Data-Driven Analysis of T-Product-Based Dynamical Systems

Xin Mao, Anqi Dong, Ziqin He, Yidan Mei, Shenghan Mei, Ren Wang, *Member, IEEE*, and Can Chen, *Member, IEEE*

***Abstract*—A wide variety of data can be represented using third-order tensors. Applications of these tensors include chemometrics, psychometrics, and image/video processing. However, traditional data-driven frameworks are not naturally equipped to process tensors without first unfolding or flattening the data, which can result in a loss of crucial higher-order structural information. In this letter, we introduce a novel framework for data-driven analysis of T-product-based dynamical systems (TPDSs), where the system evolution is governed by the T-product between a third-order dynamic tensor and a third-order state tensor. In particular, we examine the data informativity of TPDSs concerning system identification, stability, controllability, and stabilizability and illustrate significant computational improvements over unfolding-based approaches by leveraging the unique properties of the T-product. The effectiveness of our framework is demonstrated through both synthetic and real-world examples.**

***Index Terms*—Computational methods, data driven control, large-scale systems.**

I. Introduction

NUMEROUS real-world systems, including those found in image/video processing, biological systems, and social sciences, exhibit complex, multi-dimensional relationships, where the states are often represented as third-order or higher-order tensors [1], [2], [3], [4]. Multi-linear dynamical systems, newly proposed in recent years, extend classical linear systems theory that offer an effective framework for modeling these tensor-based systems that cannot be adequately captured by traditional methods [5], [6], [7]. The T-product is a powerful tool for working with multi-linear dynamical systems, capable of defining unique inverses, eigenvalue decompositions, and singular value decompositions for third-order tensors, analogous to their counterparts in linear algebra [8], [9].

The T-product framework provides a versatile approach to analyzing and controlling multi-linear dynamical systems in a variety of fields, including physics [10], [11], engineering [12], and biology [9], [13]. Specifically, the T-product empowers researchers to perform sophisticated operations on multidimensional data, e.g., images/videos, which are often represented as third-order tensors. Traditional matrix-based methods struggle to capture the complex relationships between different dimensions of images including height, width, and color channels. In contrast, T-product-based methods preserve the inherent multidimensional structure and have potential applications in tasks such as image denoising, image compression, image segmentation, and feature extraction, as demonstrated by recent research [13], [14], [15], [16], [17].

T-product-based dynamical systems (TPDSs) are systems whose evolution is governed by the T-product between a third-order dynamic tensor and a third-order state tensor. The concept was first proposed by Hoover et al. [18] as a generalization of linear time-invariant (LTI) systems. TPDSs offer a powerful framework for capturing complex interactions in three-dimensional data. The development of tensor decomposition techniques and circulant algebra has enabled a seamless extension of linear systems theory to TPDSs, covering fundamental concepts like explicit solutions, stability, controllability, and observability. Nevertheless, the lack of computational tools for their data-driven analysis has limited their use in practical applications. This gap is particularly evident in areas requiring system identification, stability, controllability, and stabilizability from observational data.

Data-driven analysis and control have garnered significant attention in recent years [19], [20], [21]. The origins of this field can be traced back to the early 1980s, with the pioneering work on the fundamental lemma (also known as persistency of excitation) [22], which laid the theoretical foundation for using input-output data to infer system properties. Recently, Van Waarde et al. [19] presented a novel data-driven analysis and control framework to investigate the data informativity

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of LTI systems, where data are not informative enough to uniquely identify the system (i.e., the fundamental lemma fails). Although data-driven research has attracted considerable interest, the exploration of data-driven approaches specifically for tensor-based systems, including the fundamental lemma, remains relatively underdeveloped.

We focus on developing a data-driven approach for discrete-time TPDSs. Tensor unfolding, which maps tensors into matrices or vectors, can be used to extend traditional linear system identification, stability, and controllability analyses to TPDSs. However, this approach faces significant challenges due to the curse of dimensionality. We use the properties of the T-product to address this challenge. We provide effective and efficient conditions of data informativity for system identification (i.e., the fundamental lemma), stability, controllability, and stabilizability of TPDSs. Additionally, we show how T-product-based conditions offer significant computational advantages over unfolding-based approaches, demonstrating their applicability through numerical experiments. The rest of this letter is structured as follows. In Section II, we begin with an overview of T-product operations. In Section III, we introduce TPDSs and examine the data informativity of system identification, stability, controllability, and stabilizability. In Section IV, we provide numerical examples for the proposed methods. Finally, we conclude with future directions in Section V.

