11.1–11.3

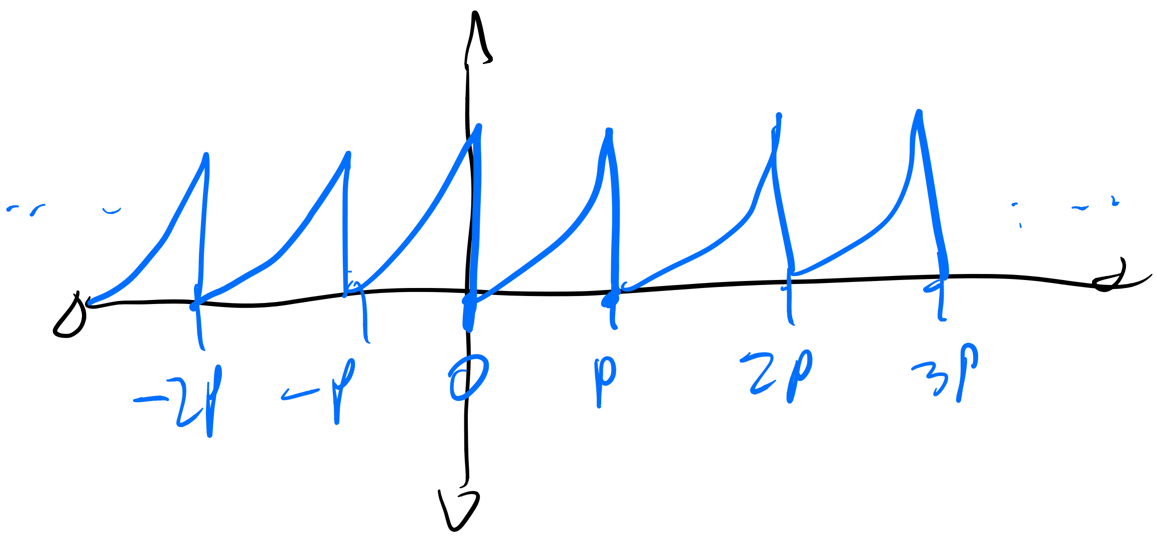
Periodic function $\Rightarrow f(x) = f(x+p)$

$p$ is the period

$$f(x+2p) = f(\overbrace{(x+p)}^{\tilde{x}}+p)$$

$$= f(\tilde{x}) = f(x+p)$$

$$= f(x)$$




Suppose $f(x)$ and $g(x)$ are

both $p$ periodic

$\Rightarrow$ $f(x)+g(x)$ is also $p$ periodic

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$$f(x) = \underbrace{a_0 + \sum_{n=1}^{\infty} \Big[ a_n \cos(nx)}_{\text{odd part}} + \underbrace{b_n \sin(nx)}_{\text{even part}} \Big]$$

Fourier Series

sin(nx) and cos(nx) $n \in \mathbb{N}$
are orthogonal to each other

$$\vec{U} \cdot \vec{V} = 0 \Rightarrow \perp$$

$$\int_{-\infty}^{\infty} f \cdot g \, dx = 0 \Rightarrow \perp$$

we can say that

$$\int_{-\pi}^{\pi} f \cdot g \, dx = 0 \Rightarrow \perp$$

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) \, dx$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0 \quad \forall m,n \in \mathbb{N}$$

$$\Rightarrow \perp$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0 \quad \forall m,n \in \mathbb{N}$$

$$\Rightarrow \perp \quad \text{except } m=n$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0 \quad \forall m,n \in \mathbb{N}$$

$$= \perp \quad \text{except } m=n$$

---

$a_0$ ?

$$\int^{\pi} f(x)\,dx$$

$$\int_{-\pi} f(x)\,dx$$

$$= \int_{-\pi}^{\pi} a_0 + \sum_{m=1}^{\infty} a_m \cos(mx) + b_m \sin(mx)\,dx$$

$$a_m \int_{-\pi}^{\pi} \cos(mx)\,dx = 0$$

$$b_m \int_{-\pi}^{\pi} \sin(mx)\,dx = 0$$

$$= \int_{-\pi}^{\pi} a_0\,dx = a_0 2\pi$$

$$\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx}$$

$a_0$ is the average value of the function over a period

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$a_m$ ?

