Chapter 4

Section 5

Nonlinear autonomous DE systems

$$\underline{\vec{y}\,' = \vec{f}(\vec{y})}$$

$$\boxed{\begin{aligned} y_1' &= 10y_1 - 5\underline{y_1 y_2} \\ y_2' &= 3y_2 + \underline{y_1 y_2} - 3\underline{y_2^2} \end{aligned}}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

Note $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$\boxed{y_1 = 0, \quad y_2 = 0}$$

$$0 = 10y_1 - 5y_1y_2 \quad ①$$

$$0 = 3y_2 + y_1y_2 - 3y_2^2 \quad ②$$

$$① \Rightarrow 0 = 5y_1(2 - y_2)$$

$$\boxed{y_1 = 0} \text{ or } \boxed{y_2 = 2}$$

if $y_1 = 0$

$$0 = 3y_2 - 3y_2^2$$

$$= 3y_2(1 - y_2)$$

$$\Rightarrow y_2 = 0 \text{ or } y_2 = 1$$

if $y_2 = 2$

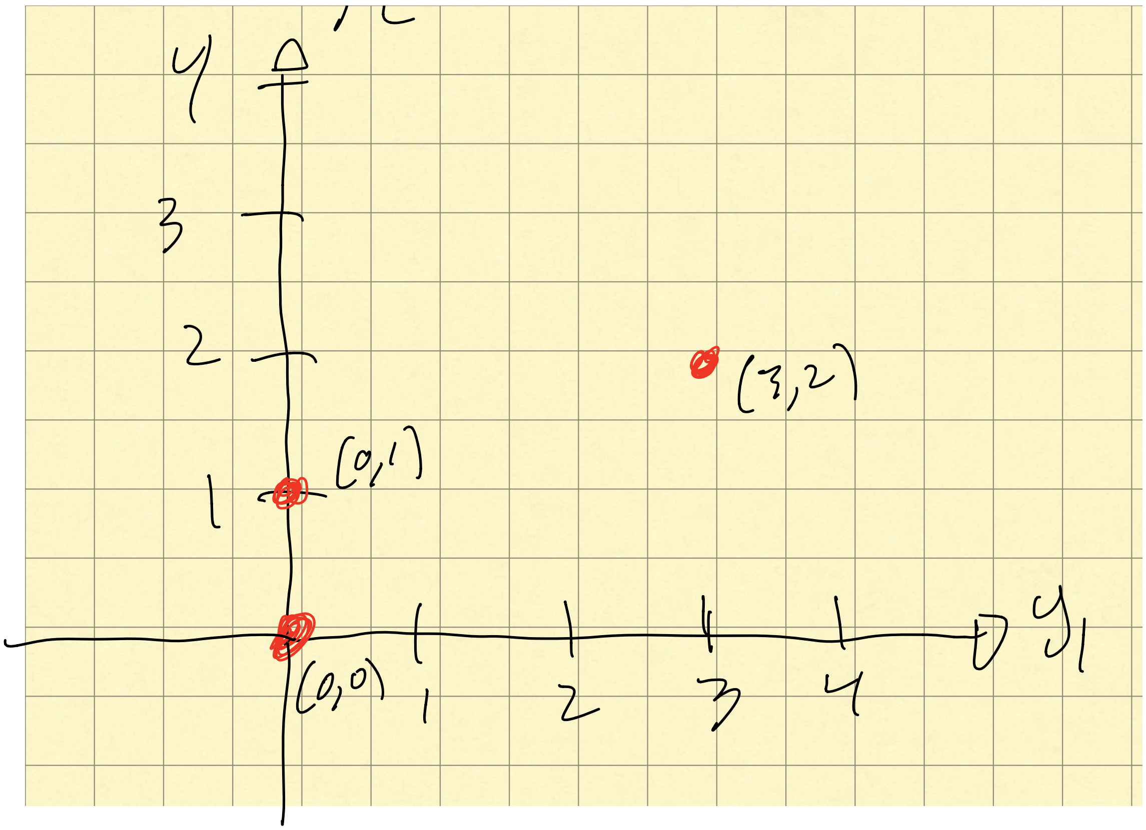
$$0 = 6 + 2y_1 - 12$$

$$\Rightarrow y_1 = 3$$

$$(y_1, y_2) = (0,0), (0,1),$$

$$(3,2)$$

y_2



Qualitative analysis tools

$$\vec{y}\,' = A\vec{y}$$

eigenvalues and eigenvector of A determined our system.

