ODEs: Forward Euler

September 13, 2021

ODEs

- Consider an *n*th order ODE of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_2y'' + a_1y' = f(t, y)$$

- We can change this into a system of $n - 1$ order ODEs by making the substitution $v = y^{(n)}$.

- Therefore we only really need to deal with first order ODEs.

Numerical ODEs

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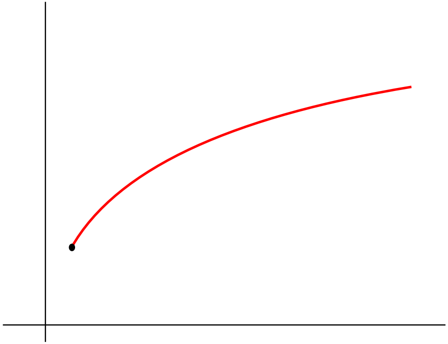
- We also know the value of the slope of the tangent line, $y'(t) = f(y,t)$. We can compute the tangent line, and step forward a small amount on the tangent line.
- At our new step, we get the approximate value $(dt, y(0) + \Delta t f(0, y(0)))$



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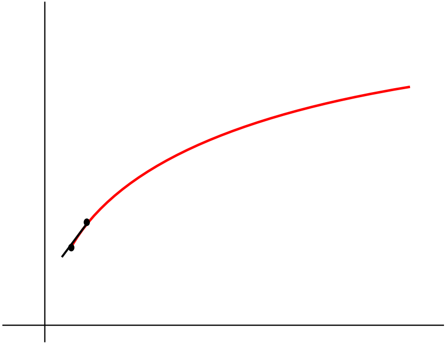

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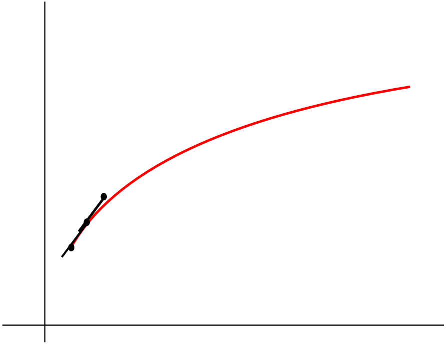

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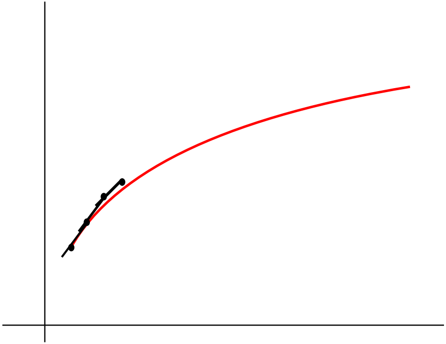

Euler's Method

- This leads us to the formulation $y_{n+1} = y_n + \Delta t f(t_n, y_n)$ where y_n is the n^{th} approximation and $t_n = n\Delta t$.
- For example, the ODE $y' = y$ with $y(0) = 1$ has the formulation

$$y_{n+1} = y_n + \Delta t y_n = y_n(1 + \Delta t)$$

Error of Euler's Method

- The *Local Truncation Error* is the error after one time step. Consider $y(t_0 + \Delta t)$:

$$
\begin{aligned}
y(t_0 + \Delta t) &= y(t_0) + \Delta t y'(t_0) + \frac{1}{2}(\Delta t)^2 y''(t_0) + \cdots \\
&= y(t_0) + \Delta t f(t_0, y_0) + \frac{1}{2}(\Delta t)^2 y''(t_0) + \cdots
\end{aligned}
$$

- Then the local truncation error is

$$
\begin{aligned}
y_1 - y(t_0 + \Delta t) &= (y_0 + \Delta t f(t_0, y_0) - (y_0 + \Delta t f(t_0, y_0) + \frac{1}{2}(\Delta t)^2 y''(t_0) + \cdots) \\
&= -\frac{1}{2}(\Delta t)^2 y''(t_0) + \cdots
\end{aligned}
$$

- So the local truncation error is second order.

Error of Euler's Method

- The *Global Truncation Error* is the error at the end of the simulation at time T.
- The total number of time steps to reach T is $\frac{T}{\Delta t}$.
- The local error at each time step is about $(\Delta t)^2$.
- Therefore, the global truncation error should be about Δt.