5.2

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$\boxed{a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m}$$

$a_0 = a_0 \quad a_m(a_0, a_1)$

$a_1 = a_1$

$$y_1 = \sum_{\text{even } m} a_m(a_0) x^m$$

$$y_2 = \sum_{\text{odd } m} a_m(a_1) x^m$$

$n=3 \quad m=3$

$\Rightarrow a_{3+2} = 0$

$\Rightarrow a_5 = 0$

$n=3, \; y_2 = \frac{1}{2}(5x^3 - 3x)$

Legendre polynomial come from the solution to (*) if n is an integer

of one of your basis solutions (y_1 or y_2) that becomes a finite polynomial instead of an infinite series).

$$\boxed{y'' + \frac{b(x)}{(x-x_0)} y' + \frac{c(x)}{(x-x_0)^2} y = 0}$$

$x = x_0 \Rightarrow y \to \infty$

$$\boxed{y = \frac{1}{x^\alpha} + \ldots\ldots}$$

guess is no good

current guess is $y = a_0 + a_1 x + a_2 x^2 + \ldots$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$

$m \in \mathbb{N}$

$r \in \mathbb{R}$

Frobenius method

Power series is special case where $r=0$

$$y'' + \frac{b(x)}{(x-x_0)} y' + \frac{c(x)}{(x-x_0)^2} y = 0$$

shift s.t. $x_0 = 0$

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \quad (*)$$

$b(0) = b_0$

$c(0) = c_0$

$$r(r-1) + b_0 r + c_0 = 0$$

This comes from plugging in Frobenius Anzats to ⊛

solve for roots of

$r_1, r_2 \qquad r_1 < r_2$

1st solution will be

$$y_1(x) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m$$

2nd solution

1. if $r_1 \neq r_2$ and $r_1 + n \neq r_2$

$$y_2(x) = x^{r_2} \sum^{\infty} A_m x^m$$

$m=0$

2. if $r_1 \neq r_2$ but $r_1 + n = r_2$

$$y_2(x) = k\,y_1(x)\ln(x) + x^{r_2}\sum_{m=0}^{\infty} A_m x^m$$

3. if $r_1 = r_2$

$$y_2(x) = y_1(x)\ln(x) + x^{r_2}\sum_{m=0}^{\infty} A_m x^m$$

$$x(x-1)y'' + (3x-1)y' + y = 0$$

$$y'' + \underbrace{\frac{(3x-1)}{x(x-1)}}_{b(x)}\,y' + \frac{y}{x^2(x-1)}\cdot x = 0 \qquad = c(x)$$

$$y'' + b(x)\,\underline{y'} + \underline{c(x)}\ \overset{*}{y} = 0$$

$$\frac{\quad}{x} \qquad \frac{\quad}{x^2}$$

$$b(x) = \frac{3x-1}{x-1} \qquad c(x) = \frac{x}{x-1}$$

$$b(0) = b_0 = 1 \qquad c(0) = c_0 = 0$$

$$r(r-1) + r = 0$$

$$r^2 - r + r = 0$$

$$r^2 = 0$$

$$r = 0, 0$$

$$y_1 = x^0 \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^m$$

$(x^2 - x) y''$

$= (x^2 - x) \sum_{m=2}^{\infty} a_m (m)(m-1) x^{m-2}$

$= \sum_{m=2}^{\infty} a_m (m)(m-1) \left(x^m - x^{m-1} \right)$

$(3x-1) y'$

$= (3x-1) \sum_{m=1}^{\infty} a_m (m) x^{m-1}$

$= \sum_{m=1}^{\infty} a_m (m) \left(3x^m - x^{m-1} \right)$

$y = \sum_{m=0}^{\infty} a_m x^m$

$A(1): -a_1 (1) + a_0 = 0$

$$a_0 = a_1$$

$$O(x): a_2(2)(1) + a_1(1)3$$

$$-a_2(2) + a_1 = 0$$

$$-4a_2 + 4a_1 = 0$$

$$a_0 = a_1 = a_2$$

If I notice that there is a simple pattern early on or I get $a_n = 0$, then I collect $O(x^m)$.