II. Preliminaries

Tensors can be considered as multidimensional arrays, which extend the concepts of vectors and matrices to higher-dimensional settings [23], [24], [25]. The order of a tensor is defined as the number of dimensions. Of particular interest in this letter are third-order tensors, which we denote by $\mathcal{T} \in \mathbb{R}^{n\times m\times s}$. A frontal slice of $\mathcal{T}$, denoted by $\mathcal{T}_{::j}$, is a matrix obtained by fixing the third mode and allowing the first two modes to vary. In the following, we introduce the notion of the T-product, an effective operation for manipulating third-order tensors, that enables their multiplication through the notion of circular convolution [9], [14], [15]. We first define four operations for a third-order tensor $\mathcal{T} \in \mathbb{R}^{n\times m\times s}$ as follows:

$$\texttt{bcirc}(\mathcal{T}) = \begin{bmatrix} \mathcal{T}_{::1} & \mathcal{T}_{::s} & \cdots & \mathcal{T}_{::2} \\ \mathcal{T}_{::2} & \mathcal{T}_{::1} & \cdots & \mathcal{T}_{::3} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{T}_{::s} & \mathcal{T}_{::(s-1)} & \cdots & \mathcal{T}_{::1} \end{bmatrix} \in \mathbb{R}^{ns\times ms},$$

$$\texttt{unfold}(\mathcal{T}) = \begin{bmatrix} \mathcal{T}_{::1}^\top & \mathcal{T}_{::2}^\top & \cdots & \mathcal{T}_{::s}^\top \end{bmatrix}^\top \in \mathbb{R}^{ns\times m},$$

$\texttt{un-bcirc}$ is the reverse operation of $\texttt{bcirc}$ such that $\texttt{un-bcirc}(\texttt{bcirc}(\mathcal{T})) = \mathcal{T}$, and $\texttt{fold}$ is the reverse operation of $\texttt{unfold}$ such that $\texttt{fold}(\texttt{unfold}(\mathcal{T})) = \mathcal{T}$.

Definition 1 (T-Product): The T-product between two third-order tensor $\mathcal{T} \in \mathbb{R}^{n\times m\times s}$ and $\mathcal{S} \in \mathbb{R}^{m\times r\times s}$, denoted by $\mathcal{T} \star \mathcal{S}$, is defined as

$$\mathcal{T} \star \mathcal{S} = \texttt{fold}\Big(\texttt{bcirc}(\mathcal{T})\texttt{unfold}(\mathcal{S})\Big) \in \mathbb{R}^{n\times r\times s}. \quad (1)$$

It is noteworthy that many fundamental matrix operations, including identity, transpose, inverse, and orthogonality can be also generalized to third-order tensors using the T-product:

1) The T-identity tensor $\mathcal{J}$ is defined as having the first frontal slice (i.e., $\mathcal{J}_{::1}$) as the identity matrix with all other frontal slices consisting of zeros.
2) The T-transpose of $\mathcal{T} \in \mathbb{R}^{n\times m\times s}$, denoted by $\mathcal{T}^\top$, is obtained by transposing each of the frontal slices and then reversing the order of the transposed frontal slices from 2 to s.
3) The T-inverse of $\mathcal{T} \in \mathbb{R}^{n\times n\times s}$, denoted by $\mathcal{T}^{-1}$, is defined as $\mathcal{T} \star \mathcal{T}^{-1} = \mathcal{T}^{-1} \star \mathcal{T} = \mathcal{J}$ (similarly for left and right T-inverse).
4) A third-order tensor $\mathcal{T} \in \mathbb{R}^{n\times n\times s}$ is called T-orthogonal is $\mathcal{T} \star \mathcal{T}^\top = \mathcal{T}^\top \star \mathcal{T} = \mathcal{J}$.

All operations explained above can be computed through the block circulant operation. For example, the T-inverse can be attained as $\mathcal{T}^{-1} = \texttt{un-bcirc}(\texttt{bcirc}(\mathcal{T})^{-1})$. Therefore, the T-product does not form a group in the space of third-order tensors because a T-inverse does not always exist (though it is associative). With a slight abuse of notation, we use the same superscript for both matrix and T-product-based operations.

Notably, eigenvalue decomposition and singular value decomposition can be defined for third-order tensors in a similar manner as matrices through the T-product [8], [9].

Definition 2 (T-Eigenvalue Decomposition): The T-eigenvalue decomposition of a third-order tensor $\mathcal{T} \in \mathbb{R}^{n\times n\times s}$ is defined as

$$\mathcal{T} = \mathcal{U} \star \mathcal{D} \star \mathcal{U}^{-1}, \quad (2)$$

where $\mathcal{U} \in \mathbb{C}^{n\times n\times s}$ and $\mathcal{D} \in \mathbb{C}^{n\times n\times s}$ is F-diagonal (i.e., each of its frontal slices is a diagonal matrix) such that the vectors $\mathcal{D}_{jj:} \in \mathbb{C}^s$ are referred to the eigentuples of $\mathcal{T}$.

