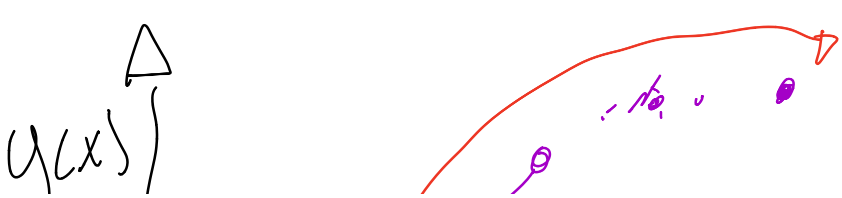
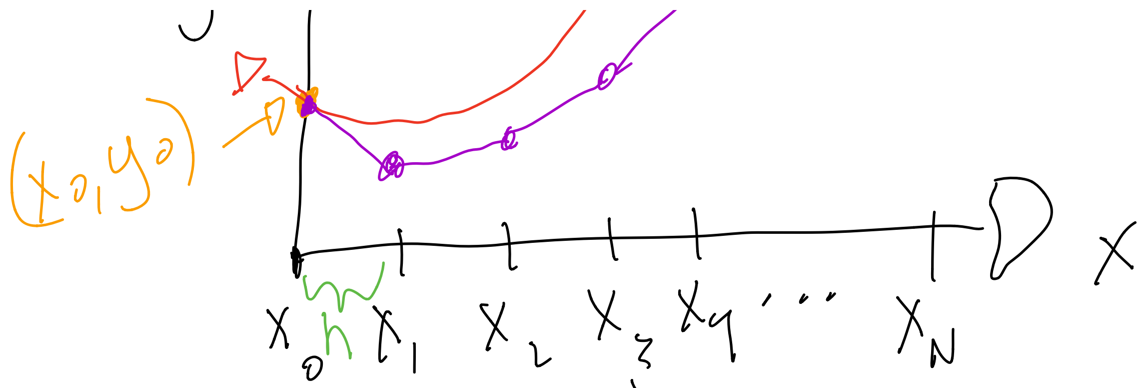
Remmber last class we covered:

- seperation of variable
  - newtons law of cooling
- Slope field analysis
  - website
  - newtons law of cooling
  - basic population model

Numerical Methods for solving DE.

- Euler's method





(y_0, x_0)

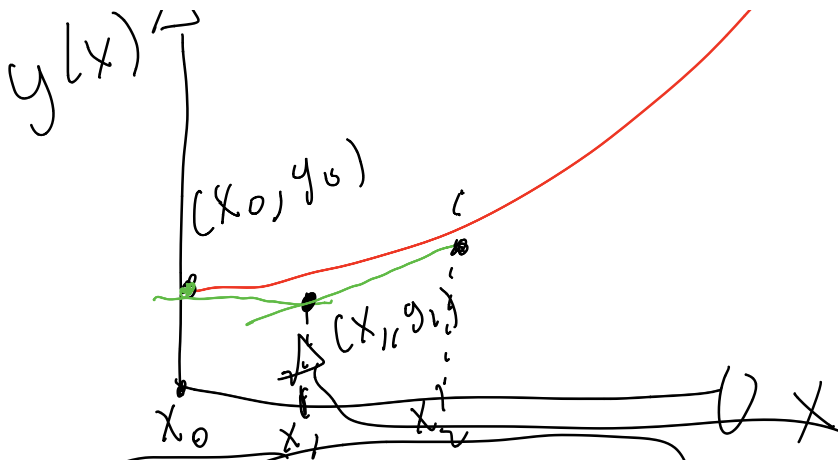
$x_n = x_0 + nh$

Algorythm

(y_1, x_1)

algorythm

(y_2, x_2)



$$\boxed{y' = \underbrace{f(x,y)}_{\text{slope}}}$$

$$x^2 + y$$

$$x_0^2 + y_0 \to y'(x_0, y_0)$$

$$\boxed{y(x_n) = y(x_{n-1}) + h\, \underbrace{y'(x_{n-1}, y_{n-1})}_{f(x_{n-1},\, y_{n-1})}}$$

$$y'(x) = f(x,y)$$

$$y(x_0 + h) = \sum^{\infty} y^{(n)}\Big|_x \frac{h^n}{n!}$$

$n = 0$ $\quad$ $i = 0$

x_0 is the point you
control at
and you are looking
at the value that
is h away

$$y(x_0 + h) = y(x_0) + y'(x_0)h + y''(x_0)\frac{h^2}{2} + y'''(x_0)\frac{h^3}{6} + \dots$$

$$y(\underbrace{x_0 + h}_{x_0 \to x_{n-1}}) \approx y(x_0) + y'(x_0)h$$

x_n

Local Error $O(h^2)$

$h=1 \to h=.25$

$.25^2 = \frac{1}{16}$

$h=1 \to h=.5$

$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

Global Error $O(h)$

Improved Euler method

— Slope from (x_1, y_1^*)

$y = x^2 + C$

$$\boxed{\tilde{y}_{n+1} = y_n + h f(x_n, y_n)}$$

$$y_{n+1} = y_n + \frac{1}{2} h \left(f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1}) \right)$$