$$\sum_{m=2}^{\infty} \underbrace{a_m(m)(m-1)(x^m}{} - \cancel{x^{m-1}})$$

$$+ \sum_{m=1}^{\infty} a_m(m)(3\underline{x^m} - x^{m-1})$$

$$+ \sum_{m=0}^{\infty} a_m x^m = 0$$

$$O(x^m): \boxed{a_m(m)(m-1) - a_{m+1}(m+1)(m) + 3a_m(m) - a_{m+1}(m+1) + a_m = 0}$$

$$\boxed{a_m((m)(m-1)+3m+1)}$$

$$+a_{m+1}\left(-(m+1)(m)-(m+1)\right)$$

$$=-a_{m+1}(m+1)(m+1)$$

$$=-a_{m+1}(m+1)^2$$

$$a_m\left(m^2-m+3m+1\right)$$

$$=a_m\left(m^2+2m+1\right)$$

$$=a_m(m+1)^2$$

$$=(a_m-a_{m+1})(m+1)^2=0$$

$$\Rightarrow a_{m+1}=a$$

$$\Rightarrow \boxed{a_m = a_{m+1}}$$

$$a_0 = a_1 = a_2 = \cdots = a_m$$

$$y_1 = a_0 \sum_{m=0}^{\infty} x^m$$

if $x > 1$ not convergent

if $x < 1$

$$\Rightarrow \boxed{y_1 = a_0 \left(\frac{1}{1-x} \right)}$$

Reminder $\left(\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = \frac{1}{1-\frac{1}{2}} \right)$

$= 2$

$$y_2 = y_1 \ln(x) + x^{r_2} \sum^{\infty} A_m x^m$$

$m=0$

If y_1 is "simple"
then you can use reduction of order instead of the y_2 Frobenius Anzats.

$$y_2 = y_1 \int \frac{e^{-\int B(x)\,dx}}{y_1^2}\,dx$$

$$B(x) = \frac{(3x-1)}{x(x-1)}$$

$$\Rightarrow -\int B(x)\,dx$$

$$= -2\ln(1-x) - \ln(x)$$

$$\Rightarrow e^{-2\ln(1-x)-\ln(x)} = (1-x)^{-2}(x)^{-1}$$

$$y_2 = y_1 \int \frac{e^{-\int B(x)\,dx}}{y_1^2}\,dx$$

$$y_2 = \frac{1}{(1-x)} \int \frac{\cancel{(1-x)^{-2}}\,(x)^{-1}}{\cancel{(1-x)^{-2}}}\,dx$$

$$y_2 = \frac{1}{1-x} \int \frac{1}{x}\,dx$$

$$\boxed{y_2 = \frac{\ln(x)}{1-x}}$$

P.9

$$2x(x-1)y'' - (x+1)y' + y = 0$$

$$y'' \underbrace{- \frac{(x+1)}{2x(x-1)}}_{b(x)} y' + \underbrace{\frac{y\,x}{2x^2(x-1)}}_{c(x)} = 0$$

$$b(x) = -\frac{(x+1)}{2(x-1)}, \quad c(x) = \frac{x}{2(x-1)}$$

$$\boxed{b(0) = b_0 = \frac{1}{2}} \qquad \boxed{c(0) = c_0 = 0}$$

$$r(r-1) + b_0 r + c_0 = 0$$

$$r^2 - r + \frac{1}{2} r = 0$$

$$r^2 - \frac{1}{2} r = 0$$

$$r\left(r - \frac{1}{2}\right) = 0$$

$$\boxed{r = 0, \frac{1}{2}}$$

$$y_n = x^{r_n} \sum_{m=0}^{\infty} a_m x^m$$

$$\boxed{\begin{aligned} y_1 &= \sum_{m=0}^{\infty} a_m x^m \\ y_2 &= x^{1/2} \sum_{m=0}^{\infty} A_m x^m \end{aligned}}$$

for y_1:

$$O(x^0): \; -a_1(1) + a_0 = 0$$

$$a_0 = a_1$$

$$O(x^1): \; 2a_2(2)(1) - \cancel{a_1(1)}$$
$$-a_2(2) + \cancel{a_1} = 0$$

$$a_2 = 0$$

$$O(x^m): \; 2a_m(m)(m-1)$$

$$-2a_{m+1}(m+1)(m)$$
$$+\,a_{m+1}(m+1) = 0$$

$$a_{m+1}\Big((m+1) - 2(m+1)(m)\Big)$$
$$= -2a_m(m)(m-1)$$

$$a_{m+1} = \frac{a_m(-2)(m)(m-1)}{(m+1) - 2(m+1)(m)}$$

$$a_{m+1} = a_m\gamma$$

$$a_0 = a_1,\ a_2 = 0$$
$$\Rightarrow a_3 = 0$$
$$\Rightarrow a_4 = 0$$
$$\Rightarrow$$

$$a_0 = a_1$$

and $a_m = 0$

$m \geq 2$

$$y_1 = a_0 + a_1 x$$

$$\boxed{y_1 = a_0(1+x)}$$

$$2x(x-1)y'' - (x+1)y' + y = 0$$

$$2x(x-1)(0) - \underbrace{(x+1)(1)}_{} + \underbrace{(1+x)}_{} = 0$$

$$0 = 0 \checkmark$$

$$y_2 = x^{1/2} \sum_{m=0}^{\infty} A_m x^m$$

or

Reduction of Order

$$\Rightarrow \boxed{y_2 = x^{1/2}}$$