally, we leverage tensor singular values to analyze the stability properties of general discrete-time HPDSs, where the state transition tensors are not odeco. We illustrate our framework with numerical examples.

Keywords Homogeneous polynomial dynamical systems · Stability · Regions of attraction · Tensor algebra · Z-eigenvalues · Tensor singular values

Mathematics Subject Classification 15A18 · 15A69 · 37N30 · 39A30 · 65P40 · 93D05 · 93D20

1 Introduction

Tensor algebra has emerged as a powerful tool for modeling and analyzing both linear and nonlinear dynamical systems (Chen 2023; Chen et al. 2021a, b; Gelß 2017; Kruppa 2017;

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Kruppa and Lichtenberg 2017; Hoover et al. 2021). It facilitates the development of more efficient and accurate numerical methods for solving differential equations and enables the derivation of new theoretical insights into the system-theoretic properties of dynamical systems. For example, Gelß (2017) employed various tensor decompositions for computing numerical solutions of master equations associated with Markov processes on extremely large state spaces, developing tensor-based representations for operators based on nearest-neighbor interactions, construction of pseudo-inverses for dimensionality reduction methods, and approximation of transfer operators of dynamical systems. Additionally, Chen et al. (2019, 2021b) formulated tensor algebraic conditions for stability, reachability, and observability for input/output discrete-time multilinear time-invariant systems, and expressed them in terms of tensor ranks and decompositions to promote efficient representation and computation.

A novel tensor-based dynamical system representation was recently introduced in (Chen et al. 2021a) to characterize the multidimensional state dynamics of hypergraphs, a generalization of graphs in which edges can connect more than two nodes (Berge 1984). This representation differs from those proposed in (Chen et al. 2019, 2021b), which can be unfolded to linear dynamical systems via tensor unfolding, an operation that transforms a tensor into a matrix. Instead, the tensor-based dynamical system evolution, inspired by hypergraphs, is captured by the tensor vector multiplication between a state transition tensor and the state vector. In fact, this representation belongs to the family of homogeneous polynomial dynamical systems (HPDSs) (Chen et al. 2021a). Significantly, HPDSs or polynomial dynamical systems have a wide range of applications, such as in robotics, ecological networks, biological processes, and more (Ghosh and Martin 2002; Grilli et al. 2017; Stigler 2007; Motee et al. 2012; Craciun 2019).

However, the stability of HPDSs is one of the most challenging problems in systems theory due to their nonlinear nature (Ahmadi and El Khadir 2019; Ahmadi and Parrilo 2013; Ji et al. 2013; Samardzija 1983; She et al. 2013). When an HPDS has degree one, its stability properties can be determined by the locations of the eigenvalues of the state transition matrix, known as linear stability. It is therefore conceivable that tensor eigenvalues might be used to determine the stability properties of HPDSs of higher degrees. Various notions of tensor eigenvalues have been proposed, such as H-eigenvalues, Z-eigenvalues, M-eigenvalues, and U-eigenvalues (Chen et al. 2021b; Lim 2006; Qi 2005, 2007), all of which generalize matrix eigenvalues in different ways. This article focuses on Z-eigenvalues and U-eigenvalues. Notably, the notion of Z-eigenvalues is intimately related to tensor orthogonal decomposition, which decomposes a tensor into a sum of rank-one tensors in the form of outer products of vectors that form an orthonormal basis (Anandkumar et al. 2014; Robeva 2016). A tensor admitting such a decomposition is termed orthogonally decomposable (odeco). Odeco tensors possess the desirable orthonormal property, which can be leveraged to elucidate the stability properties of HPDSs (Chen 2023).

We present the stability results for discrete-time HPDSs in this article, which are complementary to our recent work (Chen 2023) on continuous-time HPDSs. The key contributions are listed as follows:

- We derive an explicit solution formula for discrete-time HPDSs with odeco state transition tensors. We refer to such HPDSs as odeco HPDSs.
- By utilizing the form of the explicit solution, we explore the stability properties of odeco HPDSs. According to the stability conditions, we obtain the regions of attraction for odeco HPDSs.

- We provide an upper bound for Z-spectral radii, which can be used to efficiently determine the asymptotic stability of odeco HPDSs.
- We investigate the stability properties of general HPDSs (i.e., those with state transition tensors that are not odeco) by exploiting tensor singular values.

This article is organized into six sections. In Sect. 2, we review tensor preliminaries including tensor products, tensor eigenvalues, and tensor decompositions. In Sect. 3, we introduce the tensor-based representation of discrete-time HPDSs and establish the stability criteria for odeco HPDSs. We investigate the stability properties of general HPDSs in Sect. 4. We verify our results with numerical examples in Sect. 5 and conclude with future research directions in Sect. 6.

2 Tensor preliminaries

Tensors are multidimensional arrays that generalize vectors and matrices (Chen and Rajapakse 2020; Chen et al. 2019, 2021b; Gelß 2017; Kolda 2006; Kolda and Bader 2009; Surana et al. 2022). The order of a tensor is the number of its dimensions, and each dimension is referred to as a mode. A kth-order tensor is usually denoted by $\mathsf{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$. When all modes share the same size, i.e., $n_1 = n_2 = \cdots = n_k$, T is called cubical.

Definition 1 A kth-order cubical tensor T is called supersymmetric if its entries $\mathsf{T}_{j_1 j_2 \cdots j_k}$ are invariant under any permutation of the indices.

Definition 2 A kth-order cubical tensor T is called almost symmetric if its entries $\mathsf{T}_{j_1 j_2 \cdots j_k}$ are invariant under any permutation of the first $k-1$ indices.