$$\Big(\quad \Big)' \quad \Big(a \quad b \Big) \Big(y_1 \Big)$$

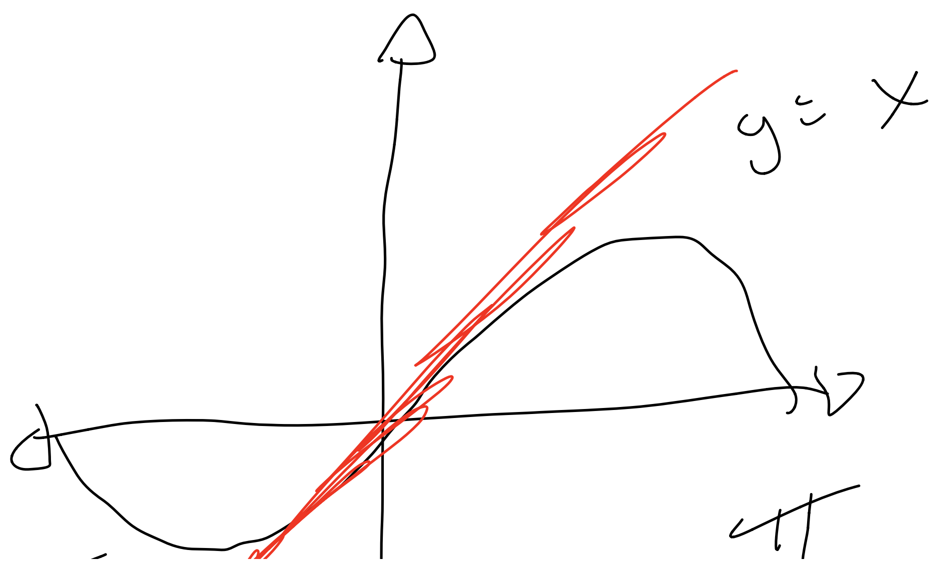
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} & \\ c & d \end{pmatrix} \begin{pmatrix} \\ y_2 \end{pmatrix}$$

Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)\big|_a}{n!} (x-a)^n$$

centered at a

if $f(x) = \sin(x)$



$$\sin(x) = \cancel{0} + x - \cancel{0 \cdot x^2} - \frac{x^3}{3!} + \cancel{0 \cdot x^4} + \frac{x^5}{5!} + \dots$$

$$\sin(x) \approx x$$

@ $x=0$

$$\boxed{\begin{aligned} y_1' &= 10y_1 - 5\underline{y_1y_2} \\ y_2' &= 3y_2 + \underline{y_1y_2} - 3\underline{y_2^2} \end{aligned}}$$

$\underline{\vec{y}' = f(\vec{y})}$

taylor expand $\vec{f}(\vec{y})$ at critical points

$$\vec{f}(\vec{y}) = \underbrace{\vec{f}(\vec{a})}_{0} + \underbrace{(\vec{y}-\vec{a})\overset{(n\times n)}{J(\vec{f})}}_{\text{linear term}}\Big|_{\vec{y}=\vec{a}} \cancel{+ \text{Hessian term}} \ldots$$

centred at $\vec{a}$

$\vec{0}$ because it is a critical point!

$\vec{y} = (y_1, y_2, \ldots y_n)$

$$J(\vec{f}) = \begin{bmatrix} \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} \\ \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{bmatrix}$$

$$\vec{f} = \begin{pmatrix} h \\ g \end{pmatrix}$$

$$\vec{y}' = J\big|_{\vec{x}=a}(\vec{y}-\vec{a})$$

$$J\big|_{\vec{x}=a} = A$$

$$\vec{y}' = A(\vec{y}-\vec{a})$$

$$\begin{cases} y_1^* = y_1 - a_1 \\ y_2^* = y_2 - a_2 \end{cases}$$

$$\Rightarrow (\vec{y}^*)' = A\vec{y}^*$$

drop $*$

$$\Rightarrow \boxed{\vec{y}' = A\vec{y}}$$

$$y_1' = y_1^2 - y_1 y_2 - y_2 = h$$

$$y_2' = \sin(y_2) e^{y_1} = g$$

$$J(\vec{f}) = \begin{bmatrix} \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} \\ \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{bmatrix}$$