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos(mx) + \sum_{m=1}^{\infty} b_m \sin(mx)$$

- $\cos(nx)$

then int. from $-\pi$ to $\pi$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx =$$

$$\int_{-\pi}^{\pi} a_0 \cos(nx)\,dx = 0$$

$$\underbrace{\phantom{x}}_{\Rightarrow = 0 \text{ except } m=n}$$

$$+ \sum_{m=1}^{\infty} a_m \int \underline{\cos(nx)\cos(mx)}\,dx$$

$$+ \boxed{\sum_{m=1}^{\infty} b_m \int \sin(mx)\cos(nx)\,dx}$$

$$\Rightarrow \text{"}0$$

$$\int_{-\pi} f(x)\cos(mx)\,dx = \sum_{m=1} a_m \int \cos(mx)\,dx$$

$$\boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx}$$

---

$b_m$?

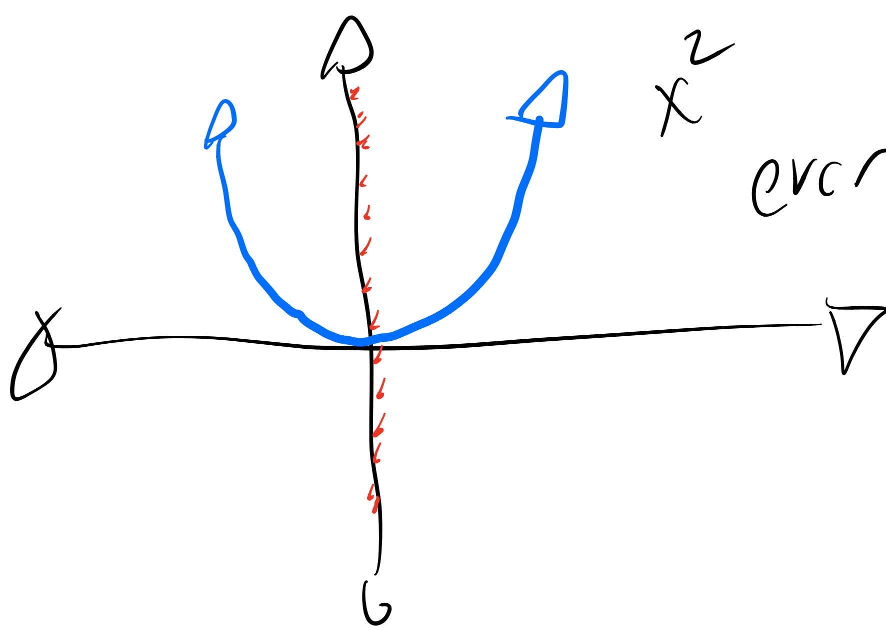
$$\boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx}$$

---

$f(x)$ is even or odd?

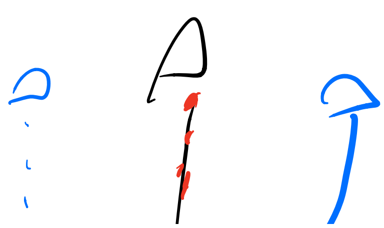
$$\Rightarrow f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos(mx) \text{ if even}$$

