Hmwk 12

1a) $$\int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau$$

$$= \int_0^t e^{bt} e^{\tau(a-b)}\, d\tau$$

$$= e^{bt} \int_0^t e^{\tau(a-b)}\, d\tau$$

$$= e^{bt}\left(\frac{e^{t(a-b)} - 1}{a-b}\right)$$

$$= \boxed{\frac{e^{at} - e^{bt}}{a-b}}$$

$$\mathcal{L}\left(\frac{\quad}{s(s+3)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\right) * \mathcal{L}\left(\frac{9}{s+3}\right)$$

$$= 9\, e^{-3t} * 1$$

$$= 9\int_0^t e^{-3\tau}\, d\tau$$

$$= 9\, \frac{e^{-3\tau}}{-3}\Bigg|_0^t = \boxed{3-3e^{-3t}}$$

1b) $\mathcal{L}\left(1\, e^{-t}\cos(t)\right)$

$$\mathcal{L}\left(t\,\underbrace{e^{-t}\cos t}_{f(t)}\right)$$

$$= -\mathcal{L}\left(\left(e^{-t}\cos(t)\right)'\right)$$

(s-shift)

$$= -\frac{d}{ds}\left(\frac{s+1}{(s+1)^2+1}\right)$$

$$= \boxed{\frac{-s(s+2)}{(s^2+2s+2)^2}}$$

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6. $\mathcal{L}\left(t^2\sin(3t)\right)$

$$= (-1)(-1)\frac{d^2}{ds^2}\mathcal{L}(\sin(3t))$$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

(mathematica)

$$= \boxed{3\left(\frac{8s^2}{(9+s^2)^3} - \frac{2}{(9+s^2)^2}\right)}$$

or ✓

$$= \boxed{\frac{18(s^2-3)}{(s^2+9)^3}}$$

$$= \ldots$$

2a) $S_1(0)=0$
$S_2(0)=0$

$$\frac{\partial S_1(t)}{\partial t} = \text{rate in} - \text{rate out}$$

$$= \frac{4\text{ oz}}{\text{min}} + \frac{S_2\text{ oz}}{5\text{ gals}}\frac{2\text{ gals}}{1\text{ min}}$$

$$- \frac{S_1\text{ oz}}{5\text{ gals}}\frac{2\text{ gals}}{1\text{ min}}$$

$$= -\frac{2}{5}S_1 + \frac{2}{5}S_2 + 4$$

similarily

$$\frac{\partial S_2(t)}{\partial t} = \frac{2}{5}S_1 - \frac{2}{5}S_2$$

$$\Rightarrow \begin{bmatrix} S_1 \end{bmatrix}' = \begin{bmatrix} -2/5 & 2/5 \end{bmatrix}\begin{bmatrix} S_1 \end{bmatrix} + \begin{bmatrix} 4 \end{bmatrix}$$

$$\begin{bmatrix} s_2 \end{bmatrix}' \quad \begin{bmatrix} \frac{2}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} s_2 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

1b) $\mathcal{L}(s_n') = s\tilde{s}_n - \cancel{s(0)}_{0}$

$$s\tilde{S}_1 = -\frac{2}{5}\tilde{s}_1 + \frac{2}{5}\tilde{s}_2 + \frac{4}{\tilde{s}_1}$$

$$\tilde{s}_2 = \frac{2}{5}\tilde{s}_1 - \frac{2}{5}\tilde{s}_2$$

$$\tilde{s}_1\left(s + \frac{2}{5}\right) - \tilde{s}_2\left(\frac{2}{5}\right) = \frac{4}{s}$$

$$\tilde{s}_1\left(-\frac{2}{5}\right) + \tilde{s}_2\left(s + \frac{2}{5}\right) = 0$$

$$\begin{bmatrix} s + \frac{2}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} \tilde{s}_1 \end{bmatrix} = \begin{bmatrix} \frac{4}{s} \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \cdots \\ -\frac{2}{5} & s+\frac{2}{5} \end{bmatrix} \begin{bmatrix} \cdots \\ \tilde{s}_2 \end{bmatrix} \quad \begin{bmatrix} \cdots \\ 0 \end{bmatrix}$$

1c) First solve for $\tilde{s}_1$ and $\tilde{s}_2$

(I used inverting the matrix)

$$A^{-1} = \frac{1}{5s^2+4s} \begin{pmatrix} (5s+2) & 2 \\ 2 & (5s+2) \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{bmatrix} = A^{-1} \begin{pmatrix} \frac{4}{s} \\ 0 \end{pmatrix}$$

$$\tilde{s}_1 = \frac{4(5s+2)}{s(5s^2+4s)}$$

$$\tilde{S}_2 = \frac{8}{s(5s^2+4s)}$$

Now solve using favorit technique

for example:

$$\mathcal{L}^{-1}\left(\frac{8}{5s^2\left(s+\frac{4}{5}\right)}\right)$$

$$= 2\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \frac{25}{2}\mathcal{L}^{-1}\left(\frac{1}{5s+4}\right)$$

$$- \frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right)$$

$$\left(-\frac{4t}{}\right)$$

$$\Rightarrow s_2 = 2t + \frac{5}{2}\left(e^{5} - 1\right)$$

(similar work ...)

$$\boxed{s_1 = 2t + \frac{5}{2}\left(1 - e^{-\frac{4}{5}t}\right)}$$

2d) $2t + \frac{5}{2}\left(e^{-4t} - 1\right) = 14$

(mathematica)

$$\boxed{t = 8.2483 \text{ minutes}}$$