$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \vec{x} = \vec{0}$$

$$\lambda_1 = -1 \,, \quad \vec{e}_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\vec{e}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 3$$

$$( \quad - \lambda = 0$$

$$\det(A - \lambda I) =$$

$$\boxed{\lambda_1 = -1}, \quad \lambda_2 = 3$$

$$(A - \lambda I)\vec{x} = 0$$

$\lambda_2 = 3$

b

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1/2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 - \frac{x_2}{2} &= 0 \\ x_2 &= x_2 \end{aligned}}$$

$$x_1 = \frac{x_2}{2}$$

$$x_2 = x_2$$

$$\Rightarrow \vec{e}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_2}{2} \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$$x_1 = 2$$

$$\vec{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

---

$$\lambda_1 = -1, \quad \vec{e}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 3, \quad \vec{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

---

$$A\vec{e}_n = \lambda_n \vec{e}_n$$

$$dy_1(x) = a y_1 + b y_2$$

$$\frac{\quad}{dx}$$

$$\frac{d\, y_2(x)}{dx} = c\, y_1 + d\, y_n$$

$$\vec{y}\,' = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

$$\boxed{\vec{y}\,' = A\,\vec{y}}$$

$$\boxed{\begin{aligned} y' &= \alpha y \\ y &= c_1 e^{\alpha x} \end{aligned}}$$

Anzats: $\vec{y} = \vec{x}\, e^{\alpha t}$

$$\alpha \vec{x} \cancel{e^{\alpha t}} = A \vec{x} \cancel{e^{\alpha t}}$$

$$\alpha \vec{x} = A \vec{x}$$

Reminder $A\vec{e}_n = \lambda_n \vec{e}_n$

$$\alpha = \lambda_n$$

$$\vec{x} = \vec{e}_n$$

$$\boxed{\vec{y} = \vec{e}_n e^{\lambda_n t}}$$

$A = [1 \ 1$

$$A = \begin{bmatrix} \cdot & \cdot \\ 4 & 1 \end{bmatrix}$$

$$\lambda_1 = -1, \quad \vec{e}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 3, \quad \vec{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

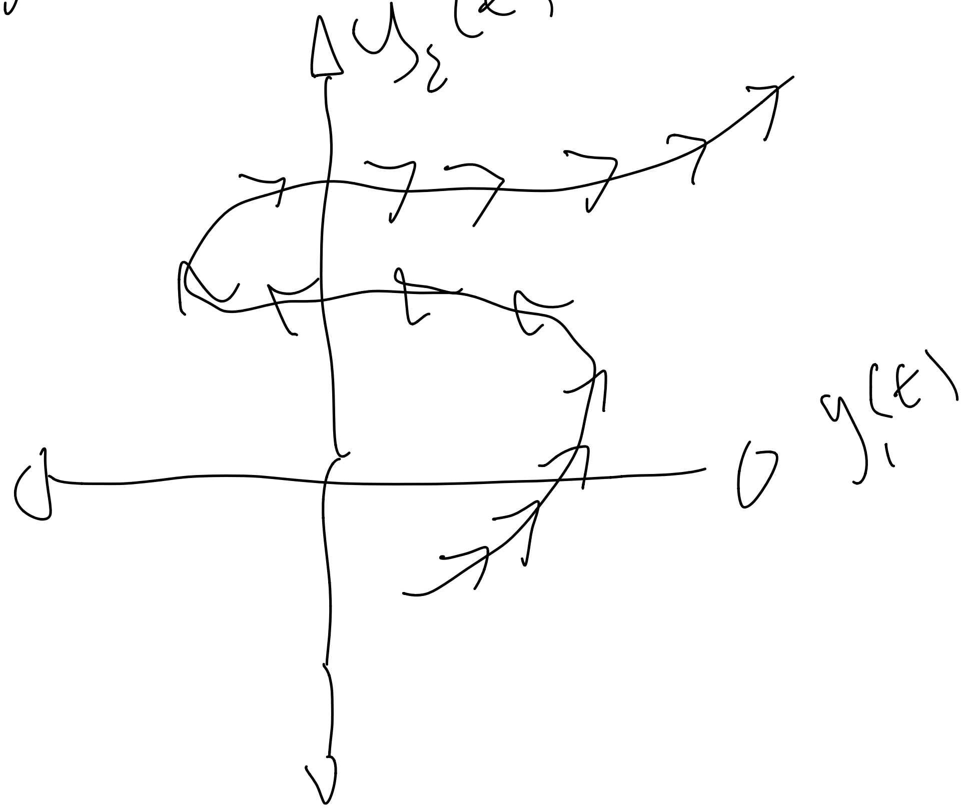
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\boxed{\vec{y}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}, \quad \vec{y}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}}$$