$$\frac{\partial h}{\partial y_1} = \frac{\partial}{\partial y_1}\left(y_1^2 - y_1 y_2 - y_2\right)$$

$$= 2y_1 - y_2 + 0$$

$$\frac{\partial h}{\partial y_2} = \frac{\partial}{\partial y_2}\left(y_1^2 - y_1 y_2 - y_2\right)$$

$$= 0 - y_1 - 1$$

$$\frac{\partial g}{\partial y_1} = \frac{\partial}{\partial y_1}\left(\sin(y_2)\, e^{y_1}\right)$$

$$= \sin(y_2)\, e^{y_1}$$

$$\frac{\partial g}{\partial y_2} = \frac{\partial}{\partial y_2}\left(\sin(y_2)\, e^{y_1}\right)$$

$$= \cos(y_2)\, e^{y_1}$$

$$T(\mathring{f}) = \begin{bmatrix} 2y_1 - y_2 & -y_1 - 1 \end{bmatrix}$$

$$\begin{bmatrix} \vdots & \vdots \\ \sin(y_2)e^{y_1} & \cos(y_2)e^{y_1} \end{bmatrix}$$

Back to our first Problem

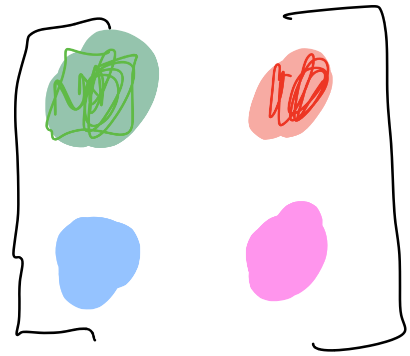
$$y_1' = 10y_1 - 5y_1y_2$$
$$y_2' = 3y_2 + y_1y_2 - 3y_2^2$$

$$(y_1, y_2) = (0,0),\ (0,1),$$
$$(3,2)$$

$$h(\vec{y}) = 10y_1 - 5y_1y_2$$

$$g(\vec{y}) = 3y_2 + y_1 y_2 - 3y_2^2$$

$$J(h(\vec{y}), g(\vec{y})) =$$



$$\frac{\partial}{\partial y_1}\left(10y_1 - 5y_1 y_2\right)$$

$$= \boxed{10 - 5y_2}$$

$$\frac{\partial}{\partial y_2}\left(10y_1 - 5y_1 y_2\right)$$

$$= \boxed{-5y_1}$$

$$\frac{\partial}{\partial y_1}\left(3y_2 + y_1 y_2 - 3y_2^2\right)$$

$$= y_2$$

$$\frac{\partial}{\partial y_2}\left(3y_2 + y_1 y_2 - 3y_2^2\right)$$

$$= 3 + y_1 - 6y_2$$

$$J(\vec{f}) = \begin{bmatrix} 10 - 5y_2 & -5y_1 \\ y_2 & 3 + y_1 - 6y_2 \end{bmatrix}$$

$(y_1, y_2) = (0,0), (0,1),$
$(3,2)$

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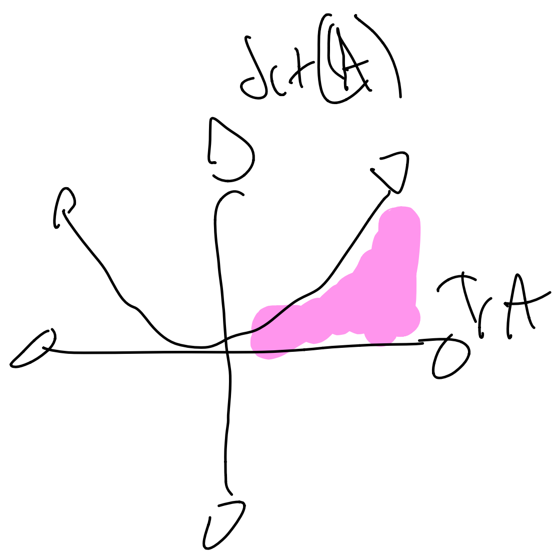
at $(0,0)$?