2.1 Tensor products

The inner product of two kth-order tensors $\mathsf{T}, \mathsf{S} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ is defined as

$$\langle \mathsf{T}, \mathsf{S} \rangle = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_k=1}^{n_k} \mathsf{T}_{j_1 j_2 \cdots j_k} \mathsf{S}_{j_1 j_2 \cdots j_k}. \tag{1}$$

The inner product (1) leads to the tensor Frobenius norm $\|\mathsf{T}\|^2 = \langle \mathsf{T}, \mathsf{T} \rangle$. The tensor vector multiplication $\mathsf{T} \times_p \mathbf{v}$ along mode p for a vector $\mathbf{v} \in \mathbb{R}^{n_p}$ is defined as

$$(\mathsf{T} \times_p \mathbf{v})_{j_1 j_2 \cdots j_{p-1} j_{p+1} \cdots j_k} = \sum_{j_p=1}^{n_p} \mathsf{T}_{j_1 j_2 \cdots j_p \cdots j_k} \mathbf{v}_{j_p}, \tag{2}$$

which can be generalized to the Tucker product, i.e.,

$$\mathsf{T} \times_1 \mathbf{v}_1 \times_2 \mathbf{v}_2 \times_3 \cdots \times_k \mathbf{v}_k = \mathsf{T}\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_k \in \mathbb{R} \tag{3}$$

for $\mathbf{v}_p \in \mathbb{R}^{n_p}$. If T is supersymmetric with $\mathbf{v}_p = \mathbf{v}$ for all p, the product (3) is also known as the homogeneous polynomial associated with T. We write it as $\mathsf{T}\mathbf{v}^k$ for simplicity. Hence, the following product:

$$\mathsf{T}\mathbf{v}^{k-1} = \mathsf{T} \times_1 \mathbf{v}_1 \times_2 \mathbf{v} \times_3 \cdots \times_{k-1} \mathbf{v} \in \mathbb{R}^n \tag{4}$$

belongs to the family of homogeneous polynomial systems. It follows immediately that if T is almost symmetric, the product (4) spans the entire homogeneous polynomial system space.

Moreover, the tensor vector multiplication (2) can be extended to tensor matrix multiplications, which are defined as

$$(\mathsf{T} \times_p \mathbf{M})_{j_1 j_2 \cdots j_{p-1} i j_{p+1} \cdots j_k} = \sum_{j_p=1}^{n_p} \mathsf{T}_{j_1 j_2 \cdots j_p \cdots j_k} \mathbf{M}_{i j_p} \tag{5}$$

for a matrix $\mathbf{M} \in \mathbb{R}^{m \times n_p}$.

2.2 Tensor Eigenvalues

Homogeneous polynomials are intrinsically linked to eigenvalue problems. The study of tensor eigenvalues for real supersymmetric tensors was independently initiated by Qi (2005, 2007) and Lim (2006). Various types of tensor eigenvalues exist, including H-eigenvalues, Z-eigenvalues, M-eigenvalues, and U-eigenvalues (Chen et al. 2021b; Qi 2005, 2007). This article particularly focuses on Z-eigenvalues and U-eigenvalues.

Definition 3 For a given kth-order n-dimensional supersymmetric tensor $\mathsf{T} \in \mathbb{R}^{n \times n \times \overset{k}{\cdots} \times n}$, the E-eigenvalues $\lambda \in \mathbb{C}$ and E-eigenvectors $\mathbf{v} \in \mathbb{C}^n$ of T satisfy the following equation:

$$\begin{cases} \mathsf{T}\mathbf{v}^{k-1} = \lambda \mathbf{v} \\ \mathbf{v}^\top \mathbf{v} = 1 \end{cases}. \tag{6}$$

If the E-eigenvalue λ is real, it is referred to as Z-eigenvalue with the corresponding Z-eigenvector $\mathbf{v}$.

Qi (2007) proved that a supersymmetric tensor always possesses Z-eigenvalues. The largest Z-eigenvalue of T can be determined by solving from the following optimization:

$$\lambda_{\max} = \max_{\mathbf{v} \in \mathbb{R}^n} \{\mathsf{T}\mathbf{v}^k : \|\mathbf{v}\|_2 = 1\}. \tag{7}$$

Due to the facts that the objective function is continuous and the feasible set is compact, the existence of a global maximizer is guaranteed (Qi 2005). The smallest Z-eigenvalue can be found similarly. Computing the E-eigenvalues or Z-eigenvalues of a tensor is NP-hard (Hillar and Lim 2013). In 2016, Chen et al. (2016) introduced numerical methods for computing E-eigenvalues and Z-eigenvalues using the homotopy continuation approach, but these methods are only effective for small-sized tensors. On the other hand, U-eigenvalues are only defined for even-order tensors.

Definition 4 For a given $2k$th-order n-dimensional tensor $\mathsf{T} \in \mathbb{R}^{n \times n \times \overset{2k}{\cdots} \times n \times n}$, the U-eigenvalues $\mu \in \mathbb{C}$ and U-eigentensors $\mathsf{V} \in \mathbb{C}^{n \times n \times \overset{k}{\cdots} \times n}$ of T satisfy the following equation:

$$\mathsf{T} * \mathsf{V} = \mu \mathsf{V}, \tag{8}$$

where "$*$" denotes the Einstein product defined as

$$(\mathsf{T} * \mathsf{V})_{i_1 i_2 \cdots i_k} = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} \mathsf{T}_{j_1 i_1 \cdots j_k i_k} \mathsf{V}_{j_1 j_2 \cdots j_k}. \tag{9}$$

