

ODEs: Runge-Kutta methods

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Forward Euler

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Forward Euler

- Problem: Solve the ordinary differential equation $y'(t) = f(y, t)$ with initial data $y(0) = y_0$ on the domain $[0, L]$
- We can derive Forward Euler's Method by using Taylor series:

$$\begin{aligned}y_{n+1} &= y(t_n + h) = y(t_n) + hy'(t_n) + \frac{1}{2}(h)^2y''(t_n) + \cdots \\&= y(t_n) + hf(t_n, y_n) + \frac{1}{2}(h)^2y''(t_n) + \cdots\end{aligned}$$

so that the scheme become

$$y_{n+1} = y_n + hf(y_n, t_n)$$

Runge-Kutta Methods

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Runge-Kutta Methods

- Forward Euler is a *first order* method. If we half the time step, we should half the error.
- Another method, Adams Bashforth, is a second order multistep method. This means if we half the error, we should decrease the error by a factor of 4.

Runge-Kutta Methods

- Recall Forward Euler gave us the method:

$$y_{n+1} = y_n + hf(t_n, y_n) = y_n + hy'(t_n)$$

- Solving for y' , we get

$$\frac{y_{n+1} - y_n}{h} \approx y'(t_n)$$

- This is a *first order* approximation of the derivative.
- Instead, we can approximate $y'(t_n + \frac{h}{2})$ by

$$\frac{y_{n+1} - y_n}{h} \approx y' \left(t_n + \frac{h}{2} \right)$$

This is a *second order* approximation.

Runge-Kutta Methods

- Idea: Replace $y'(t_n)$ in Euler's Method with $y'(t_n + \frac{h}{2})$ to get a more accurate method:

$$y_{n+1} = y_n + hy' \left(t_n + \frac{h}{2} \right) = y_n + hf \left(t_n + \frac{h}{2}, y \left(t_n + \frac{h}{2} \right) \right)$$

- We don't know the value $y(t_n + \frac{h}{2})$, so Taylor Expand!:

$$y \left(t_n + \frac{h}{2} \right) = y(t_n) + \frac{h}{2}y'(t_n) + \cdots \approx y(t_n) + \frac{h}{2}f(t_n, y(t_n))$$

- This gives us the scheme

$$y_{n+1} = y_n + hf \left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n) \right)$$

This is a second order Runge-Kutta method (also called the midpoint method).

Runge-Kutta Methods

- Second order Runge-Kutta method:

$$y_{n+1} = y_n + hf \left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n) \right)$$

- There are a whole family of Runge-Kutta methods. Another popular one is a *fourth order* method

$$y_{n+1} = y_n + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f \left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1 \right)$$

$$k_3 = f \left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2 \right)$$

$$k_4 = f(t_n + h, y_n + hk_3)$$