$$\begin{bmatrix} 10 & 0 \\ 0 & 3 \end{bmatrix} = A$$

$(\vec{y}\,' = A\vec{y})$

$\det(A) = 30$



$Tr(A) = 13$

$\Delta = (Tr A)^2 - 4 \det(A)$

$= + > 0$

$\boxed{\text{Source at } (0,0)}$

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at $(0,1)$?

$$\begin{bmatrix} 10 - 5y_2 & -5y_1 \\ y_2 & 3 + y_1 - 6y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 1 & -3 \end{bmatrix}$$

$\det(A) = -16 \Rightarrow$ saddle

$Tr(A) = 2$

$$\boxed{\text{saddle at } (0,1)}$$

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at $(3,2)$?

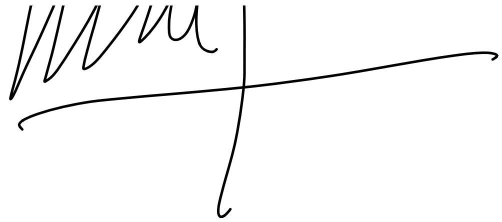
$$\begin{bmatrix} 10-5y_2 & -5y_1 \\ y_2 & 3+y_1-6y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -15 \\ 2 & -6 \end{bmatrix}$$

$\det(A) = 30$

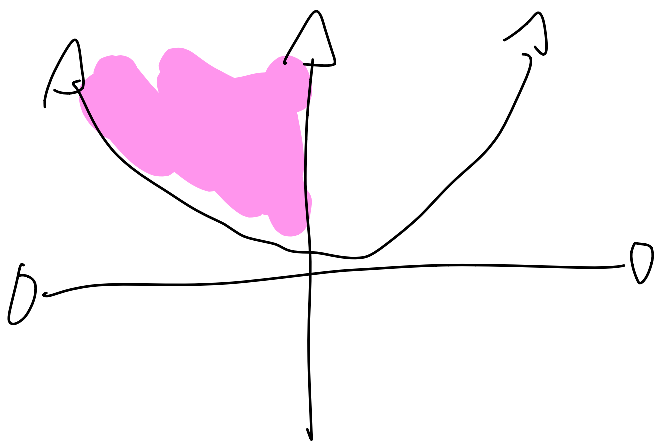
$\mathrm{Tr}(A) = -6$

[illegible]