The T-eigenvalue decomposition can be computed using circulant operation $\texttt{bcirc}$ and matrix eigenvalue decomposition. However, employing the discrete Fourier transform can significantly expedite the process. In particular, a circulant matrix can be block diagonalized via left and right multiplication by a block diagonal discrete Fourier transform matrix. The Fourier transform $\mathcal{F}\{\texttt{bcirc}(\mathcal{T})\}$ is defined as

$$\begin{aligned} \mathcal{F}\{\texttt{bcirc}(\mathcal{T})\} &= (\mathbf{F}_n \otimes \mathbf{I}_s)\texttt{bcirc}(\mathcal{T})\big(\mathbf{F}_n^* \otimes \mathbf{I}_s\big) \\ &= \texttt{blkdiag}(\mathbf{T}_1, \ldots, \mathbf{T}_s), \end{aligned}$$

where the operation $\texttt{blkdiag}$ denotes the MATLAB block diagonal function, the superscript $*$ denotes the conjugate transpose, $\mathbf{I}_s \in \mathbb{R}^{s\times s}$ is the identity matrix, and $\mathbf{F}_n \in \mathbb{C}^{n\times n}$ is the discrete Fourier transform matrix defined as

$$\mathbf{F}_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix},$$

with $\omega = \exp\{\frac{-2\pi i}{n}\}$ (note that i denotes the imaginary number here), and $\otimes$ denotes the Kronecker product. In practice, the matrix $\mathbf{F}_n$ does not need to be explicitly formed, and its effect can be efficiently implemented using a fast Fourier transform operation. The T-eigenvalue decomposition of $\mathcal{T}$ can be constructed through the eigenvalue decompositions of $\mathbf{T}_j$. Given that $\mathbf{T}_j = \mathbf{U}_j\mathbf{D}_j\mathbf{U}_j^{-1}$, $\mathcal{U}$ can be recovered by

$\mathtt{bcirc}(\mathcal{U}) = (\mathbf{F}_n^* \otimes \mathbf{I}_s)\mathtt{blkdaig}(\mathbf{U}_1, \dots, \mathbf{U}_s)(\mathbf{F}_n \otimes \mathbf{I}_s)$. The F-diagonal tensor $\mathcal{D}$ can be obtained in a similar manner.

Definition 3 (T-Singular Value Decomposition): T-singular value decomposition (T-SVD) of a third-order tensor $\mathcal{T} \in \mathbb{R}^{n\times m\times s}$ is defined as

$$\mathcal{T} = \mathcal{U} \star \mathcal{S} \star \mathcal{V}^\top, \tag{3}$$

where $\mathcal{U} \in \mathbb{R}^{n\times n\times s}$ and $\mathcal{V} \in \mathbb{R}^{m\times m\times s}$ are T-orthogonal, and $\mathcal{S} \in \mathbb{R}^{n\times m\times s}$ is a F-(rectangle) diagonal tensor such that $\mathcal{S}_{jj:} \in \mathbb{R}^s$ are referred to the singular tuples of $\mathcal{T}$.

The T-SVD can be computed analogously by applying the Fourier transform $\mathcal{F}\{\mathtt{bcirc}(\mathcal{T})\}$ and then performing singular value decomposition on the block diagonal matrices $\mathbf{T}_j$. We will show that both T-eigenvalue decomposition and T-SVD play critical roles in data-driven analysis of TPDSs.

III. Data-Driven Analysis of TPDSs

We are now positioned to conduct data-driven analysis of discrete-time TPDSs, which are defined as

$$\mathcal{X}(t+1) = \mathcal{A} \star \mathcal{X}(t), \tag{4}$$

with $\mathcal{A} \in \mathbb{R}^{n\times n\times r}$ represents the state transition tensor, and $\mathcal{X}(t) \in \mathbb{R}^{n\times h\times r}$ denotes the state. Significantly, the TPDS (4) can be transformed into LTI systems using bcirc, unfold, and vec (i.e., the vectorization operation), resulting in three equivalent linear representations, i.e.,

$$\mathtt{unfold}(\mathcal{X}(t+1)) = \mathtt{bcirc}(\mathcal{A})\mathtt{unfold}(\mathcal{X}(t)), \tag{5}$$

$$\mathtt{bcirc}(\mathcal{X}(t+1)) = \mathtt{bcirc}(\mathcal{A})\mathtt{bcirc}(\mathcal{X}(t)), \tag{6}$$

$$\mathtt{vec}(\mathcal{X}(t+1)) = (\mathbf{I}_{hr} \otimes \mathtt{bcirc}(\mathcal{A}))\mathtt{vec}(\mathcal{X}(t)). \tag{7}$$