$$\vec{y} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

$$y_1(t) = -C_1 e^{-t} + C_2 e^{3t}$$

$$y_2(t) = 2C_1 e^{-t} + 2C_2 e^{3t}$$

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for a givn $(n \times n)$ matrix $A$

$$\begin{bmatrix} y_1 \end{bmatrix}' \qquad \begin{bmatrix} y_1 \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ y_n' \end{bmatrix} = A \begin{bmatrix} \vdots \\ y_n \end{bmatrix}$$

$\lambda_1 \ldots \lambda_n$

$\vec{e}_1, \ldots e_n$

$$\vec{y} = c_1 \vec{e}_1 e^{\lambda_1 t} + c_2 \vec{e}_2 e^{\lambda_2 t} + \ldots + c_n \vec{e}_n e^{\lambda_n t}$$

works for any distinct eigenvalues ($\lambda_1 \neq \lambda_2 \ldots \neq \lambda_n$)

---

$$\lambda_1 = 3-5i, \quad \vec{e}_1 = \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$$

Flip sign in front of imaginary parts

$$\lambda_2 = 3+5i, \quad \vec{e}_2 = \begin{bmatrix} 2 \\ -1-i \end{bmatrix}$$

$$\vec{y} = e^{3t}\left(c_1 \begin{bmatrix}2\\-1+i\end{bmatrix} e^{-5it} + c_2 \begin{bmatrix}2\\-1-i\end{bmatrix} e^{5it}\right)$$

$$\vec{y}_1 = e^{3t}\begin{bmatrix}2\\-1+i\end{bmatrix}\boxed{e^{-5it}}$$

$$\vec{y}_1 = e^{3t}\begin{bmatrix}2\\-1+i\end{bmatrix}\left(\cos(5t) - i\sin(5t)\right)$$

$$= e^{3t}\left(\begin{bmatrix}2\cos(5t)\\-\cos(5t)+\sin(5t)\end{bmatrix} + \right.$$

$$i\begin{bmatrix} -2\sin(5t) \\ \cos(5t)+\sin(5t) \end{bmatrix}\Bigg)$$

$$= e^{3t}\left(\vec{u}(t) + i\,\vec{v}(t)\right)$$

$$\boxed{\vec{y} = e^{3t}\left(c_1\vec{u}(t) + c_2\vec{v}(t)\right)}$$

---

$$\lambda_1 = \lambda_2$$

$$\vec{y}_1 = \vec{e}_1 e^{\lambda_1 t}$$

$$\text{try } \vec{y}_2 = \vec{e}_2\, t\, e^{\lambda_2 t} \quad ?$$

$$\vec{y}_2{}' = A\vec{y}_2$$

$$\vec{e}_2 \cancel{e^{\lambda_2 t}} + \vec{e}_2 \lambda_2 t \cancel{e^{\lambda_2 t}} \quad \text{LHS}$$

$$= A\vec{e}_2 t \cancel{e^{\lambda_2 t}}$$

$$\vec{e}_2 + \vec{e}_2 \lambda_2 t = A\vec{e}_2 t$$

$$\Theta(1): \vec{e}_2 = \vec{0}$$

$$\Theta(t): \vec{e}_2 \lambda_2 = A\vec{e}_2$$

$$\boxed{5x^2} + \boxed{6x} + \boxed{2} = \boxed{ax^2} + \boxed{bx} + \boxed{c}$$

$$\boxed{\vec{y}_2(t) = t e^{\lambda t}\vec{e}_1 + e^{\lambda t}\vec{e}_2}$$

$$\vec{y}_2\,' = A\vec{y}_2$$

$$\vec{e}_1 \cancel{e^{\lambda t}} + \lambda\vec{e}_1 t \cancel{e^{\lambda t}} + \lambda\vec{e}_2 \cancel{e^{\lambda t}}$$

