$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$\nu \in \mathbb{R}$$

$$y'' + \frac{y'}{x} + \frac{(x^2 - \nu^2)}{x^2} y = 0$$

$b(0) = b_0 = 1$

$b(x) = 1$

$c(0) = c_0 = -\nu^2$

$c(x) = x^2 - \nu^2$

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

$$r(r-1) + b_0 r + c_0 = 0$$

$$r^2 - r + r - \nu^2 = 0$$

$$r^2 = \nu^2$$

...

$\Rightarrow \quad r = \pm \nu$

$\nu = 0 \Rightarrow$ double root case

$\nu = \frac{n}{2}, n \quad n \in \mathbb{N}$

$\Rightarrow \nu_1 - \nu_2 = m \quad m \in \mathbb{N} \Rightarrow$

seperated by an integer case

not one of these $\Rightarrow$ it is the other case

Distinct roots not differing by an integer

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_m x^m$$

$$y_1 = x^{\nu_1} \sum_{m=0}^{\infty} a_m x^m$$

$$a_1 = a_3 = a_{2k+1} = 0 \quad k \in \mathbb{N}$$

$$a_{2k} = \boxed{-\frac{(1)^m \, a_0}{2^{2m} m! (\nu+1)(\nu+2)\cdots(\nu+m)}}$$

$$y_1 = \underline{a_0} \, x^{\nu_1} \sum_{m=0}^{\infty} x^{2m}$$

$$a_0 = \frac{1}{2^n n!}$$

$$y_1 = x^n \sum^{\infty} \frac{(-1)^m x^{2m}}{\quad}$$

$$\sum_{m=0} \quad \frac{\quad}{2^{2m+n}\, m!\,(n+m)!}$$

(Bessel of the first kind

$$J_{\nu_1}(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi \nu_1}{2} - \frac{\pi}{4}\right)$$

---

distinct case (not sep. by int.)

$$y_1 = x^{\nu_1} \sum_{m=0}^{\infty} a_m x^m = x^{\nu} \sum_{m=0}^{\infty} a_m x^m$$

$$y_2 = x^{\nu_2} \sum_{m=0}^{\infty} A_m x^m = x^{-\nu} \sum_{m=0}^{\infty} A_m x^m$$

$$\Rightarrow y_2 = J_{-\nu}(x)$$

$$c \; J \; (x) + c \; J \; (x)$$

$$y = C_1 J_v(x) + C_2 J_{-v}(x)$$

$$y_1 = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\, m!\, (n+m)!}$$

$v \in \mathbb{N}$

Only work for $v \in \mathbb{N}$

$$y_1 = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+2}\, m!\, \Gamma(v+m+1)}$$

$\Gamma(x)$ if $x = n+1$ $n \in \mathbb{N}$

$\Gamma(x) = (x-1)! = n!$

$$\boxed{\Gamma(\nu+1) = \int_0^\infty e^{-t} t^{\nu}\, dt}$$

$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$

roots of the ind. eqn.

are ν and $-\nu$

$\Gamma($

$$\boxed{y_1 = c_1 J_\nu(x)}$$

if distinct and not differing by an integer

$$\Rightarrow \boxed{\hat{y}_2 = c_2 J_{-\nu}(x)}$$

if $\nu_1 = \nu_2 = 0$

$$\Rightarrow \quad y_2(x) = y_1(x)\ln(x) + \sum_{m=0}^{\infty} A_m x^m$$

if $(\nu_1 - \nu_2) \in \mathbb{N}$

$$y_2(x) = k y_1(x)\ln(x) + \sum A_m x^m$$

$m=0$

$$\boxed{y_2 = Y_\nu(x)}$$

special function called the bessel fn. of the second kind.

In general if you have the D.E.:

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

The solution is just

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

a short form way of writing

$$\boxed{Y_\nu(x) = \frac{\left[J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x) \right]}{\sin(\nu\pi)}}$$

$$(2x+1)^2 \frac{d^2y}{dx^2} + 2(2x+1)\frac{dy}{dx} + 16x(x+1)y = 0$$

$$\boxed{z = 2x+1} \Rightarrow \left(\frac{z-1}{2} = x\right)$$

$$z^2 \frac{d^2y}{dx^2} + 2(z)\frac{dy}{dx} + 16\left(\frac{z-1}{2}\right)\left(\frac{z-1}{2}+1\right)y = 0$$

$$4(z-1)(z-1+2)$$

$$4(z-1)(z+1)$$

$$=4(z^2-1^2)$$

$$z^2\frac{d^2y}{dx^2}+2z\frac{dy}{dx}+4(z^2-1)y=0$$

$$\frac{d}{dx}=\frac{d}{dz}\boxed{\frac{dz}{dx}}=\boxed{\frac{d}{dz}\cdot 2}$$

$$z=2x+1 \Rightarrow \frac{dz}{dx}=2$$

$$z^2\left(\cancel{\frac{d}{dx}}^{\frac{d}{dz}\cdot 2}\right)\left(\cancel{\frac{d}{dx}}^{\frac{d}{dz}\cdot 2}\right)y+2z\left(\cancel{\frac{d}{dx}}\right)y+4(z^2-1)y=0$$

$$= \cancel{\left(\frac{d}{dx}\right)}\cancel{\left(\frac{d}{dx}\right)}$$

$$\frac{d}{dz} \cdot 2$$

$$\cancel{4}z^2\frac{d^2y}{dz^2} + \cancel{4}z\frac{dy}{dz} + \cancel{4}(z^2-1)y=0$$

$$\boxed{z^2y'' + zy' + (z^2-1^2)y = 0}$$

$$\boxed{\frac{d}{dx} = \frac{d}{dz}\cdot\frac{dz}{dx} = \frac{d}{dz}2}$$

$$z(x) = 2x+1$$

$$\frac{dz(x)}{dx} = \frac{d}{dx}(2x+1)$$

$$\boxed{\frac{dz(x)}{dx} = 2}$$

$$\int \frac{1}{(x+2)}\,dx$$

$$u(x) = x+2 \qquad \frac{du(x)}{dx} = 1$$

$$\Rightarrow du(x) = dx$$

$$\int \frac{1}{u}\,dx$$

$$\int \frac{1}{u}\,du$$

$$v = 1$$

$$\boxed{z^2 y'' + z y' + (z^2 - 1^2) y = 0}$$

$$\boxed{y(x) = c_1 J_1(z) + c_2 Y_1(z)}$$

$$\boxed{= c_1 J_1(2x+1) + c_2 Y_1(2x+1)}$$

$$\lim_{\nu \to \infty} \Big[\underbrace{P_\nu}_{\text{legendre polynomial}} \left(\cos\left(\frac{x}{\nu}\right)\right) \Big] = \underbrace{J_0}_{\text{bessel fn}}(x)$$

$$\lim \left[\nu^\mu P_\nu^{-\mu} \left(\cos\left(\frac{x}{\nu}\right)\right) \right] = J_\mu(x)$$

$n \to \infty$