As a result, data-driven analysis techniques from LTI systems can be effectively extended to TPDSs. The state data tensors are constructed by assembling the data as shown below:

$$\mathcal{X}_0 = \left[\mathcal{X}(0)\ \mathcal{X}(1)\ \cdots\ \mathcal{X}(l-1)\right] \in \mathbb{R}^{n\times lh\times r},$$
$$\mathcal{X}_1 = \left[\mathcal{X}(1)\ \mathcal{X}(2)\ \cdots\ \mathcal{X}(l)\right] \in \mathbb{R}^{n\times lh\times r}.$$

While the unfolded linear representations are useful for studying the data-driven analysis of TPDSs, the full computational benefits are achieved by leveraging the properties of the T-product. In the following, we first examine the data informativity of TPDSs with respect to system identification, stability, controllability, and stabilizability. The efficiency of T-product-based computations in expressing these conditions is also illustrated, along with numerical examples.

A. System Identification

The data informativity for system identification of TPDSs involves determining the conditions under which the state transition tensor $\mathcal{A}$ can be uniquely identified. It plays a crucial role in extracting valuable information from time-series data of third-order tensors, such as distinguishing foregrounds and backgrounds in image/video data.

Definition 4 (System Identification): We say the data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for system identification if the state transition tensor $\mathcal{A}$ can be uniquely identified.

Proposition 1: The data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for system identification if and only if the rank of $\mathtt{bcirc}(\mathcal{X}_0)$ is equal to nr.

Proof: Substituting data $(\mathcal{X}_0, \mathcal{X}_1)$ into the TPDS (4) gives $\mathcal{X}_1 = \mathcal{A} \star \mathcal{X}_0$. Due to the properties of block circulant matrix operation, it follows that

$$\mathtt{bcirc}(\mathcal{A}) = \mathtt{bcirc}(\mathcal{X}_1)\mathtt{bcirc}(\mathcal{X}_0)^\dagger.$$

According to linear matrix theory, $\mathtt{bcirc}(\mathcal{A})$ can be uniquely determined if and only if $\mathtt{bcirc}(\mathcal{X}_0)$ has full row rank, i.e., nr. The result thus follows immediately. ∎

We choose to use the second linear representation (6) because the rank of $\mathtt{bcirc}(\mathcal{X}_0)$ can be determined through T-product-based computations. Specifically, we can leverage the T-SVD of $\mathcal{X}_0$ and the associated block diagonal matrices in the Fourier domain.

Corollary 1: The data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for system identification if and only if all entries of the singular tuples of $\mathcal{X}_0$ in the Fourier domain are nonzero.

Proof: Based on the finding in [11], the singular values of $\mathtt{bcirc}(\mathcal{X}_0)$ are the union of elements from the singular tuples of $\mathcal{X}_0$ in the Fourier domain. According to linear matrix theory, the rank of $\mathtt{bcirc}(\mathcal{X}_0)$ is determined by the number of its nonzero singular values. Therefore, the result follows from Proposition 1. ∎

Corollary 2: The data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for system identification if and only if the sum of the ranks of the block diagonal matrices of $\mathcal{F}\{\mathtt{bcirc}(\mathcal{X}_0)\}$ is equal to nr.

Proof: The singular tuples of $\mathcal{X}_0$ in the Fourier domain can be computed from the SVDs of the block diagonal matrices of $\mathcal{F}\{\mathtt{bcirc}(\mathcal{X}_0)\}$. Therefore, each block diagonal matrix has full rank if and only if the corresponding singular tuples in the Fourier domain contains nonzero entries. The result then follows from Proposition 1. ∎

Remark 1: The time complexity of directly computing the rank of $\mathtt{bcirc}(\mathcal{X}_0)$ is about $\mathcal{O}(n^2r^3lh)$ (assuming $n < lh$). Note that the complexity reduces to $\mathcal{O}(n^2r^2lh)$ by using the linear representation (5). On the other hand, both Corollaries 1 and 2 only require $\mathcal{O}(n^2rlh)$ operations for determining the data informativity for system identification of TPDSs. Therefore, T-product-based computations offer computational benefits over the unfolding-based approach.