Similar to Z-eigenvalues, the largest U-eigenvalue of an even-order supersymmetric tensor T can be determined by solving the following optimization problem:

$$\mu_{\max} = \max_{\mathsf{V} \in \mathbb{R}^{n \times n \times \overset{k}{\cdots} \times n}} \{\mathsf{V}^\top * \mathsf{T} * \mathsf{V} : \|\mathsf{V}\| = 1\}, \tag{10}$$

where "$\top$" denotes the transpose operation defined in (Chen et al. 2021b) for even-order tensors. The smallest U-eigenvalue can be found similarly. Significantly, U-eigenvalues can be computed from the eigenvalues of its unfolded matrix defined as

$$\mathbf{A} = \psi(\mathsf{A}) \text{ such that } \mathsf{A}_{j_1 i_1 \cdots j_k i_k} \xrightarrow{\psi} \mathbf{A}_{ji}, \tag{11}$$

where $j = j_1 + \sum_{p=2}^{k}(j_p - 1)n^{p-1}$ and $i = i_1 + \sum_{p=2}^{k}(i_p - 1)n^{p-1}$. Consequently, an even-order cubical tensor always possesses U-eigenvalues.

2.3 Tensor decompositions

Tensor decompositions are powerful tools for tensor computation and application. There are various kinds of tensor decompositions, including higher-order singular value decomposition (HOSVD), CANDECOMP/PARAFAC decomposition (CPD), Tucker decomposition, and tensor train decomposition (De Lathauwer et al. 2000; Kolda 2006; Kolda and Bader 2009; Oseledets and Tyrtyshnikov 2009; Oseledets 2011).

Definition 5 The HOSVD of a kth-order tensor $\mathsf{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ is defined as

$$\mathsf{T} = \mathsf{S} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \times_3 \cdots \times_k \mathbf{U}_k, \tag{12}$$

where $\mathbf{U}_p \in \mathbb{R}^{n_p \times n_p}$ are orthogonal matrices, and $\mathsf{S} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ is the core tensor. The sub-tensors $\mathsf{S}_{j_p=\alpha}$, obtained by fixing the pth index to α, exhibit the following "diagonal" properties: (i) all-orthogonality: two subtensors $\mathsf{S}_{j_p=\alpha}$ and $\mathsf{S}_{j_p=\beta}$ are orthogonal for all possible values of p, α and β subject to $\alpha \neq \beta$; (ii) ordering: $\|\mathsf{S}_{j_p=1}\| \geq \cdots \geq \|\mathsf{S}_{j_p=n_p}\| \geq 0$ for all possible values of p. The Frobenius norms $\|\mathsf{S}_{j_p=j}\|$, denoted by $\gamma_j^{(p)}$, are the p-mode singular values of T.

The number of non-vanished p-mode singular values is equal to the p-rank of T (De Lathauwer et al. 2000). Additionally, the low-rank approximation of HOSVD is quasi-optimal, with an error bound reported in (De Lathauwer et al. 2000).

Definition 6 The CPD of a kth-order tensor $\mathsf{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$ is defined as

$$\mathsf{T} = \sum_{j=1}^{r} \lambda_j \mathbf{v}_j^{(1)} \circ \mathbf{v}_j^{(2)} \circ \cdots \circ \mathbf{v}_j^{(k)}, \tag{13}$$

where "$\circ$" denotes the outer product operation (defined as $(\mathbf{x} \circ \mathbf{y} \circ \mathbf{z})_{jik} = x_j y_i z_k$ for vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$), $\mathbf{v}_j^{(p)} \in \mathbb{R}^{n_p}$ have unit length with corresponding weights λ_j, and r is called the CP rank of T if it is the minimum integer that realizes (13).

Every tensor has a CP decomposition, and it is unique up to scaling and permutation under a weak condition (Kolda and Bader 2009). While the best CP rank approximation is ill-posed, carefully truncating the CP rank will produce a good approximation of the original tensor (Chen et al. 2021b). Furthermore, tensor orthogonal decomposition is a special case of CPD.

Definition 7 A kth-order n-dimensional supersymmetric tensor $\mathsf{T} \in \mathbb{R}^{n \times n \times \overset{k}{\cdots} \times n}$ is called orthogonally decomposable (odeco) if it can be written as

$$\mathsf{T} = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \circ \mathbf{v}_j \circ \overset{k}{\cdots} \circ \mathbf{v}_j, \tag{14}$$

where $\lambda_j \in \mathbb{R}$ in the descending order, and $\mathbf{v}_j \in \mathbb{R}^n$ are orthonormal

Robeva (2016) proved that λ_j are the Z-eigenvalues of T with the corresponding Z-eigenvectors $\mathbf{v}_j$. However, λ_j do not include all the Z-eigenvalues of T. Additionally, odeco tensors are conjectured to satisfy a set of polynomial equations that vanish on the odeco variety, which is the Zariski closure of the set of odeco tensors inside the space of kth-order n-dimensional complex supersymmetric tensors (Robeva 2016). This conjecture has been proven for the case $n = 2$, and strong evidence has been provided for its validity for all values of n. While the exact characterization of odeco tensors is intricate, we can numerically compute the "nearest" orthogonal decomposition for any supersymmetric tensors, i.e., $\mathsf{T} = \mathsf{T}_{\text{odeco}} + \mathsf{E}$ where $\mathsf{T}_{\text{odeco}}$ is odeco and E is the error tensor.

3 Stability of odeco HPDSs

In this article, we consider a discrete-time HPDS represented by

$$\mathbf{x}_{t+1} = \mathsf{A} \times_1 \mathbf{x}_t \times_2 \mathbf{x}_t \times_3 \cdots \times_{k-1} \mathbf{x}_t = \mathsf{A}\mathbf{x}_t^{k-1}, \tag{15}$$

where $\mathsf{A} \in \mathbb{R}^{n \times n \times \overset{k}{\cdots} \times n}$ is an almost symmetric state transition tensor, and $\mathbf{x}_t \in \mathbb{R}^n$ is the state variable. Every HPDS can be represented in the form of (15) (Chen 2023).