$$= \sum_{m=1}^{\infty} b_m \sin(mx) \text{ if odd}$$



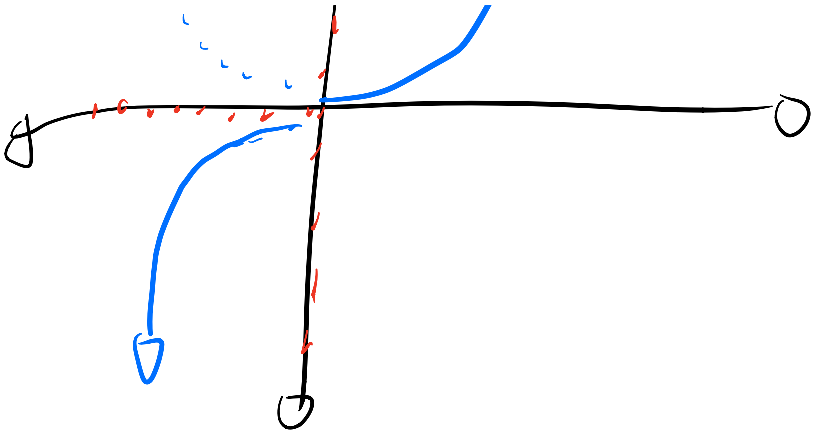
$x^2$

even



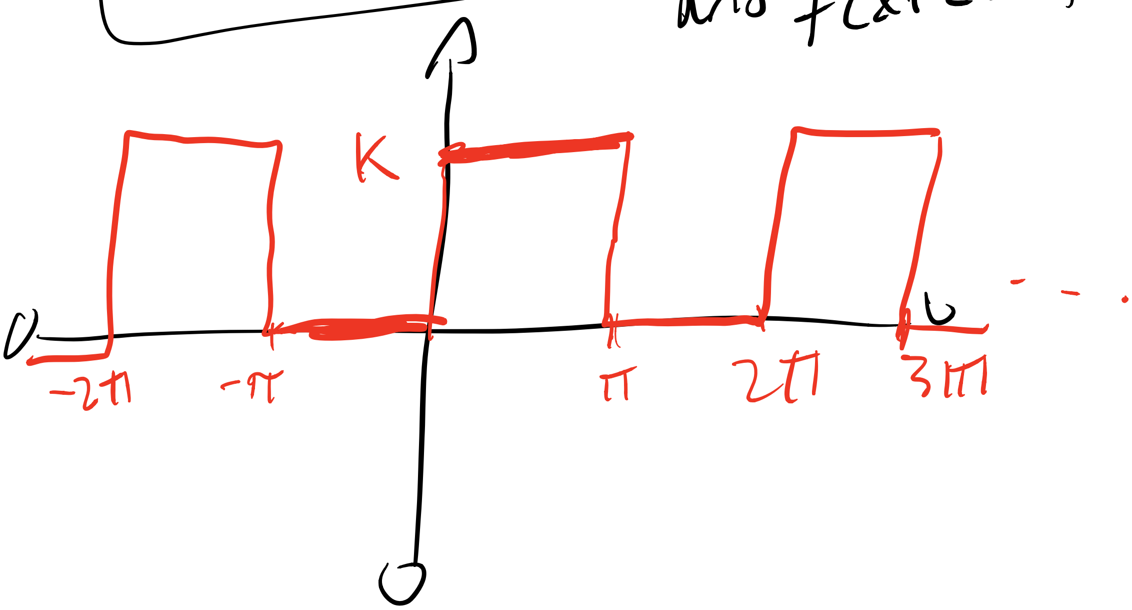
$x^3$

odd




$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$

and $f(x+2\pi) = f(x)$




$| \quad \wedge^{\pi}$

$$a_0 = \frac{1}{2\pi} \int_{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi} \int_{\pi}^{\pi} K \, dx$$

$$= \frac{K}{2}$$

---

$a_m$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx$$

$+$

$$a_m = \frac{1}{\ldots} \int^{\pi} K \cos(mx) \, dx$$

$$\pi \int_0$$

$$\cancel{+ \frac{1}{\pi} \int_{-\pi}^{0} 0 \cos(mx)\, dx} \quad 0 \quad 0$$

$$= \frac{1}{\pi} k \left. \frac{\sin(mx)}{m} \right|_0^{\pi}$$

$$= 0$$

$$b_m = \frac{1}{\pi} \int^{\pi} f(x) \sin(mx)\, dx$$

$$= \frac{1}{\pi}\left(\cancel{\int_{-\pi}^{0} 0 \cdot \sin(mx)\,dx}^{\,0} + \int_{0}^{\pi} k \sin(mx)\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{-k\cos(mx)}{m}\right)\Bigg|_{0}^{\pi}$$