B. Stability

The data informativity for the stability of TPDSs involves determining whether any state transition tensor $\mathcal{A}$ identified from the data is stable. Specifically, a state transition tensor $\mathcal{A}$ is considered stable if $\mathtt{bcirc}(\mathcal{A})$ is stable, meaning that the eigenvalues of $\mathtt{bcirc}(\mathcal{A})$ lie within the unit circle. As an illustrative example, assessing the stability of image/video data is essential for ensuring the reliability and consistency of visual information over time across different conditions (e.g., distortions, noise, or transformations).

Definition 5 (Stability): We say the data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for stability if any state transition tensor $\mathcal{A}$ identified from the data is stable.

Proposition 2: The data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for stability if the following conditions: (i) the rank of $\mathtt{bcirc}(\mathcal{X}_0)$

is equal to nr; (ii) $\texttt{bcirc}(\mathcal{X}_1 \star \mathcal{X}_0^\dagger)$ is stable for any right T-inverse $\mathcal{X}_0^\dagger$, are satisfied.

Proof: Based on the finding of the data informativity for stability of LTI systems [19], the matrix data $(\texttt{bcirc}(\mathcal{X}_0), \texttt{bcirc}(\mathcal{X}_1))$ are informative for stability if and only if the rank of $\texttt{bcirc}(\mathcal{X}_0)$ is equal to nr and $\texttt{bcirc}(\mathcal{X}_1)\texttt{bcirc}(\mathcal{X}_0)^\dagger$ is stable. Additionally, according to the properties of block circulant matrices, it follows that

$$\texttt{bcirc}(\mathcal{X}_1)\texttt{bcirc}(\mathcal{X}_0)^\dagger = \texttt{bcirc}\left(\mathcal{X}_1 \star \mathcal{X}_0^\dagger\right).$$

Therefore, the result follows immediately. ■

Similar to LTI systems, the data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for stability only if the system can be uniquely identified (i.e., the data are informative for system identification). In the following, we exploit the T-eigenvalue decomposition/T-SVD and the corresponding block diagonal matrices in the Fourier domain to articulate the aforementioned conditions.

Corollary 3: The data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for stability if and only if the following conditions: (i) all entries of the singular tuples of $\mathcal{X}_0$ in the Fourier domain are nonzero; (ii) all entries of the eigentuples of $\mathcal{X}_1 \star \mathcal{X}_0^\dagger$ in the Fourier domain lie within the unit circle for any right T-inverse $\mathcal{X}_0^\dagger$, are satisfied.

Proof: The first condition follows Corollary 1. For the second condition, based on the finding in [11], the eigenvalues of $\texttt{bcirc}(\mathcal{X}_1 \star \mathcal{X}_0^\dagger)$ are the union of elements from the eigentuples of $\mathcal{X}_1 \star \mathcal{X}_0^\dagger$ in the Fourier domain. Therefore, the result follows from Proposition 2. ■

Corollary 4: The data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for stability if and only if the following conditions: (i) the sum of the ranks of the block diagonal matrices of $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_0)\}$ is equal to nr; (ii) the block diagonal matrices of $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_1 \star \mathcal{X}_0^\dagger)\}$ are stable for any right T-inverse $\mathcal{X}_0^\dagger$,are satisfied.

Proof: The first condition follows Corollary 2. For the second condition, the eigentuples of $\mathcal{X}_1 \star \mathcal{X}_0^\dagger$ in the Fourier domain can be computed from the eigenvalue decomposition of the block diagonal matrices of $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_1 \star \mathcal{X}_0^\dagger)\}$. Therefore, each block diagonal matrix is stable if and only if the corresponding eigentuple in the Fourier domain contains nonzero entries. The result follows from Proposition 2. ■

Remark 2: The time complexity of directly computing the eigenvalues of $\texttt{bcirc}(\mathcal{X}_1 \star \mathcal{X}_0^\dagger)$ is estimated as $\mathcal{O}(n^3r^3)$. On the contrary, both Corollaries 3 and 4 only involve $\mathcal{O}(n^3r)$ operations for determining the data informativity for stability of TPDSs (excluding the time for system identification). Hence, the T-product-based computations are advantageous compared to the unfolding-based approach.

