Hmwk 17

1a $\quad y'' - 2y' + (\lambda+1)y = 0, \quad y(0)=0, \quad y(1) = 0$

$$x^2 - 2x + (\lambda+1) = 0$$

(characteristic polynomial)

$$\Rightarrow \frac{2 \pm \sqrt{4 - 4(\lambda+1)}}{2} = x_{1,2}$$

$$= 1 \pm \sqrt{1-(\lambda+1)}$$

$$= 1 \pm \sqrt{-\lambda}$$

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case 1 : $\lambda = 0$

$$\Rightarrow x_1, x_2 = 1$$

$$\Rightarrow y = c_1 e^x + c_2 x e^x$$

$$y(0)=0 \Rightarrow c_1 = 0$$

$y(0)$

$$y(1)=0 \Rightarrow 0 = C_2 e^{1}$$

$$\Rightarrow C_2 = 0$$

trivial soln!

$$\lambda < 0 \Rightarrow \lambda = -\alpha^2 \quad \alpha > 0$$

$$\Rightarrow \quad x_{1,2} = 1 \pm \sqrt{\alpha^2}$$

$$\Rightarrow \quad x_{1,2} = 1 \pm \alpha$$

$$y = C_1 e^{x(1+\alpha)} + C_2 e^{x(1-\alpha)}$$

$$y(0)=0 \Rightarrow C_1 = -C_2$$

$$y(1)=0$$

$$\Rightarrow 0 = C_1\left(e^{(1+\alpha)} - e^{(1-\alpha)}\right)$$

$$\underbrace{\qquad\qquad\qquad}_{\neq 0 \text{ for } \alpha > 0}$$

$\Rightarrow c_1 = 0 \Rightarrow c_2 = 0$

trivial soln!

$\lambda > 0 \Rightarrow \lambda = \alpha^2 \quad \alpha > 0$

$\Rightarrow \quad x_{1,2} = 1 \pm \sqrt{-\alpha^2}$

$\Rightarrow \quad x_{1,2} = 1 \pm i\alpha$

$y = e^x \left(c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \right)$

$y(0) = 0 \Rightarrow c_1 = 0$

$y(1) = 0 \Rightarrow 0 = e^1 c_2 \sin(\alpha)$

true if $\alpha = n\pi \quad n \in \mathbb{Z}$

$\Rightarrow \alpha^2 = (n\pi)^2 = \lambda$

$\Rightarrow \alpha = (n\pi)$

eigenvalues $\lambda = (n\pi)^2$

eigenfunctions $y = c_2 e^x \sin(n\pi x)$

or $\operatorname{span}\{e^x \sin(n\pi x)\}$

1b (using problem 6)

$$y'' + fy' + (g + \lambda h)y = 0$$

$$y'' - 2y' + (\lambda + 1)y = 0$$

$$\Rightarrow f = -2$$

$$g = 1$$

$h = 1$

$$p(x) = e^{\int -2\,dx} = e^{-2x}$$

$$q(x) = p(x) g = e^{-2x}$$

$$r(x) = h \cdot p(x) = e^{-2x}$$

$\Rightarrow$ S.L. form:

$$\boxed{\left[e^{-2x} y'\right]' + \left[e^{-2x} + x e^{-2x}\right] y = 0}$$

(just to check)

$$= -2\cancel{e^{-2x}} y' + \cancel{e^{-2x}} y'' + \left(\cancel{e^{-2x}} + x \cancel{e^{-2x}}\right) y = 0$$

$$= y'' - 2y' + (1+x) y = 0$$

$$\int_0^1 e^{-2x} e^{x} \sin(n\pi x) e^{x} \sin(m\pi x)\, dx$$

$$= \int_0^1 \sin(n\pi x) \sin(m\pi x)\, dx$$

(various ways to show this)

$$= \frac{1}{2} \int_0^1 \cos((m-n)\pi)\, dx$$

$$- \frac{1}{2} \int_0^1 \cos((m+n)\pi)\, dx$$

$$= 0 \quad m \neq n$$

$$= 1 \quad m = n$$

2a) $a_m = \frac{2m+1}{2} \int_{-1}^{1} \cos(\pi x) P_m(x)\, dx$ (pg 505)

$$a_0 = \frac{1}{2} \int_{-1}^{1} \cos(\pi x) P_0(x)\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \cos(\pi x)\, 1\, dx$$

(mathematica)

$$= \frac{1}{2}(0) = \boxed{0}$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} \cos(\pi x) P_1(x)\, dx$$

$$= \frac{3}{2} \int^{1} \cos(\pi x)\, x\, dx$$

$$2 \int_{-1} \cos(\ldots$$ (mathematica)

$$= \frac{3}{2}(0) = 0$$

$$a_2 = \frac{5}{2}\int_{-1}^{1} \cos(\pi x) P_2(x)\, dx$$

$$= \frac{5}{2}\int_{-1}^{1} \cos(\pi x)\left(\frac{1}{2}(-1+3x^2)\right) dx$$ (mathematica)

$$= \left(-\frac{6}{\pi^2}\right)\frac{5}{2} = \frac{-15}{\pi^2}$$

$$a_3 = \frac{7}{2}\int_{-1}^{1} \cos(\pi x) P_3(x)\, dx$$

$$\ldots \left(\frac{1}{2}(-3x+5x^3)\right) dx$$

$$= \frac{7}{2}\int_{-1}^{1} \cos(\pi x)\,\left(\tfrac{1}{2}(\text{[illegible]})\right)$$

(muthcmatica)

$$= \left(\frac{7}{2}\right)\cdot 0 = 0$$

$$a_4 = \frac{9}{2}\int_{-1}^{1} \cos(\pi x)\, P_4(x)\, dx$$

$$a_3 = \frac{7}{2}\int_{-1}^{1} \cos(\pi x) P_3(x)\, dx$$

$$= \frac{7}{2}\int_{-1}^{1} \cos(\pi x)\left(\tfrac{1}{2}(-3x+5x^3)\right) dx$$

(muthcmatica)

$$= \left(\frac{7}{2}\right)\cdot 0 = 0$$

$$= \frac{9}{2}\int_{-1}^{1} \cos(\pi x)\,\frac{1}{8}\left(3-30x^2+35x^4\right) dx$$

(muthcmatica)

$$= \left(\frac{9}{2}\right)\cdot\left(\frac{210-20\pi^2}{\pi^4}\right)$$

$$\boxed{\cos(\pi x) \approx \underbrace{\frac{-15}{\pi^2} P_1(x) + \frac{9}{2}\left(\frac{210-20\pi^2}{\pi^4}\right) P_4(x)}_{A(x)}}$$

2b. $A(1) = -0.937371$

$\cos(\pi) = -1$

difference is $\approx -.063$