if m is even = 1
odd = -1

$$\frac{-k}{\pi}\left(\frac{\cos(m\pi)}{m} - \frac{\overset{1}{\cos(0)}}{m}\right)$$

if $m$ is even $= 0$

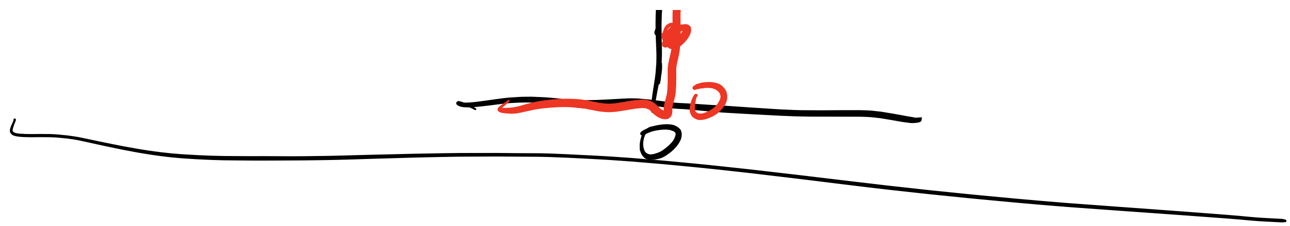
odd $= \frac{-2}{m}$

$$= \boxed{\frac{2k}{m\pi} \text{ if } m \text{ is odd} \quad \theta\left(\frac{1}{m}\right)}$$

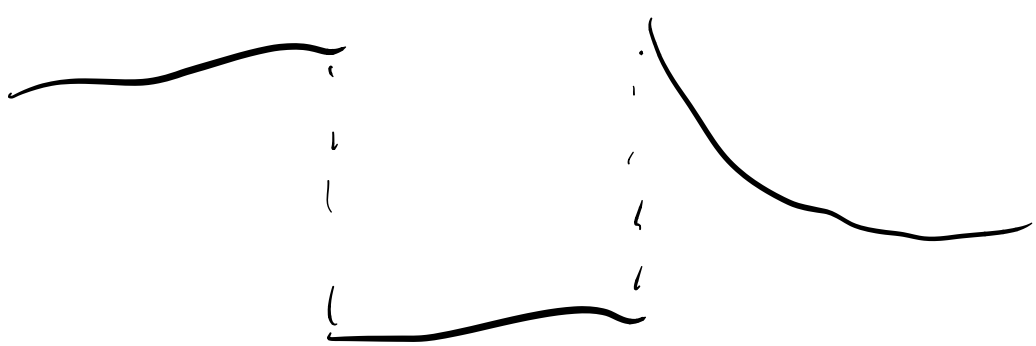
$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos(mx) + b_m \sin(mx)$$

$$\boxed{= \frac{k}{2} + \sum_{m=1}^{\infty} \frac{2k}{\pi} \sin((2m+1)x)}$$

$i \to k$



A fourier series can always be found (and will converge to the function) if the function is periodic and at worst piecewise.