Assumption 1 We assume throughout this section that the state transition tensor A in (15) is odeco with the following orthogonal decomposition:

$$\mathsf{A} = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \circ \mathbf{v}_j \circ \overset{k}{\cdots} \circ \mathbf{v}_j, \tag{16}$$

where λ_j are the Z-eigenvalues of A with the corresponding Z-eigenvectors $\mathbf{v}_j$.

While the class of odeco HPDSs is not exhaustive, it still can capture a certain amount of structured population dynamics with higher-order interactions, such as those arising in neuronal networks, chemical reaction networks, and ecological networks (Chen 2023). For example, in the context of higher-order ecological networks, the Z-eigenvectors $\mathbf{v}_j$ can represent interaction types (e.g., promotion and inhibition) between species, the Z-eigenvalues λ_j can represent interaction magnitudes, and the order k can represent interaction orders. Our approach therefore holds promise for investigating the stability properties of such higher-order networks.

3.1 Explicit solutions

Finding an explicit solution of an HPDS is usually challenging due to its nonlinear nature. However, if the state transition tensor A is odeco, we can write down the solution of (15) explicitly in a simple fashion.

Proposition 1 *Suppose that Assumption 1 holds with $k \geq 3$. Let the initial condition $\boldsymbol{x}_0 = \sum_{j=1}^{n} c_j \boldsymbol{v}_j$. The explicit solution of the discrete-time odeco HPDS (15) at time q, given the initial condition $\boldsymbol{x}_0$, can be computed as*

$$\boldsymbol{x}_q = \sum_{j=1}^{n} \lambda_j^{\alpha} c_j^{\beta} \boldsymbol{v}_j, \tag{17}$$

where $\alpha = \sum_{j=0}^{q-1}(k-1)^j = \frac{(k-1)^q-1}{k-2}$ *and* $\beta = (k-1)^q$. *Here,* λ_j *are the Z-eigenvalues from the orthogonal decomposition of* A *with the corresponding Z-eigenvectors* $\boldsymbol{v}_j$.

Proof Based on the property of tensor vector multiplications and tensor orthogonal decomposition, we can write down the explicit solution $\mathbf{x}_1$ as follows:

$$\begin{aligned}
\mathbf{x}_1 &= \mathsf{A} \times_1 \Big(\sum_{j=1}^{n} c_j \mathbf{v}_j\Big) \times_2 \cdots \times_{k-1} \Big(\sum_{j=1}^{n} c_j \mathbf{v}_j\Big) \\
&= \Big(\sum_{j=1}^{n} \lambda_j \mathbf{v}_j \circ \mathbf{v}_j \circ \cdots \circ \mathbf{v}_j\Big) \times_1 \Big(\sum_{j=1}^{n} c_j \mathbf{v}_j\Big) \times_2 \cdots \times_{k-1} \Big(\sum_{j=1}^{n} c_i \mathbf{v}_j\Big) \\
&= \sum_{j=1}^{n} \lambda_j \Big\langle \mathbf{v}_j, \sum_{i=1}^{n} c_i \mathbf{v}_i \Big\rangle^{k-1} \mathbf{v}_j = \sum_{j=1}^{n} \lambda_j \mathbf{c}_j^{k-1} \mathbf{v}_j.
\end{aligned}$$

Similarly, the solution $\mathbf{x}_2$ can be computed as

$$\begin{aligned}
\mathbf{x}_2 &= \mathsf{A} \times_1 \Big(\sum_{j=1}^{n} \lambda_j c_j^{k-1} \mathbf{v}_j\Big) \times_2 \cdots \times_{k-1} \Big(\sum_{j=1}^{n} \lambda_j c_j^{k-1} \mathbf{v}_j\Big) \\
&= \sum_{j=1}^{n} \lambda_j \Big\langle \mathbf{v}_j, \sum_{i=1}^{n} \lambda_i c_i^{k-1} \mathbf{v}_i \Big\rangle^{k-1} \mathbf{v}_j = \sum_{j=1}^{n} \lambda_j^k c_j^{(k-1)^2} \mathbf{v}_j.
\end{aligned}$$

Consequently, we can continue to compute $\mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_q$ similarly, and the result follows immediately. □

The coefficient c_j can be determined by taking the inner product between $\mathbf{x}_0$ and $\mathbf{v}_j$. When $k = 2$, Proposition 1 reduces to the classical solution formula for linear dynamical systems, i.e.,

$$\lim_{k\to 2} \lambda_j^{\frac{(k-1)^q-1}{k-2}} c_j^{(k-1)^q} = \lim_{k\to 2} \lambda_j^{q(k-1)^{q-1}} c_j^{(k-1)^q} = c_j \lambda_j^q.$$

Furthermore, by exploiting the form of the explicit solution, we are able to establish the stability criteria for discrete-time odeco HPDSs.

3.2 Stability

In linear systems theory, it is common to investigate so-called (internal) stability. The equilibrium point $\mathbf{x}_e = \mathbf{0}$ of an odeco HPDS is called stable if $\|\mathbf{x}_t\| \leq \gamma \|\mathbf{x}_0\|$ for the initial condition $\mathbf{x}_0$ and $\gamma > 0$, asymptotically stable if $\mathbf{x}_t \to \mathbf{0}$ as $t \to \infty$, and unstable otherwise. Note that if $\lambda_j = 0$ for some $j = 1, 2, \ldots, n$, the odeco HPDS will exhibit infinitely many equilibrium points. Yet, those non-zero equilibrium points behave in the exactly same manner as the equilibrium point at the origin (Chen 2023). We demonstrate that the stability properties of discrete-time odeco HPDSs depend on both Z-eigenvalues and initial conditions, and resemble those of discrete-time linear dynamical systems.

