ODEs: Adams-Bashforth

September 20, 2021

Forward Euler

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- We can derive Forward Euler's Method by using Taylor series:

$$\begin{aligned} y_{n+1} = y(t_n + h) &= y(t_n) + hy'(t_n) + \frac{1}{2}(h)^2 y''(t_n) + \cdots \\ &= y(t_n) + hf(t_n, y_n) + \frac{1}{2}(h)^2 y''(t_n) + \cdots \end{aligned}$$

so that the scheme become

$$y_{n+1} = y_n + hf(y_n, t_n)$$

Forward Euler

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- We saw last class that this was a *first order method*. This means that if we half the step size, we expect to half the error.
- In a lot of applications, this means that we need to use very small step sizes. Is there a way to get around this?

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- We saw last class that this was a *first order method*. This means that if we half the step size, we expect to half the error.
- In a lot of applications, this means that we need to use very small step sizes. Is there a way to get around this?
- We are only approximating y_{n+1} using data from y_n, but what if we use more points?
- Instead, our scheme might look like

$$y_{n+1} = y_n + \alpha f(t_n, y_n) + \beta f(t_{n-1}, y_{n-1})$$

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- Our scheme looks like

$$y_{n+1} = y_n + \alpha f(t_n, y_n) + \beta f(t_{n-1}, y_{n-1}) = y_n + \alpha y_n' + \beta y_{n-1}'$$

- Suppose we know the values y_n and y_{n-1}, and we want to find y_{n+1}. Let's Taylor Expand again:

$$\begin{aligned} y_{n+1} &= y_n + h y_n' + \frac{h^2}{2} y_n'' + \cdots \\ y_{n-1}' &= y_n' - h y_n'' + \cdots \end{aligned}$$

- Combining the above results, we have

$$y_n + h y_n' + \frac{h^2}{2} y_n'' + \cdots = y_n + \alpha y_n' + \beta \left(y_n' - h y_n'' + \cdots \right)$$

Adams-Bashforth

- We have

$$y_n + hy_n' + \frac{h^2}{2}y_n'' + \cdots = y_n + \alpha y_n' + \beta\left(y_n' - hy_n'' + \cdots\right)$$

which reduces to

$$hy_n' + \frac{h^2}{2}y_n'' + \cdots = \alpha y_n' + \beta\left(y_n' - hy_n'' + \cdots\right)$$

- If we compare different orders of h, we get $\beta = -\frac{h}{2}$ and $\alpha = \frac{3h}{2}$.
- Therefore, our scheme should be

$$y_{n+1} = y_n + \frac{3h}{2}f(t_n, y_n) - \frac{h}{2}f(t_{n-1}, y_{n-1})$$

Adams-Bashforth

- Adams-Bashforth of order 2

$$y_{n+1} = y_n + \frac{3h}{2} f(t_n, y_n) - \frac{h}{2} f(t_{n-1}, y_{n-1})$$

- This is called a *multistep method* because we use multiple values behind us.
- We could use more values behind us, and get a higher order method!
- This is a *second order method* which means if we half our time step, we should decrease our error by a factor of four.

Adams-Bashforth

- Adams-Bashforth of order 2

$$y_{n+1} = y_n + \frac{3h}{2} f(t_n, y_n) - \frac{h}{2} f(t_{n-1}, y_{n-1})$$

- What happens when $n = 1$ on our first step?
- We need two points in order to use this method.
- One option: Use Euler's method as a first step, and then use Adams-Bashforth.

Error Analysis

- This method is of order 2 which means the error term looks like

$$e_h = y_N - y_{\text{exact}}(t_N) \approx Ch^2$$

- If we take the log of both sides, we get

$$\log e_h = \log Ch^2 = \log C + \log h^2 = \log C + 2\log h$$

- This is the equation of a line with slope 2. Therefore, if we plot $\log h$ on the x-axis, and $\log e_h$ and the y-axis, we should have a line with slope 2.
- Forward Euler is first order which means the error term looks like

$$e_h \approx Ch$$

- This results in a line of slope 1 (after taking the log of both sides).