Hmwk 14

1a) $xy'' + y = 0$

(standard form)

$$y'' + \frac{y}{x} = 0$$

$$b(x) = 0 \Rightarrow b(0) = b_0 = 0$$

$$\frac{c(x)}{x^2} = \frac{1}{x} \Rightarrow c(x) = x$$

$$c(0) = c_0 = 0$$

$$\Rightarrow r(r-1) = 0$$

$$\Rightarrow r = 0, 1$$

$$(r_1 - r_2) > 0 \Rightarrow \boxed{r_1 = 1, \quad r_2 = 0}$$

$$y_1(x) = x\sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+1}$$

$$\Rightarrow y_1''(x) = \sum_{m=1}^{\infty} a_m (m+1)(m) x^{m-1}$$

$$\Rightarrow x y_1'' = \sum_{m=1}^{\infty} a_m (m+1) m x^m$$

$$\sum_{m=1}^{\infty} a_m (m+1) m x^m + \sum_{m=0}^{\infty} a_m x^{m+1} = 0$$

$O(x^0)$: $0=0$, $a_0 = a_0$

$O(x^1)$: $a_1(2)(1) + a_0 = 0$

$$a_1 = \frac{-a_0}{2}$$

$$O(x^2): \quad a_2(3)(2) + a_1 = 0$$

$$a_2 = \frac{-a_1}{6} = \frac{a_0}{12}$$

$$O(x^3): \quad a_3(4)(3) + a_2 = 0$$

$$a_3 = \frac{-a_2}{(4\cdot 3)} = \frac{-a_0}{(12^2)}$$

$$O(x^4): \quad a_4(5)(4) + a_3 = 0$$

$$a_4 = \frac{-a_3}{(5\cdot 4)} = \frac{a_0}{20\cdot 12^2}$$

$$O(x^5): \quad a_5(6)(5) + a_4 = 0,$$

$$a_5 = -a_4 \quad = \quad -a_0$$

$$a_5 = \frac{\ }{(6\cdot 5)} = \boxed{\frac{\ }{30\cdot 20\cdot 12^2}}$$

$$O(x^6): a_6 (7)(6) + a_5 = 0$$

$$a_6 = \frac{-a_5}{7\cdot 6} = \boxed{\frac{a_0}{42\cdot 30\cdot 20\cdot 12^2}}$$

You could also notice pattern

$$\boxed{a_m = \frac{a_0 (-1)^m}{(m+1)!\, m!}}$$

$$\Rightarrow \boxed{y_1(x) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!\, m!} x^m}$$

or you could write up to $O(x^6)$

$$y_2(x) = k\, y_1(x) \ln(x) + (A_0 + A_1 x + A_2 x^2 + \dots)$$

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1 b) $xy'' + (1-x)y' - y = 0$

$$y'' + \frac{(1-x)}{x}y' - \frac{y}{x} = 0$$

$$\frac{b(x)}{x} = \frac{(1-x)}{x} \Rightarrow b(x) = (1-x)$$

$$b(0) = b_0 = 1$$

$$\frac{c(x)}{x^2} = \frac{-1}{x} \Rightarrow c(x) = -x$$

$$c(0) = c_0 = 0$$

$$r(r-1) + r = 0$$

$$\Rightarrow r^2 - r + r = 0$$

$$\Rightarrow \boxed{r_{1,2} = 0, 0}$$

$$y_1(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$\Rightarrow \quad y_1'(x) = \sum_{m=1}^{\infty} a_m (m) x^{m-1}$$

$$\Rightarrow \quad y_1''(x) = \sum_{m=2}^{\infty} a_m (m)(m-1) x^{m-2}$$

$$\Rightarrow \quad x y_1'' + (1-x) y_1' - y_1 = 0$$

$$\sum_{m=2}^{\infty} a_m (m)(m-1) x^{m-1} + \sum_{m=1}^{\infty} a_m (m)\left(x^{m-1} - x^m\right)$$

$$- \sum_{m=0}^{\infty} a_m x^m = 0$$

$$-\sum_{m=0} a_m x \quad - \quad$$

$$a_0 = a_0$$

$$\theta(x^0): \; a_1(1) - a_0 = 0$$

$$a_1 = a_0$$

$$\theta(x^1): a_2(2)(1) + a_2(2) - a_1(1)$$

$$-a_1 = 0$$

$$\Rightarrow 4a_2 = 2a_0$$

$$\Rightarrow a_2 = \frac{a_0}{2}$$

$$\theta(x^m): a_m(m)(m-1) + a_m(m) - a_{m-1}(m-1)$$

$$- a_{m-1}$$

$$\Rightarrow a_m m^2 - a_{m-1}(m) = 0$$

$$\Rightarrow a_m = \frac{a_{m-1}}{}$$

m

$$\Rightarrow \quad a_3 = \frac{a_2}{3} = \frac{a_0}{2} = \frac{a_0}{6}$$

$$\Rightarrow \quad a_4 = \frac{a_3}{4} = \frac{a_0}{6} = \frac{a_0}{24}$$

$\vdots$ (noticing the pattern!)

$$a_m = \frac{a_0}{m!}$$

$$\boxed{y_1(x) = a_0 \sum_{m=0}^{\infty} \frac{x^m}{m!} = a_0 e^x}$$

$$\boxed{y_2(x) = y_1(x) \ln(x) + \left(A_1 x + A_2 x^2 + \dots\right)}$$

from above

Using table
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