Proposition 2 *Suppose that Assumption 1 holds with* $k \geq 3$. *Let the initial condition* $\boldsymbol{x}_0 = \sum_{j=1}^{n} c_j \boldsymbol{v}_j$. *The equilibrium point* $\boldsymbol{x}_e = \boldsymbol{0}$ *of the discrete-time odeco HPDS (15) is ("$|\cdot|$" denotes the absolute value operation):*

- *stable if and only if* $|c_j \lambda_j^{\frac{1}{k-2}}| \leq 1$ *for all* $j = 1, 2, \ldots, n$;
- *asymptotically stable if and only if* $|c_j \lambda_j^{\frac{1}{k-2}}| < 1$ *for all* $j = 1, 2, \ldots, n$;
- *unstable if and only if* $|c_j \lambda_j^{\frac{1}{k-2}}| > 1$ *for some* $j = 1, 2, \ldots, n$,

where λ_j *are the Z-eigenvalues from the orthogonal decomposition of* A *with the corresponding Z-eigenvectors* $\mathbf{v}_j$.

Proof According to Proposition 1, the solution at time q, given the initial condition $\mathbf{x}_0$, is computed as $\mathbf{x}_q = \sum_{j=1}^{n} \lambda_j^{\alpha} c_j^{\beta} \mathbf{v}_j$, where $\alpha = \sum_{j=0}^{q-1}(k-1)^j = \frac{(k-1)^q - 1}{k-2}$ and $\beta = (k-1)^q$. Consequently, it can be shown that

$$\lambda_j^{\alpha} c_j^{\beta} = \lambda_j^{\frac{(k-1)^q - 1}{k-2}} c_j^{(k-1)^q} = \lambda_j^{-\frac{1}{k-2}} (\lambda_j^{\frac{1}{k-2}} c_j)^{(k-1)^q}.$$

If $|c_j \lambda_j^{\frac{1}{k-2}}| \leq 1$ for all j, by triangular inequality, it can be shown that

$$\begin{aligned} \|\mathbf{x}_q\| &= \|\sum_{j=1}^{n} \lambda_j^{-\frac{1}{k-2}} (\lambda_j^{\frac{1}{k-2}} c_j)^{(k-1)^q} \mathbf{v}_j\| \\ &\leq \sum_{j=1}^{n} \|\lambda_j^{-\frac{1}{k-2}} (\lambda_j^{\frac{1}{k-2}} c_j)^{(k-1)^q} \mathbf{v}_j\| \leq \sum_{j=1}^{n} |\lambda_j|^{\frac{1}{k-2}}. \end{aligned}$$

Therefore, the equilibrium point $\mathbf{x}_e = \mathbf{0}$ is stable. On the other hand, the fact that $\|\mathbf{x}_q\|$ is bounded implies that the quantity $(\lambda_j^{\frac{1}{k-2}} c_j)^{(k-1)^q}$ must also be bounded for any q. Thus, we have $|c_j \lambda_j^{\frac{1}{k-2}}| \leq 1$ for all j. The other two cases can be shown similarly. □

Clearly, we can write $|c_j \lambda_j^{\frac{1}{k-2}}|$ as $|\langle \mathbf{x}_0, \lambda_j^{\frac{1}{k-2}} \mathbf{v}_j \rangle|$. In addition, the inequalities obtained from the asymptotic stability condition can provide us with the exact region of attraction for the odeco HPDS (15), i.e.,

$$R = \Big\{ \mathbf{x} : |c_j| < |\lambda_j|^{-\frac{1}{k-2}} \text{ where } \mathbf{x} = \sum_{j=1}^{n} c_j \mathbf{v}_j \Big\}. \tag{18}$$

Furthermore, if the product between $\max_j |c_j|$ and $\max_j |\lambda_j|^{\frac{1}{k-2}}$ is less than one, the odeco HPDS (15) will be asymptotically stable.

Definition 8 The Z-spectral radius of a supersymmetric tensor is the maximum of the absolute values of all its Z-eigenvalues.

Corollary 1 *Suppose that Assumption 1 holds with* $k \geq 3$. *Let* $\boldsymbol{x}_0$ *be some initial conditions. For the odeco HPDS (15), the equilibrium point* $\boldsymbol{x}_e = \boldsymbol{0}$ *is asymptotically stable if* $\lambda^{\frac{1}{k-2}} \|\boldsymbol{x}_0\| < 1$ *where* $\lambda = \max\{|\lambda_1|, |\lambda_n|\}$ *is the Z-spectral radius of* A.

Proof By the Cauchy-Schwarz inequality, $|c_j| \leq \|\mathbf{x}_0\|$ for all $j = 1, 2, \ldots, n$. In addition, $\max_j |\lambda_j| \leq \lambda$. Moreover, it has been proved that the Z-spectral radius $\lambda = \max\{|\lambda_1|, |\lambda_n|\}$ where λ_1 and λ_n are the largest and the smallest Z-eigenvalues from the orthogonal decomposition of A, respectively (Chen et al. 2022). Therefore, the result follows immediately from Proposition 2. □

3.3 Upper bounds for Z-spectral radii

Computing the orthogonal decomposition or Z-eigenvalues of a supersymmetric tensor is known to be NP-hard (Hillar and Lim 2013; Robeva 2016). Therefore, establishing upper bounds for the Z-spectral radii of the state transition tensor can provide a computationally efficient approach for analyzing the stability properties of odeco HPDSs.