C. Controllability & Stabilizability

The data informativity for controllability/stabilizability of TPDSs entails determining the conditions under which any system identified from the data is controllable/stabilizable. Both notions have significant implications for optimal control design, e.g., driving image/video data to a desired state for specific applications like object tracking or scene manipulation. First, we introduce the model of TPDSs with control

$$\mathcal{X}(t+1) = \mathcal{A} \star \mathcal{X}(t) + \mathcal{B} \star \mathcal{U}(t), \tag{8}$$

where $\mathcal{B} \in \mathbb{R}^{n\times m\times r}$ represents the control matrix, and $\mathcal{U}(t) \in \mathbb{R}^{m\times h\times r}$ denotes the control input. Let T denote the finite time horizon. The system (8) is said to be controllable if for any initial state $\mathcal{X}(0)$ and target state $\mathcal{X}(T)$, there exists a sequence of inputs $\mathcal{U}(t)$ that drives the system from $\mathcal{X}(0)$ to $\mathcal{X}(T)$ [23]. The system (8) is considered stabilizable if there exists a sequence of inputs of the form $\mathcal{U}(t) = \mathcal{K} \star \mathcal{X}(t)$ for $\mathcal{K} \in \mathbb{R}^{m\times n\times r}$, such that the new system $\mathcal{A} + \mathcal{B} \star \mathcal{K}$ is stable. Finally, suppose that the input data is collected as

$$\mathcal{U}_0 = \left[\mathcal{U}(0)\ \mathcal{U}(1)\ \cdots\ \mathcal{U}(l-1)\right] \in \mathbb{R}^{m\times lh\times r}.$$

The data informativity of controllability and stabilizability for controlled TPDSs can be defined as follows.

Definition 6 (Controllability): We say data $(\mathcal{U}_0, \mathcal{X}_0, \mathcal{X}_1)$ are informative for controllability if any pair $(\mathcal{A}, \mathcal{B})$ identified by the data is controllable.

Definition 7 (Stabilizability): We say data $(\mathcal{U}_0, \mathcal{X}_0, \mathcal{X}_1)$ are informative for stabilizability if any pair $(\mathcal{A}, \mathcal{B})$ identified by the data is stabilizable.

Proposition 3: The data $(\mathcal{U}_0, \mathcal{X}_0, \mathcal{X}_1)$ are informative for controllability if and only if the rank of $\texttt{bcirc}(\mathcal{X}_1 - \lambda\mathcal{X}_0)$ is equal to nr for any $\lambda \in \mathbb{C}$.

Proof: Based on the finding of the data informativity for controllability of LTI systems [19], the matrix data $(\texttt{bcirc}(\mathcal{X}_0), \texttt{bcirc}(\mathcal{X}_1))$ are informative for controllability if and only if the rank of $\texttt{bcirc}(\mathcal{X}_1) - \lambda\texttt{bcirc}(\mathcal{X}_0)$ is equal to nr for any $\lambda \in \mathbb{C}$. Moreover, according to the properties of block circulant matrices, it follows that

$$\texttt{bcirc}(\mathcal{X}_1) - \lambda\texttt{bcirc}(\mathcal{X}_0) = \texttt{bcirc}(\mathcal{X}_1 - \lambda\mathcal{X}_0),$$

and the result follows immediately. ■

Similar to the result in [19], the above condition is equivalent to the rank of $\texttt{bcirc}(\mathcal{X}_1)$ being equal to nr, and the rank of $\texttt{bcirc}(\mathcal{X}_1 - \lambda\mathcal{X}_0)$ being equal to nr for all $\lambda \neq 0$, with $\lambda^{-1} \in \sigma(\texttt{bcirc}(\mathcal{X}_0)\texttt{bcirc}(\mathcal{X}_1)^\dagger)$, where σ denotes the spectrum of a matrix. The following two corollaries can be proven similarly as Corollaries 1 and 2 with T-SVD and block diagonal matrices in the Fourier domain.

Corollary 5: The data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for controllability if and only if all entries of the singular tuples of $\mathcal{X}_1 - \lambda\mathcal{X}_0$ in the Fourier domain are nonzero for any $\lambda \in \mathbb{C}$.

Corollary 6: The data $(\mathcal{X}_0, \mathcal{X}_1)$ are informative for stabilizability if and only if the sum of the ranks of the block diagonal matrices of $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_1 - \lambda\mathcal{X}_0)\}$ is equal to nr for any $\lambda \in \mathbb{C}$.

For the data informativity regarding the stabilizability of TPDSs, an additional condition of $|\lambda| \geq 1$ is required, as established by [19]. The computational complexity analysis follows similar principles as those presented in Remark 1.

IV. Numerical Examples

We proceed to illustrate our framework with the following numerical experiments. All experiments in this section were conducted on a platform equipped with an M1 Pro CPU and 16GB of memory. The code used for these experiments is available at https://github.com/dytroshut/TPDSs.

