Reminder: Hmwk 1 due tonight;
- Keon's OH Today
- Survey for my office hours

2.1, 2.2, 2.6

2nd order linear DE

- homogeneous
- constant coefficient

$$y'' + a(x)y' + b(x)y = c(x)$$

homogeneous $\Rightarrow c(x) = 0$

constant coefficients

$$a(x) = a$$

$$b(x) = b$$

$$y'' + ay' + by = 0$$

2nd order, linear, ordinary, homogeneous, constant coeff. differential equation

~~$y'' + y' = 0 \Rightarrow Y' + Y = 0$~~

~~$y' = Y$~~

$$y'' + y = 0$$

span $\{\sin(x), \cos(x)\}$

$$y^* = c_1 \sin(x) + c_2 \cos(x)$$

$$y^* = \sin(x) \Rightarrow -\sin(x) + \sin(x)$$
$$= 0 \checkmark$$

$$y^* = 35\sin(x) + 34\cos(x)$$

$$(y^*)'' = -35\sin(x) - 34\cos(x)$$

$$y^{*''} + y^* = 0 \checkmark$$

$$y'' - y = 0$$

$$\text{span}\{e^x, e^{-x}\}$$

$$y_s = c_1 e^x + c_2 e^{-x}$$

linearly indp.

$$c_1 y_1 + c_2 y_2 \neq 0$$

$$c_1 e^{x} + c_2 e^{-x} \neq 0$$

$\forall x \in \mathbb{R}$

$$W(y_1, y_1) \neq 0$$

$$= \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$= y_1 y_2' - y_1' y_2$$

$y_1 = e^{x}$

$y_2 = 5e^{-x}$

wanna show lin. indp.

$$-e^x 5e^{-x} - e^x 5e^{-x}$$

$$= -2e^x 5e^{-x}$$

$$= -10e^x e^{-x} = -10 \neq 0$$

$\Rightarrow$ linearly indp.

$$y_1 = 6^x \quad y_2 = 6^{x+2}$$

$$W(6^x, 6^{x+2}) = 0 \ \forall x \in \mathbb{R}?$$

$y_1 y_2' - y_1' y_2$

$$\underbrace{6^x 6^{x+2} \ln(6)} - \underbrace{6^{x+2} 6^x \ln(6)} = 0$$

$$0 = 0$$

$\Rightarrow$ lin. dep.

$$y_1 = x\sin(wx) \qquad w \neq 0$$
$$y_2 = \cos(wx)$$

$W(y_1, y_2)$.

$y_1 = x\sin(wx)$	$y_2 = \cos(wx)$
$y_1' = wx\cos(wx) + \sin(wx)$	$y_2' = -w\sin(wx)$

$$\overbrace{-wx\sin(wx)\sin(wx)}^{y_1 y_2'}$$

$$\underbrace{-wx\cos(wx)\cos(wx) - \sin(wx)\cos(wx)}_{y_1' y_2}$$

$= \omega x \left(\sin^2(\omega x) + \cos^2(\omega x) \right)$

$- \sin(\omega x)\cos(\omega x)$

$= \omega x - \sin(\omega x)\cos(\omega x) \quad \omega \neq 0$

$\neq 0$ except on x in Ω

s.t. Wronskian is not equal to zero

$\Rightarrow$ lin. indp.

Solving:

2nd order, linear, ordinary, homogeneous, constant coeff. differential equation

$$y = e^{\lambda x} \qquad \lambda \in \mathbb{C}$$

$$y'' + ay' + by = 0$$

$$\lambda^2 \cancel{e^{\lambda x}} + a\lambda \cancel{e^{\lambda x}} + b \cancel{e^{\lambda x}} = 0$$

$$\lambda^2 + a\lambda + b = 0$$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Case 1: (a)

$$\boxed{\begin{array}{l} a^2 - 4b \neq 0 \\ \lambda_1 \neq \lambda_2 \end{array}}$$

$\lambda_1, \lambda_2 \in \mathbb{R}$

$e^{\lambda_1 x} \qquad u = e^{\lambda_2 x}$

$y_1 = e \quad \lambda_2$

$$\boxed{y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

b) $\lambda_1, \lambda_2 \in \mathbb{C}$

if $a^2 - 4b < 0$

$\Rightarrow \lambda_1, \lambda_2 \in \mathbb{C}$

$$\boxed{\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = A \pm iB}$$

$$y_h = c_1 e^{(A+iB)x} + c_2 e^{(A-iB)x}$$

Euler's Formula!