Lemma 1 *Suppose that* $\mathsf{A} \in \mathbb{R}^{n\times n\times \overset{2k}{\cdots} \times n\times n}$ *is a* $2k$*th-order* n*-dimensional supersymmetric tensor. The Z-spectral radius of* A *is upper bounded by its U-spectral radius, i.e., the maximum of the absolute values of all its U-eigenvalues.*

Proof Based on (7), the largest Z-eigenvalue of A can be solved from an equivalent optimization, which can be computed as

$$\max_{\mathsf{V}\in\mathbb{R}^{n\times n\times \overset{k}{\cdots}\times n}} \{\mathsf{V}^\top * \mathsf{A} * \mathsf{V} : \|\mathsf{V}\| = 1 \text{ and } \mathsf{V} = \mathbf{v}\circ\mathbf{v}\circ\cdots\circ\mathbf{v}\}.$$

Therefore, the largest Z-eigenvalue of A is always less than or equal to its largest U-eigenvalue. Similarly, we can show that the smallest Z-eigenvalue of A is always greater than or equal to its smallest U-eigenvalue. Hence, the result follows immediately. □

The U-spectral radius of an even-order supersymmetric tensor can be efficiently obtained by computing the spectral radius of its unfolded matrix (11), which requires $\mathcal{O}(n^{3k})$ operations. While the complexity grows exponentially with the order k, it remains significantly more efficient than computing Z-spectral radii. Once an upper bound for the Z-spectral radius of the state transition tensor is obtained, we can determine the asymptotic stability of the odeco HPDS (15) without the need for computing the orthogonal decomposition or Z-eigenvalues.

Corollary 2 *Suppose that Assumption 1 holds with even* $k \geq 4$*. Let* $\boldsymbol{x}_0$ *be some initial conditions. For the odeco HPDS (15), the equilibrium point* $\boldsymbol{x}_e = \boldsymbol{0}$ *is asymptotically stable if* $\mu^{\frac{1}{k-2}}\|\boldsymbol{x}_0\| < 1$ *where* μ *is the U-spectral radius of* A*.*

Proof The result follows immediately from Lemma 1 and Corollary 1. □

The condition offers a conservative region of attraction for the odeco HPDS (15) without requiring knowledge of the orthogonal decomposition of A, i.e.,

$$R = \{\mathbf{x} : \|\mathbf{x}\| < \mu^{-\frac{1}{k-2}}\}.$$

Several other upper bounds for Z-spectral radii of supersymmetric tensors exist (Chang et al. 2013; He and Huang 2014; Ma and Song 2019; Wu et al. 2018). For instance, He and Huang (2014) proposed that for a given positive kth-order supersymmetric tensor A, its Z-spectral radius is upper bounded by

$$\lambda \leq g - l\left(1 - \left(\frac{r}{g}\right)^{\frac{1}{k}}\right), \tag{19}$$

where l is the minimum entry of A,

$$r = \min_j \left(\sum_{j_2=1}^n \sum_{j_3=1}^n \cdots \sum_{j_k=1}^n \mathsf{A}_{jj_2\ldots j_k}\right),$$

$$g = \max_j \left(\sum_{j_2=1}^n \sum_{j_3=1}^n \cdots \sum_{j_k=1}^n \mathsf{A}_{jj_2\ldots j_k}\right).$$

Hence, we can also use this upper bound to determine the stability of an odeco HPDS if the state transition tensor contains all positive entries. Note that a tighter upper bound on the Z-spectral radius yields stronger stability conditions.

4 Stability of general HPDSs

As noted in (Robeva 2016), not all supersymmetric tensors are odeco. In this section, we extend the stability results to general HPDSs (i.e., the state transition tensor A is almost symmetric) by utilizing tensor singular values. To begin, we adapt a lemma from (Jiang et al. 2017) that provides an upper bound on the Frobenius norm of tensor vector multiplications. This result is analogous to that for matrix vector multiplications.

Lemma 2 (Jiang et al. 2017) Suppose that $\mathsf{A} \in \mathbb{R}^{n\times n\times \overset{k}{\cdots}\times n}$ is a kth-order n-dimensional tensor and $\mathbf{v} \in \mathbb{R}^n$. The following inequality holds:

$$\|\mathsf{A} \times_p \mathbf{v}\| \leq \|\mathsf{A}\|\,\|\mathbf{v}\|. \tag{20}$$

According to Lemma 2, we can obtain a relative weaker but more general stability condition for all HPDSs.

Proposition 3 *Suppose that $k \geq 3$. Let $\boldsymbol{x}_0$ be some initial conditions. The equilibrium point $\boldsymbol{x}_e = \boldsymbol{0}$ of the HPDS (15) is asymptotically stable if*

$$\|\mathsf{A}\|^{\frac{1}{k-2}}\|\boldsymbol{x}_0\| < 1. \tag{21}$$

Equivalently, the equilibrium point $\boldsymbol{x}_e = \boldsymbol{0}$ of the HPDS (15) is asymptotically stable if

$$\Big(\sum_{j=1}^{n}(\gamma_j^{(p)})^2\Big)^{\frac{1}{k-2}}\|\boldsymbol{x}_0\| < 1, \tag{22}$$

where $\gamma_j^{(p)}$ are the p-mode singular values of A, for any p.