Example

$$x y' = 2y + x^3 e^x$$

$$\boxed{y' = \frac{2y}{x} + x^2 e^x}$$

1 2.718

$y_n = y(x_n)$ $\quad y(1) = e$ $= 2\ldots$

$x_0 = 1 \quad h = .5$

$\Rightarrow x_1 = 1.5$

Method	$y(x_0)$	$y(x_1)$
Euler	e^1	6.796
I.E.	e^1	9.543
Exact	e^1	10.083

Eulers

$$y(x_1) = y(x_0) + h\, y'(x_0, y_0)$$

$y(x_0)$

$$y(x_1) = e^1 + (.5)\left(\frac{2 \cdot e^1}{1} + 1^2 e^1\right)$$

$$= 6.796$$

I.E.

$$\tilde{y}_1 = 6.796$$

$$\ldots \left(f(x_0, y_0) + f(x_1, \tilde{y}_1)\right)$$

$$y_1 = y_0 + \frac{1}{2} h \left(f(\right.$$

$$y_1 = e^1 + \left(\frac{1}{2} \cdot \frac{1}{2}\right)\left(\left(\frac{2 \cdot e^1}{1} + 1^2 e^1\right)\right.$$

$$\left. + \left(\frac{2 \cdot 6.796}{1.5} + (1.5)^2 e^{1.5}\right)\right)$$

$$= 9.543$$

Exact

$$y' = \frac{2y}{x} + x^2 e^x$$

$$y' - \frac{2}{x} y = x^2 e^x$$

$$y' + a(x) y = b(x)$$

$$e^{\int -\frac{2}{x}\,dx} = e^{-2\ln(x)}$$

$$= \frac{1}{x^2}$$

$$\times \frac{1}{x^2}$$

$$= \boxed{\frac{1}{x^2} y' - \frac{2}{x^3} y} = e^x$$

$$\boxed{\left(\frac{1}{x^2} y\right)' = e^x}$$

$$\frac{y}{x^2} = e^x + C$$

$$y = x^2 e^x + Cx^2$$

$$y(1) = e^1$$

$$e^1 = e^1 + c$$

$$\Rightarrow c = 0$$

$$y = x^2 e^x$$

	GE	LE
IE	$O(h^2)$	$O(h^3)$
Euler	$O(h)$	$O(h^2)$

$$y' = f(x, y)$$

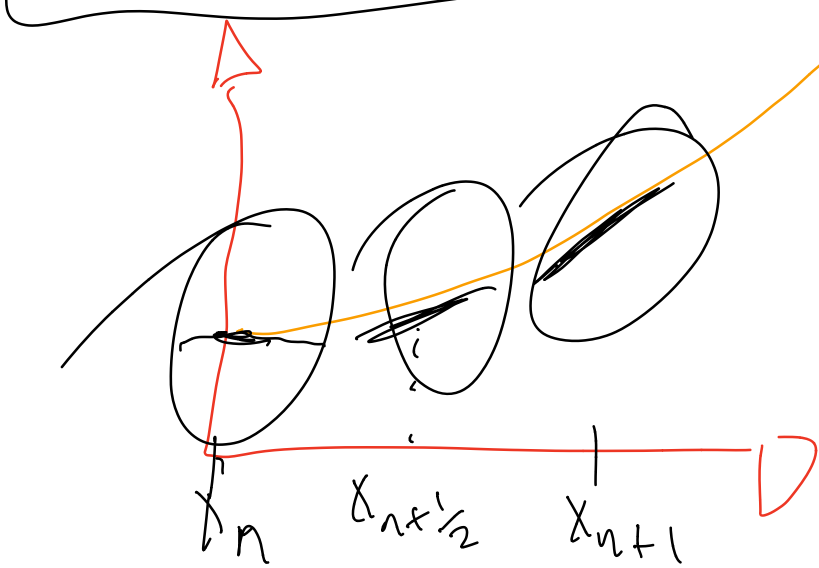
$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$

$$k_3 = h f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$

$$\boxed{k_4 = hf(x_n + h,\ y_n + k_3)}$$



Backward Euler method

$$\boxed{y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})}$$

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$$\boxed{y' = -20y + 20x^2 + 2x}\ , \quad y(0) = 1$$

$$y_{n+1} = y_n + h\left(-20y_{n+1} + 20x_{n+1}^2 + 2x_{n+1}\right)$$

$$+ 20hx_{n+1}^2$$

$$-20h\,y_{n+1}$$

$$2h\,x_{n+1}$$

$y_{n+1} = y_n$

$y_{n+1} = y_n + 20h\,x_{n+1}^2 + 2h\,x_{n+1}$

$$\boxed{y_{n+1} = \frac{y_n}{(1+20h)}}$$

$(y_{n+1} + 20h\,y_{n+1})$

$(y_{n+1}(1+20h)$

$= y_{n+1}(1+20h)$

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RK4 GE $O(h^4)$ LE $O(h^5)$

GE 2 → $\frac{1}{8}$

$h = 1 \to 0.5$

GE $2 \cdot (.5)^4 = \frac{1}{16}$