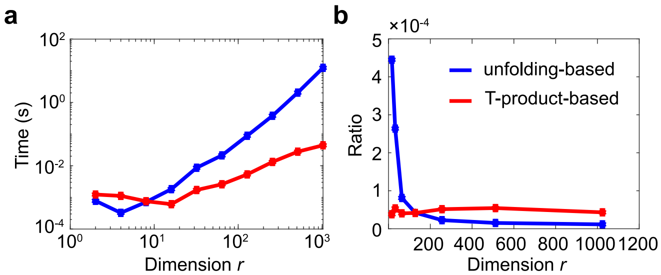

Fig. 1. Computational time comparison in determining the data informativity for system identification between the unfolding-based and T-product-based approaches. **a.** Log-log plot of computational time with respect to the dimension of the third mode r. **b.** Ratio of time to the dimension of the third mode r (i.e., time/r^3 for the unfolding-based approach and time/r for the T-product-based approach).

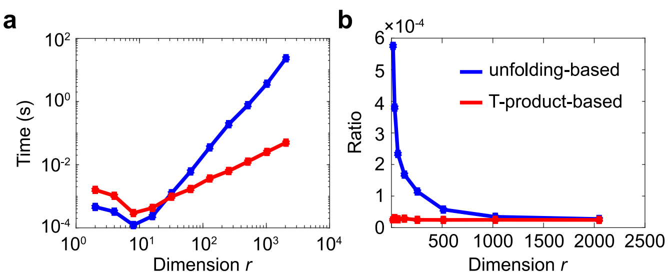

Fig. 2. Computational time comparison in determining the data informativity for stability between the unfolding-based and T-product-based approaches. **a.** Log-log plot of computational time with respect to the dimension of the third mode r. **b.** Ratio of time to the dimension of the third mode r (i.e., time/r^3 for the unfolding-based approach and time/r for the T-product-based approach).

A. System Identification

In this example, we evaluated our approach for determining the data informativity for system identification of TPDSs. We first generated image data $\mathcal{X}(0), \mathcal{X}(1), \dots, \mathcal{X}(l) \in \mathbb{R}^{2\times 2\times r}$ from a predefined TPDS and constructed the state data tensor $\mathcal{X}_0 \in \mathbb{R}^{2\times 2l\times r}$ with $l = 10$ and $r = 2^p$ for $p = 2, 3, \dots, 10$. By Corollary 2, the sum of the ranks of the block diagonal matrices of $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_0)\}$ is equivalent to the rank of $\texttt{bcirc}(\mathcal{X}_0)$. We then compared the computational efficiency of computing the ranks of all block matrices in $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_0)\}$ with Proposition 1 that directly computes the rank of $\texttt{bcirc}(\mathcal{X}_0)$. The computation time for each rank relative to the corresponding dimension r is shown in Fig. 1a, demonstrating that our approach significantly outperforms the direct rank computation. Fig. 1b also shows that our results are consistent with the complexity analysis.

B. Stability

In this example, we evaluated our approach for determining the data informativity for stability of TPDSs. We first generated image data $\mathcal{X}(0), \mathcal{X}(1), \dots, \mathcal{X}(l) \in \mathbb{R}^{2\times 2\times r}$ from a predefined TPDS and constructed the state data tensors $\mathcal{X}_0, \mathcal{X}_1 \in \mathbb{R}^{2\times 2l\times r}$ with $l = 10$ and $r = 2^p$ for $p = 2, 3, \dots, 11$. By Corollary 4, the eigenvalues of the block diagonal matrices of $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_1 \star \mathcal{X}_0^\dagger)\}$ are equivalent to the eigenvalues of $\texttt{bcirc}(\mathcal{X}_1 \star \mathcal{X}_0^\dagger)$. After ensuring the first condition, we computed the eigenvalue decompositions of all block matrices in $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_1 \star \mathcal{X}_0^\dagger)\}$ and compared the computational efficiency with Proposition 2 which directly computes the eigenvalue decomposition of $\texttt{bcirc}(\mathcal{X}_1 \star \mathcal{X}_0^\dagger)$. As with the first example, the computational time for using Corollary 4 is significantly less than that of the unfolding-based approach as r increases, see Fig. 2a. Moreover, this result is consistent with our complexity analysis, see Fig. 2(b).