$\tilde{f}(x) \to f(x)$ where $f(x)$

$\lim_{n \to \infty} f(x_n) \to f(x)$ ... is continuous

$\lim_{n \to \infty} \tilde{f}(x_n) \to \left( \frac{f(x^+) + f(x^-)}{2} \right)$ where $f(x)$ is discontinuous

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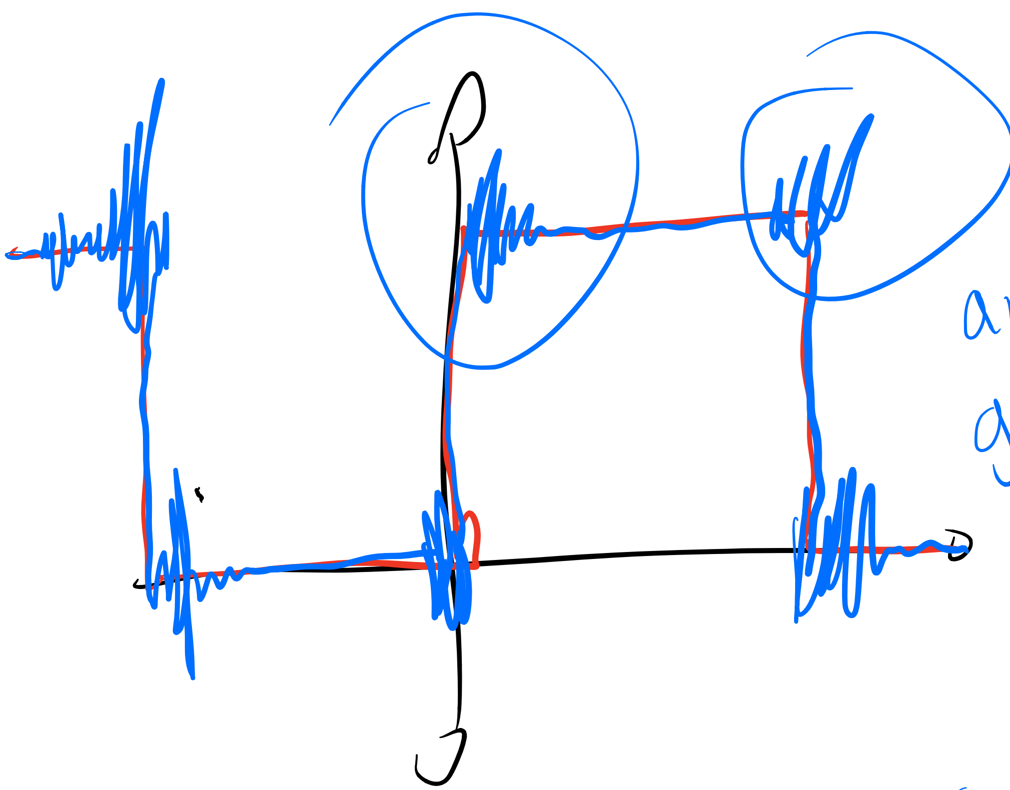

are called gibbs phenomenon

Fourier series converge "slowly"

near regions of discontinuities

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Extend the theory to any period "p".

Suppose we have a function

$$f(x) = f(x + np)$$

$$p \in \mathbb{R}$$

$$L = \frac{p}{2}$$

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{\pi n}{L} x\right)$$

$$+ \sum^{\infty} b_n \sin\left(\frac{\pi n}{\ } x\right.$$

$$\sum_{m=1} \quad m \quad \left(\left(\frac{}{L}\right)\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