$$\Delta = \operatorname{Tr}(A)^2 - 4\det(A)$$

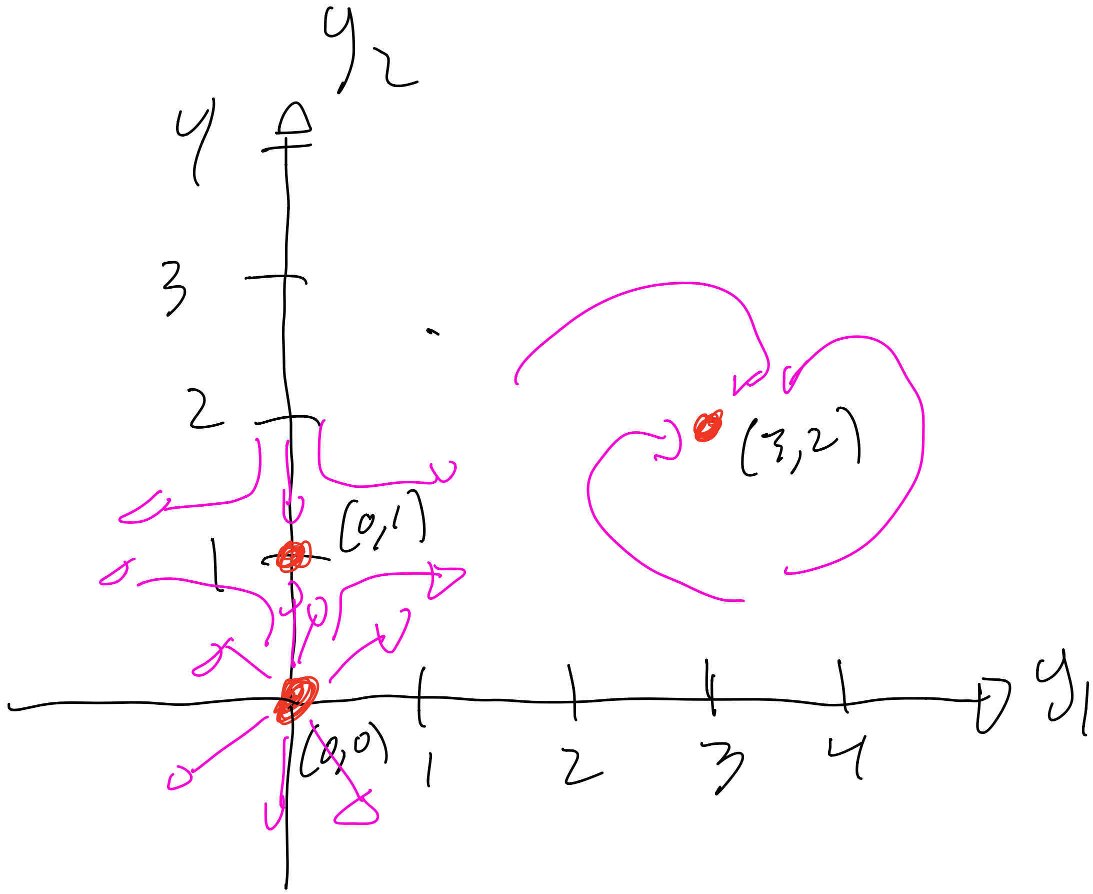
$$= - < 0$$



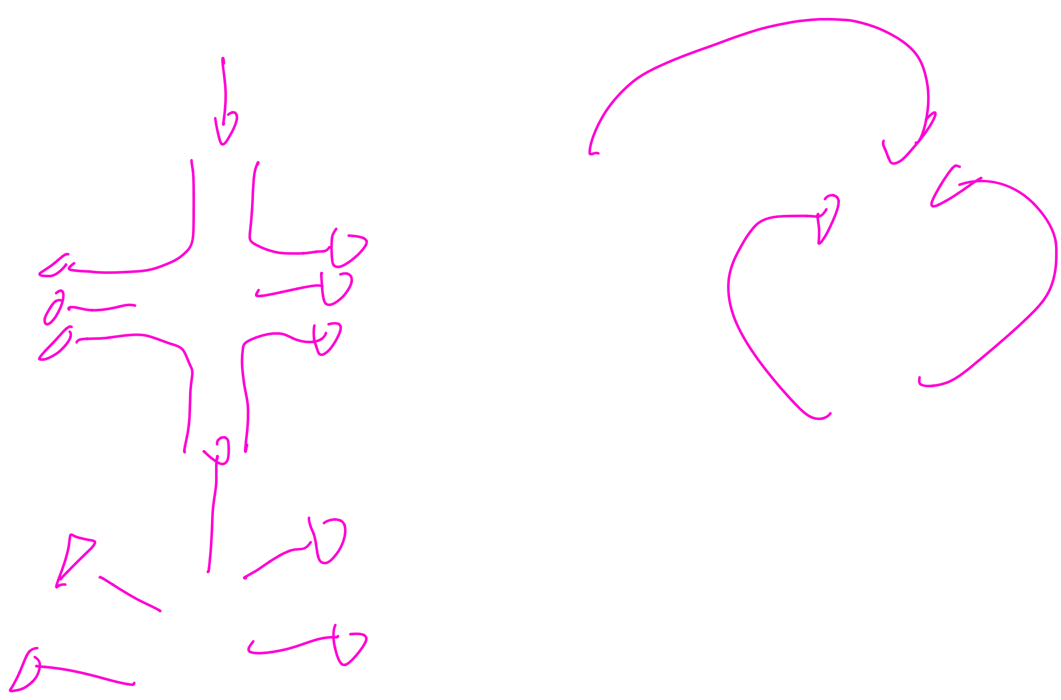
Spiral sink at $(3,2)$

$\vec{y}' = \vec{f}(\vec{y})$

- first find critical points (where $y_1' = 0$ and $y_2' = 0$)
- find Jacobinan of $\vec{f}$
- Evulate $J(\vec{f})$ at critical points to form A matrix
- look at $Tr(A)$ and $det(A)$ in conjunction with Poincare Diagram to classify them
- Graph using info