$$= A t \cancel{e^{\lambda t}}\vec{e}_1 + A \cancel{e^{\lambda t}}\vec{e}_2$$

$$O(1): \ \vec{e}_1 + \lambda\vec{e}_2 = A\vec{e}_2$$

$$O(t): \ \lambda\vec{e}_1 = A\vec{e}_1 \ \checkmark$$

$$(A - \lambda I)\vec{e}_2 = \underline{\vec{e}_1}$$

$\boxed{\text{Ex}}$ $\vec{y}\,' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \vec{y}$

$$\det \begin{pmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{pmatrix} = 0$$

$$= \lambda^2 - 10\lambda + 25 = 0$$

$$= (\lambda - 5)^2$$

$$\boxed{\lambda_1 = \lambda_2 = 5}$$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{\lambda_1 = 5 \,,\ \vec{e}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

$$(A - \lambda I)\vec{e}_2 = \vec{e}_1$$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1/2 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1/2 \\ \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$0 \quad 0$$

$$x_1 + \frac{x_2}{2} = \frac{1}{2}$$

$$x_2 = x_2$$

$$\vec{e}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

Note if $x_2 = -2$, $\vec{e}_1$

$$x_2 = 0$$

$$\vec{e}_2 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$\vec{y}_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} t e^{5t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{5t}$$

$$\vec{y}(t) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{5t} + c_2 \left( \begin{bmatrix} 1 \\ -2 \end{bmatrix} t e^{5t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{5t} \right)$$

---

for homogenous, autonomous nth order DE

$$y''' + \boxed{ay'' + by' + cy} = 0$$

$y_1 = y \qquad \vec{y}' = A\vec{y}$

$y_2 = y'$

$y_3 = y''$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix}$$

$$= \begin{bmatrix} y_2 \\ y_3 \\ -ay'' - by' - cy \end{bmatrix}$$

$$= \Big[\, 0 + y_2 + 0 \,\Big]$$

$$= \begin{bmatrix} 0 + 0 + y_3 \\ -c y_1 - b y_2 - a y_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\vec{y}\,' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix} \vec{y}$$

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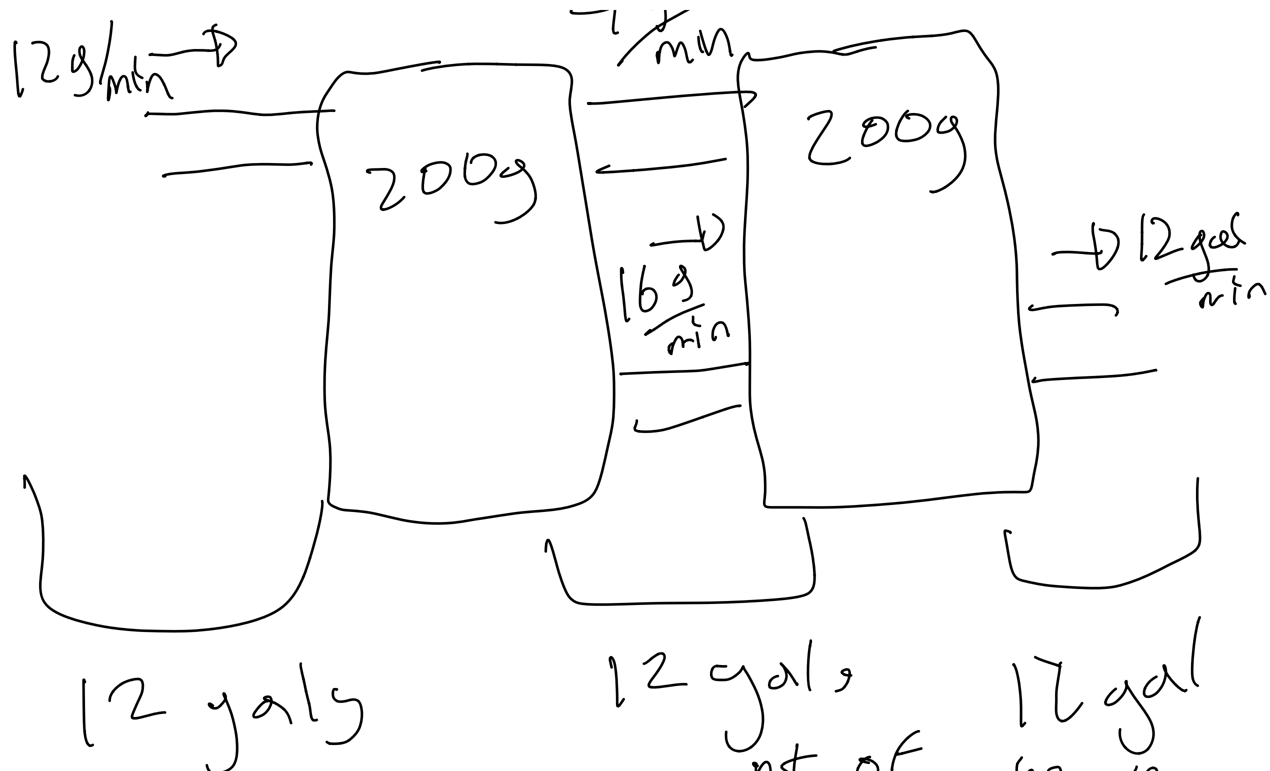
Tank Problems