Proof Based on Lemma 2, we have

$$\|\mathbf{x}_{t+1}\| \leq \|\mathsf{A}\|\,\|\mathbf{x}_t\|^{k-1}.$$

Thus, it can be shown similarly as Proposition 1 that at the qth step, we have

$$\|\mathbf{x}_q\| \leq \|\mathsf{A}\|^{\alpha}\|\mathbf{x}_0\|^{\beta},$$

where α and β are the same quantities as defined in Proposition 1. Moreover, the Frobenius norm of a tensor is equal to the sum of its p-mode singular values' square for any p. Therefore, the result follows immediately. □

Similarly, Proposition 3 can be used to obtain a conservative region of attraction, which is computed as

$$R = \Big\{\mathbf{x} : \|\mathbf{x}\| < \Big(\sum_{j=1}^{n}(\gamma_j^{(p)})^2\Big)^{-\frac{1}{k-2}}\Big\}. \tag{23}$$

Table 1 Stability results for the HPDS with different initial conditions

IC	$\max\{\lvert c_r\lambda_r\rvert\}$	$\sum_{j=1}^{n}(\gamma_j^{(p)})^2\lVert\mathbf{x}_0\rVert$	**Stability**
a	0.9735	28.7712	Asym. stable
b	0.6032	0.9413	Asym. stable
c	1	53.9410	Stable
d	1.0053	1.5688	Unstable

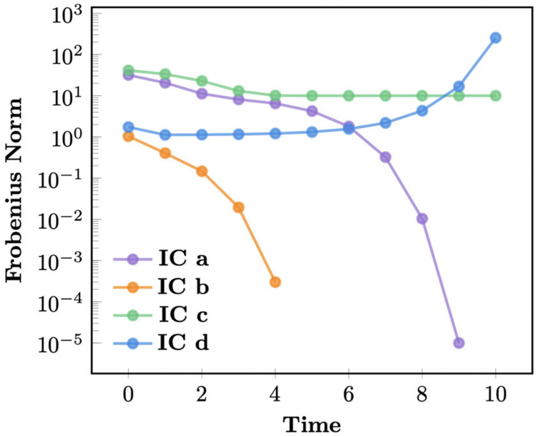

Fig. 1 Stability results for the different initial conditions, corresponding to Table 1. When the norm of $\mathbf{x}_t$ is less than 10^{-5}, we omitted the point

5 Numerical examples

All numerical examples presented were performed on a Macintosh machine with 16 GB RAM and a 2 GHz Quad-Core Intel Core i5 processor in MATLAB R2020b using the MATLAB tensor toolbox (Bader and Kolda 2006).

5.1 Stability of odeco HPDSs

In this example, we verified the stability results presented in Proposition 2. For a discrete-time odeco HPDS of the form (15), the orthogonal decomposition of the state transition tensor $\mathsf{A} \in \mathbb{R}^{3\times3\times3}$ is given by (columns of $\mathbf{V}$ are $\mathbf{v}_j$ in (16))

$$\mathbf{V} = \begin{bmatrix} -0.8482 & -0.5212 & 0.0947 \\ -0.4840 & 0.6899 & -0.5382 \\ 0.2152 & -0.5024 & -0.8374 \end{bmatrix} \text{ and } \boldsymbol{\lambda} = \begin{bmatrix} 0.9 \\ 0.1 \\ 0.02 \end{bmatrix}.$$

We computed the trajectories $\mathbf{x}_t$ for the following four initial conditions:

$$\mathbf{x_a} = \begin{bmatrix} 3 \\ 10 \\ 30 \end{bmatrix}, \ \mathbf{x_b} = \begin{bmatrix} 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}, \ \mathbf{x_c} = \begin{bmatrix} -2.2720 \\ -15.1148 \\ -38.3064 \end{bmatrix}, \ \mathbf{x_d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The results are shown in Table 1 and Fig. 1. For each initial condition, we calculated the quantities $\max\{\lvert c_r\lambda_r\rvert\}$ and $\sum_{j=1}^{n}(\gamma_j^{(p)})^2\lVert\mathbf{x}_0\rVert$, and compared them to one. Clearly, the locations of $c_r\lambda_r$ determine the stability of the HPDS. The region of attraction R of the HPDS can be obtained by

$$R = \left\{ \mathbf{x} : \begin{array}{l} |-0.8482x_1 - 0.4840x_2 + 0.2152x_3| < \frac{10}{9} \\ |-0.5212x_1 + 0.6899x_2 - 0.5024x_3| < 10 \\ |0.0947x_1 - 0.5382x_2 - 0.8374x_3| < 50 \end{array} \right\},$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$. In addition, the stability condition stated in Proposition 3 is weaker than that in Proposition 2, see Fig. 1 IC a and b.

5.2 Stability using upper bounds of Z-spectral radii

In this example, we applied the upper bound of Z-spectral radii defined in Corollary 2 to obtain a conservative region of attraction for a discrete-time odeco HPDS of the form (15). Suppose that the state transition tensor $\mathsf{A} \in \mathbb{R}^{2\times2\times2\times2}$ is given by

$$\mathsf{A}_{::11} = \begin{bmatrix} 0.2285 & 0.0376 \\ 0.0376 & 0.2243 \end{bmatrix}, \ \mathsf{A}_{::12} = \begin{bmatrix} 0.0376 & 0.2243 \\ 0.2243 & 0.0124 \end{bmatrix},$$
$$\mathsf{A}_{::21} = \begin{bmatrix} 0.0376 & 0.2243 \\ 0.2243 & 0.0124 \end{bmatrix}, \ \mathsf{A}_{::22} = \begin{bmatrix} 0.2243 & 0.0124 \\ 0.0124 & 0.2229 \end{bmatrix}.$$

The U-spectral radius of A can be computed from the spectral radius of the following unfolded matrix

$$\mathbf{A} = \begin{bmatrix} 0.2285 & 0.0376 & 0.0376 & 0.2243 \\ 0.0376 & 0.2243 & 0.2243 & 0.0124 \\ 0.0376 & 0.2243 & 0.2243 & 0.0124 \\ 0.2243 & 0.0124 & 0.0124 & 0.2229 \end{bmatrix}.$$

The U-spectral radius of A is $\mu = \frac{1}{2}$, and thus the conservative region of attraction of the HPDS is an open disk with radius $\sqrt{2}$ centered at the origin (note that the second upper bound (19) produces $\lambda \leq 1.0263$, which will give an even more conservative region of attraction). We tested the following four initial conditions to verify the region of attraction:

$$\mathbf{x_a} = \begin{bmatrix} -1.4 \\ 0 \end{bmatrix}, \ \mathbf{x_b} = \begin{bmatrix} 0.9 \\ -0.9 \end{bmatrix}, \ \mathbf{x_c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{x_d} = \begin{bmatrix} 1.2 \\ 1.2 \end{bmatrix}.$$

The results are shown in Fig. 2. It is clear to see that the trajectories of the HPDS with the initial conditions started within the open disk converge to the origin, see IC a and b. Additionally, since the region of attraction is conservative, we observed that a trajectory started on the circle also converge to the origin, see IC c.