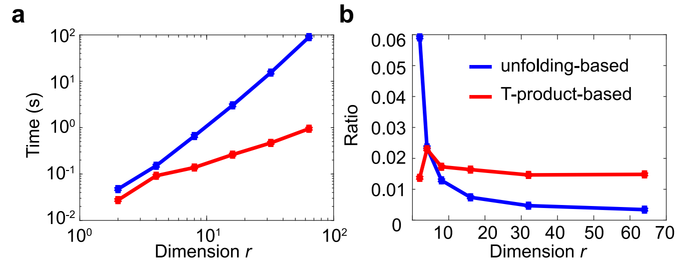

Fig. 3. Computational time comparison in determining the data informativity for controllability between the unfolding-based and T-product-based approaches. **a.** Log-log plot of computational time with respect to the dimension of the third mode r. **b.** Ratio of time to the dimension of the third mode r (i.e., time/r^3 for the unfolding-based approach and time/r for the T-product-based approach).

C. Controllability

In this example, we evaluated our approach for determining the data informativity for controllability of TPDSs. We first generated image data $\mathcal{X}(0), \mathcal{X}(1), \dots, \mathcal{X}(l) \in \mathbb{R}^{2\times 2\times r}$ from a predefined TPDS and constructed the state data tensors $\mathcal{X}_0, \mathcal{X}_1 \in \mathbb{R}^{2\times 2l\times r}$ with $l = 10$ and $r = 2^p$ for $p = 2, 3, \dots, 9$. By Corollary 6, the sum of the ranks of the block diagonal matrices of $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_1 - \lambda\mathcal{X}_0)\}$ is equivalent to the rank of $\texttt{bcirc}(\mathcal{X}_1 - \lambda\mathcal{X}_0)$. We then computed the ranks of all block matrices in $\mathcal{F}\{\texttt{bcirc}(\mathcal{X}_1 - \lambda\mathcal{X}_0)\}$ and compared the computational efficiency with Proposition 3 that directly computes the rank of $\texttt{bcirc}(\mathcal{X}_1 - \lambda\mathcal{X}_0)$. Here, we used the MATLAB symbolic computation to compute the ranks (i.e., symbolic ranks). As with the first two examples, our approach is significantly faster in determining the data informativity for controllability when r is large compared to the unfolding-based approach, see Fig. 3a. Again, our results align with the complexity analysis, see Fig. 3b.

D. Case Study on Video Data

We conducted a case study on a video dataset to illustrate the efficacy of our proposed framework. The video consists of a static noisy background overlaid with a single white square that moves horizontally from left to right at a constant speed. Two snapshots from the video at different time stamps are shown in Fig. 4. The video comprises 70 frames, each defined at a specific time instant and having a resolution of 70 × 70 pixels. Thus, it is represented as a three-dimensional tensor, while each individual frame is represented as a two-dimensional matrix. The spatial dimensions (height and width) correspond to pixel locations, and the temporal dimension captures the sequence of frames. We compared the computational efficiency of the T-product-based methods with the corresponding unfolding-based approaches

Fig. 4. Snapshots of the video at two time stamps. The white square moves horizontally across a static noisy background. The left image shows the initial position of the square, while the right image illustrates its position after horizontal motion.

TABLE I

COMPUTATIONAL TIME COMPARISON (IN SECONDS) IN DETERMINING THE DATA INFORMATIVITY FOR SYSTEM IDENTIFICATION, STABILITY, AND CONTROLLABILITY OF THE VIDEO DATASET

	unfolding-based	T-product-based
System Identification	32.48	0.33
Stability	68.00	1.15
Controllability	23567.02	201.10

in determining data informativity for system identification, stability, and controllability of the underlying video dynamics. The results are shown in Table I, where the T-product-based methods significantly reduce computation time. Additionally, our findings have practical implications. For instance, our ongoing work has demonstrated that data informativity for system identification is crucial for effectively distinguishing the static noise background from the foreground (i.e., the white square) in this video data.

V. CONCLUSION

In this letter, we introduced a data-driven analysis framework for TPDSs, where the system evolution is governed by the T-product. We established effective and efficient criteria for determining the data informativity for system identification, stability, controllability, and stabilzability of TPDSs by leveraging the unique properties of the T-product. We further offered detailed complexity analyses for the proposed criteria and verified them with numerical examples. Our T-product-based framework can handle tensors with up to 2^{30} double-precision floating-point numbers (equivalent to 8 GB data) on standard laptops with 16 GB of memory, demonstrating its effectiveness in managing large-scale datasets. The results can also be readily generalized to continuous-time TPDSs by approximating the continuous-valued dynamics through sampling. In the future, it will be valuable to explore data-driven control of TPDSs, e.g., data informativity for state feedback and quadratic regulators. Additionally, applying the data-driven framework of TPDSs to real-world tasks where data are often imperfect and noisy is essential. Generalizing the results to nonlinear and hybrid systems with higher-order tensors (integrated with tensor decompositions, such as tensor trains) will also be an important direction for future work.

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