Pb 18




$y_n$ is amount of substance in tank $n$.

$$\frac{dy_1}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dy_2}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = \frac{16}{\text{min}}\left(\frac{\text{Quantity}}{\text{time}}\right)$$

$$\frac{y_2 \text{ lb}}{200 \cancel{\text{gal}}} \frac{4 \cancel{\text{gals}}}{\text{min}} + \cancel{\frac{0 \text{ lb}}{1 \text{ gal}} \frac{12 \text{ gal}}{\text{min}}}$$

$$\text{rate in} = \frac{y_2}{200} 4 = \frac{y_2}{50}$$

$$\text{rate out} = \frac{y_1}{200} \cdot 16$$

$$\frac{dy_1}{dt} = \text{rate in} - \text{rate out}$$

$$= \frac{y_2}{50} - \frac{y_1}{200} \cdot 16$$

$$dy_2 = y_1 16 - \frac{y_2 \, 4}{\phantom{200}}$$

$\overline{\gamma t}$ $\overline{200}$ 200

$$m y'' + \gamma y' + k y = 0$$

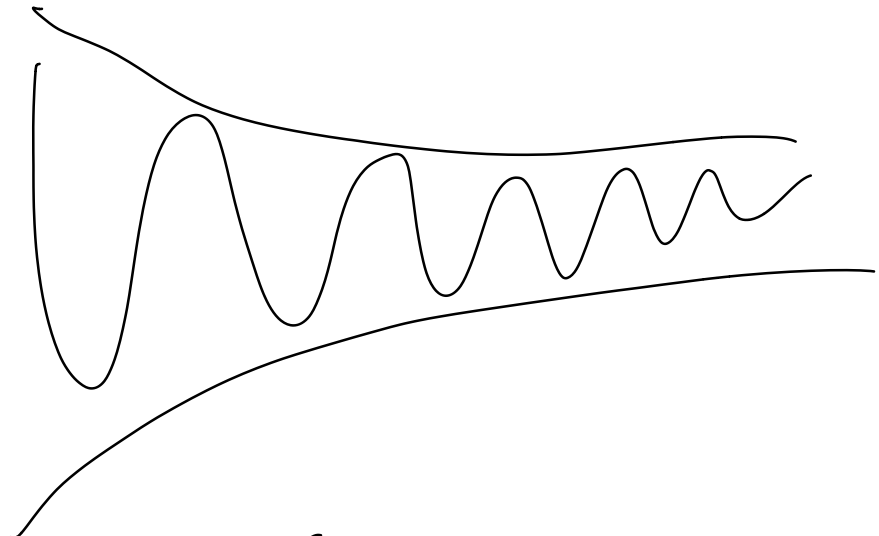
$$m \lambda^2 + \gamma \lambda + k = 0$$

$$\frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}$$

0

A

$\sin(A t)$



$$e^{Re(\lambda)t}\left(c_1 \sin(im(\lambda)t) + c_2 \cos(im(\lambda)t)\right)$$

$im(2i) = 2$

$\sin(\omega t)$ $\frac{\text{cycles}}{\text{second}}$

$\omega t_0 = 2\pi$

$$t_0 = \frac{2\pi}{\omega} = 2\ sec$$

$$F = \frac{\omega}{2\pi}$$ frequency

$$\frac{1}{\omega} = \frac{2\ sec}{2\pi}$$

$\omega$ = angular frequency

$F$ = frequency

$$\omega = \frac{2\pi}{2\ sec}$$

$$\frac{m}{m} = \frac{2\pi}{2}$$