5.3 Stability of ecological networks

In this example, we employed our framework to study the stability of a higher-order ecological network with three species x, y, and z, where its evolution is described by the following discrete-time odeco HPDS:

$$\begin{cases} x_{t+1} &= 0.083x_t^3 + 0.167z_t^3 + 0.25x_t^2 z_t + 0.25x_t y_t^2 + 0.25y_t^2 z_t + 0.5x_t y_t z_t \\ y_{t+1} &= 0.083y_t^3 + 0.167z_t^3 + 0.25x_t^2 y_t + 0.25x_t^2 z_t + 0.25y_t^2 z_t + 0.5x_t y_t z_t \\ z_{t+1} &= 0.083x_t^3 + 0.083y_t^3 - 0.167z_t^3 + 0.25x_t^2 y_t + 0.25x_t y_t^2 + 0.5x_t z_t^2 + 0.5y_t z_t^2 \end{cases},$$

where x_t, y_t, and z_t denote the species abundances. Based on the dynamics equations, it is clear that species x is positively regulated by itself, species y and z individually, and a

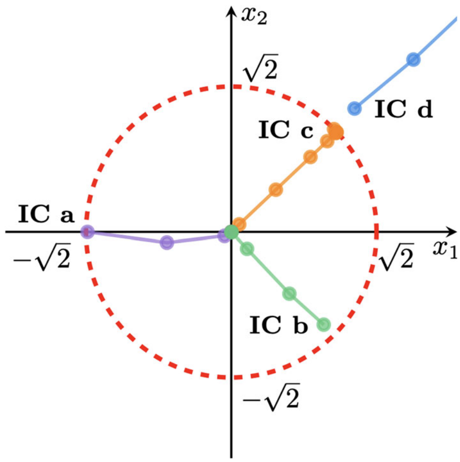

Fig. 2 A conservative region of attraction (red dashed line) of the HPDS, and four initial conditions with their trajectories

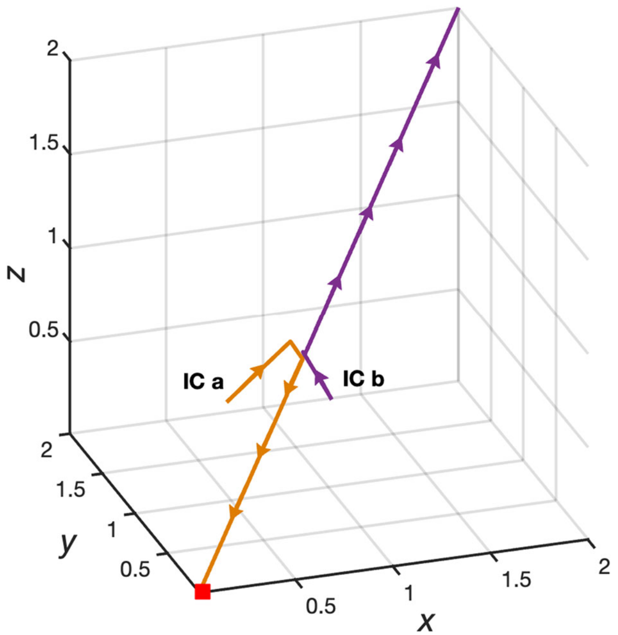

Fig. 3 Trajectories of the HPDS with two distinct initial conditions, where IC a is situated within the region of attraction, whereas IC b is not

combination effect of species y and z (which is a higher-order interaction). Similarly, species y is positively regulated by itself, species x and z individually, and a combination effect of species x and z. Finally, species z is negatively regulated by itself and is positively regulated by species x and y individually and their combination. The Z-eigenvalues of the state transition tensor can be computed as 1/6, 1/2, and 1/2. According to Proposition 2, we can determine the stability of the higher-order ecological network. When the initial abundances of the three species are situated within the region of attraction, the higher-order ecological network is asymptotically stable. Otherwise, it is unstable, see Fig. 3.

6 Conclusion

In this article, we investigated the stability properties of discrete-time HPDSs. In contrast to linear dynamical systems, the stability of HPDSs depends on both the spectrum of the state transition tensor A and initial conditions. In particular, when the state transition tensor A is odeco, we can obtain necessary and sufficient conditions by exploiting tensor Z-eigenvalues. We also provided an upper bound for the Z-spectral radii of even-order supersymmetric tensors, which can be used to determine the asymptotic stability of odeco HPDSs efficiently. In addition, we extended the stability results to general HPDSs, where the state transition tensors are almost symmetric, using tensor singular values.

While we applied odeco HPDSs to model higher-order ecological networks, the thorough interpretation of this class remains elusive due to the complex nature of odeco tensors. Therefore, it is crucial to delve into the interpretation of odeco HPDSs. One promising avenue is to investigate necessary and sufficient conditions on network structures for which the corresponding state transition tensors are odeco. Additionally, it will be worthwhile to explore stronger stability conditions regarding general HPDSs. We further intend to analyze the stabilizability and reachability of discrete-time HPDSs using tensor algebra in future work.

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Data availability Not applicable.

Declarations

Conflict of interest The author has no conflicts of interest.

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