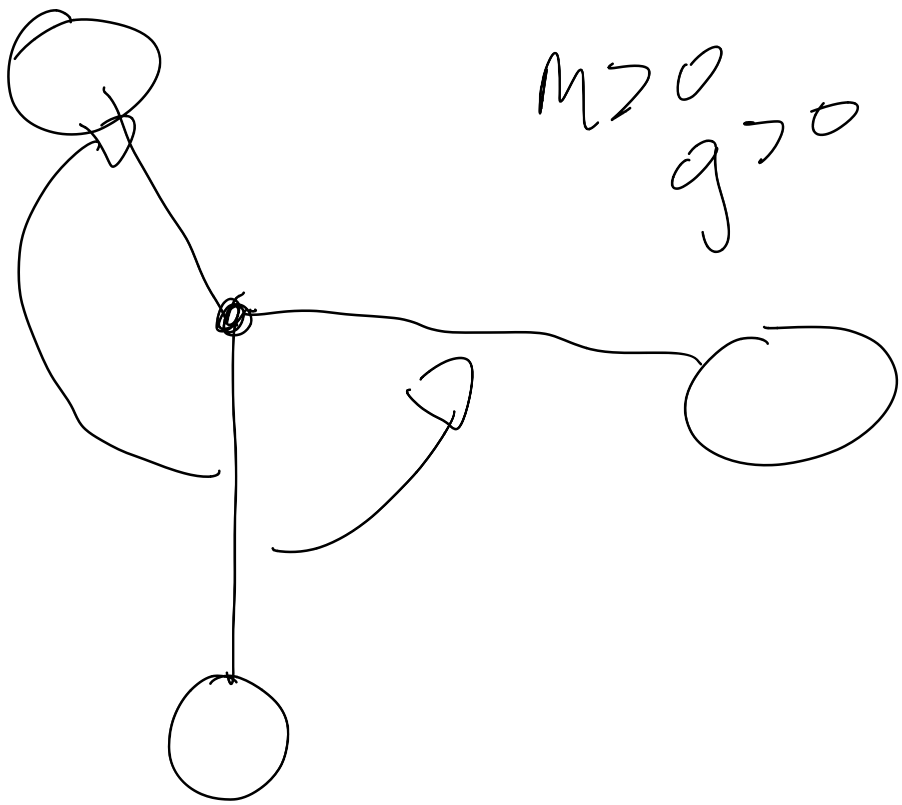
y2
4
3
2
1
(0,1)
(3,2)
(0,0)
1
2
3
4
y1

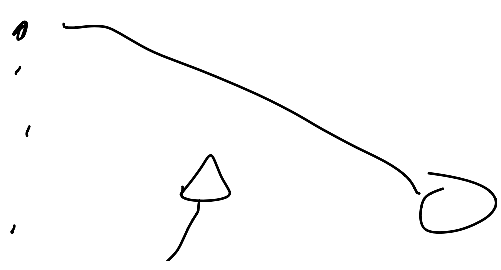


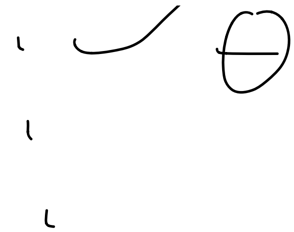
Ex: $\boxed{mL\ddot{\theta} + mg\sin\theta = 0}$

$m > 0$

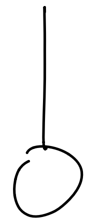
$g > 0$



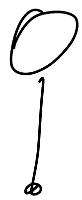




$\theta = 0$



$\theta = \pi$



$$\ddot{\theta} + \cancel{\frac{g}{l}} \sin\theta = 0$$

K

$$\ddot{\theta} + K \sin\theta = 0$$

$K > 0$

$$\theta_1 = \theta$$

$$\theta_2 = \dot{\theta}$$

$$\dot{\vec{\theta}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \theta_2 \\ -k \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \theta_2 \\ -k\sin(\theta_1) \end{bmatrix}$$

$$\dot{\vec{\theta}} = \begin{bmatrix} \theta_2 \\ -k\sin(\theta_1) \end{bmatrix}$$

$$\theta_2 = 0$$

$$-k\sin(\theta_1) = 0$$

$$\Rightarrow \sin(\theta_1) = 0$$

$$\theta_1 = \pm n\pi \quad n \in \mathbb{N}$$

Critical points are at $(0, \pm n\pi)$