# Root Finding: Newton’s Method

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- Suppose we have some initial guess $x_0$ of the root.

- Taylor expand $f(x)$ around $x_0$:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2}f''(x_0) + \cdots$$

- If $x_0 - x$ is small enough, then $(x - x_0)^2$ is even smaller.

# Root Finding

- Neglecting the terms $(x - x_0)^n$ for $n > 1$ and letting $f(x) = 0$:

$$0 \approx f(x_0) + (x - x_0) f'(x_0)$$

- Solving for $x$:

$$x \approx x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Let this new approximation to the root be $x_1$ and continue iteratively.

# Newton's Method

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

# Error Analysis

- How good is this method for computing the root?
- Let $x^\star$ be the root of $f(x)$, i.e. $f(x^\star) = 0$.
- Taylor expand $f(x^\star)$ around $x_n$, the $n$th iteration of the method:

$$0 = f(x^\star) = f(x_n) + f'(x_n)(x^\star - x_n) + \frac{(x^\star - x_n)^2}{2} f''(\xi_n)$$

where $\xi_n$ is between $x_n$ and $x^\star$.

- Rearranging terms, we get that

$$\frac{f(x_n)}{f'(x_n)} + x^\star - x_n = \frac{-f''(\xi_n)}{2f'(x_n)}(x^\star - x_n)^2$$

# Error Analysis

$$\frac{f(x_n)}{f'(x_n)} + x^\star - x_n = \frac{-f''(\xi_n)}{2f'(x_n)}(x^\star - x_n)^2$$

- Recall that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- This gives us

$$x^\star - x_{n+1} = \frac{-f''(\xi_n)}{2f'(x_n)}(x^\star - x_n)^2$$

  or by letting $e_n = x^\star - x_n$ be the error at the $n$th step:

$$e_{n+1} = \frac{-f''(\xi_n)}{2f'(x_n)} e_n^2$$

- The error decreases quadratically at each step.

# Newton's Method

- When do we stop the iterations? Let $x^\star$ be the root, i.e. $f(x^\star) = 0$.

- Three ways to determine when we've done enough:
  1) $|x^\star - x_n| < \text{tol}$. If we are close enough to the root, we can stop.
  2) $|x_n - x_{n-1}| < \text{tol}$. Iterations are getting really close together.
  3) $N$ is too big. Finding the root is taking to long.

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  3) $N$ is too big. Finding the root is taking to long.
- We, in general, do not know $x^\star$, so we need to use cases 2 and 3.