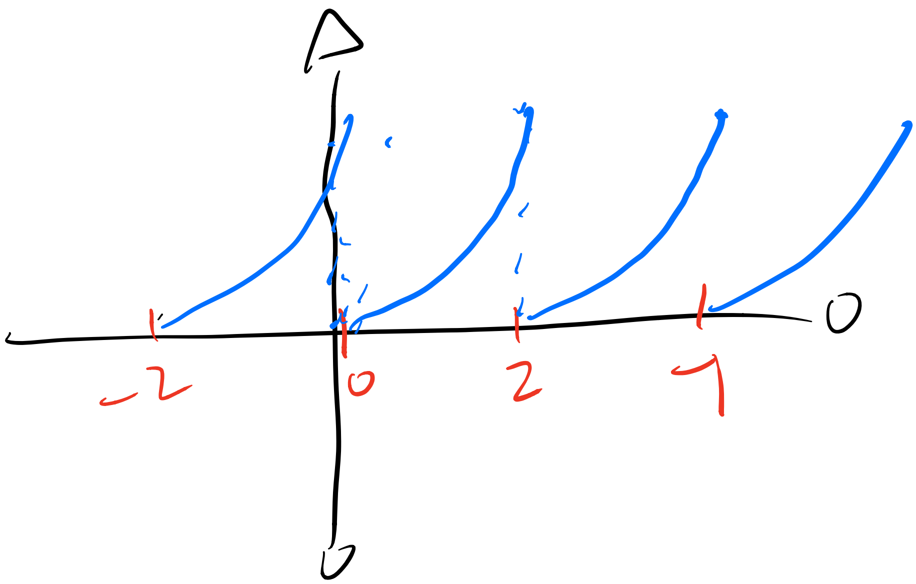
---

suppose we have

$$\boxed{f(x) = x^2 \text{ for } 0 < x < 2}$$

$$f(x) = f(x+2)$$

$f(x) =$




$$\Rightarrow L = \frac{p}{2} = 1$$

$$a_0 = 4/3$$

$$a_n = \frac{4}{n^2\pi^2} \qquad O\left(\frac{1}{n^2}\right)$$

$$b_n = -\frac{4}{\ldots} \qquad O\left(\frac{1}{n}\right)$$

$n\pi$

$$\boxed{g(x) = ax^3 + bx^2 + c}$$

$$a = \frac{2}{\pi^2}, \quad b = \frac{-3}{\pi}$$

$$c = \pi$$

$$f(x + 2\pi) = f(x)$$

$$\Rightarrow \quad b_n = 0$$

$$a_0 = \frac{\pi}{2}$$

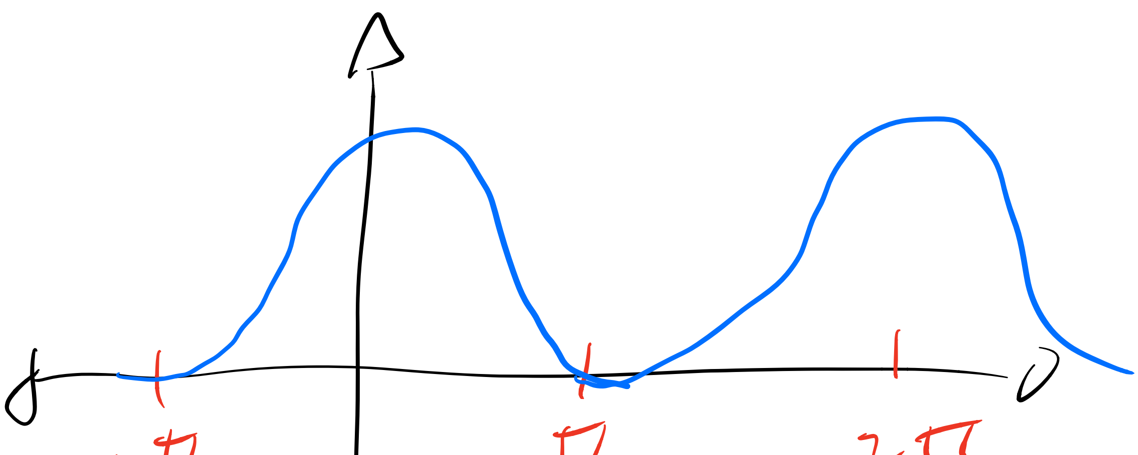
$$a_n = \frac{24}{\pi^3 n^4} \underbrace{(1 - \cos(n\pi))}_{n \text{ is odd}}$$

$= 2$

$m$ is even

$= 0$

$$a_m = \frac{48}{\pi^3 m^4}$$




$$f(x) = \frac{\pi}{2} + \frac{48}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4}$$

$\cos((2m-1)x)$

$O\left(\frac{1}{m^4}\right)$

Rule of thumb the "smoother" the function the faster the convergence.