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

(Euler Identity: $e^{i\pi} + 1 = 0$)

$$e^{Ax}\left(C_1 e^{iBx} + C_2 e^{-iBx}\right)$$

$$[\cos(Bx) + i\sin(Bx)$$

$$[\cos(-Bx) + i\sin(-Bx)$$

$$\cos(Bx) - i\sin(Bx)$$

$$\boxed{= c_1 e^{ax}\sin(Bx) + c_2 e^{ax}\cos(Bx)}$$

Case 2: $\lambda_1 = \lambda_2$

$$a^2 - 4b = 0$$

$$\boxed{\lambda_1, \lambda_2 = -\frac{a}{2}}$$

$$\boxed{y_1 = e^{-\frac{a}{2}x}, \quad y_2 = xe^{-\frac{a}{2}x}}$$

~~Constant coeff~~

Now have: $a(x)$, $b(x)$

$$y'' + a(x)y' + b(x)y = 0$$

$$x^2y'' - 5xy' + 9y = 0$$

$$2x^2y'' = 6x$$

$y_1 = x^3 \qquad y_1' = 3x^2 \qquad y_1''$

$$6x^3 - 15x^3 + 9x^3 = 0$$

Reduction of Order:

$$y'' + a(x)y' + b(x)y = 0$$

y_1 is a solution

$$y_2 = uy_1 = u(x)y_1(x)$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

$$u''[y_1] + u'[2y_1' + a(x)y_1]$$

0

$$\cancel{u\left[\underbrace{y_1'' + a(x)y_1' + b(x)y_1}_{=0}\right]}$$

$$u''[y_1] + u'[2y_1' + a(x)y_1] = 0$$

$$U = u'$$

$$U'[y_1] + U[2y_1' + a(x)y_1] = 0$$

$$\frac{U'}{U} = \frac{-[2y_1' + a(x)y_1]}{y_1}$$

$$\ln(U)' = \ldots$$

$$\ln(U) = \int \ldots dx$$

$$U = e^{\int \ldots dx}$$

$U = $

$$\frac{-[2y_1' + a(x) y_1]}{y_1}$$

$$-2\ln(y_1)' - a(x)$$

$$\int -2\ln(y_1)' - a(x)\,dx$$

$$U = e^{\int -2\ln(y_1)'} e^{-\int a(x)\,dx}$$

$$\boxed{U = \frac{1}{y_1^2} e}$$

$\boxed{y_2 = u y_1 \quad \text{Reminder}}$

$$u' = \text{RHS}$$

$$u = \int U\,dx$$

$$y_2 = y_1 \int U \, dx$$

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$$x^2 y'' - 5xy' + 9y = 0$$

$$y'' \underbrace{- \frac{5}{x}}_{a(x)} y' + \frac{9}{x^2} y = 0$$

$$y_1 = x^3$$

$$y_2 = x^3 \int \frac{1}{(x^3)^2} e^{-\int \left(-\frac{5}{x}\right) dx}$$

$$= x^3 \int \frac{e^{5 \ln(x)}}{x^6} \, dx$$

$$= x^3 \int \frac{\cancel{x^3}}{x^6}\, dx$$

$$= x^3 \int \frac{1}{x}\, dx$$

$$= \boxed{\ln(x)\, x^3 = y_2}$$

$$y_1 = x^3$$

$$\boxed{y_H = c_1 x^3 + c_2 \ln(x)\, x^3}$$

$$y_1 = e^{\lambda x} = e^{-\frac{a}{2} x} \qquad \lambda_1 = \lambda_2$$

$$y'' + ay' + by = 0$$

...tion of order formula

$$\int -a\, dx$$

Reduct…

$$\Rightarrow y_2 = e^{-\frac{a}{2}x} \int \frac{1}{\left(e^{-\frac{a}{2}x}\right)^2}\, e \ldots$$

$$e^{-ax} \qquad e^{-ax}$$

$$y_2 = e^{-\frac{a}{2}x} \int \cancel{\frac{e^{-ax}}{e^{-ax}}}\, dx$$

$$\boxed{y_2 = x\, e^{-\frac{a}{2}x}}$$