HW9 Solution

1(a)
$$\begin{cases} \frac{dp_1}{dt} = a p_1 - b p_1 p_2 = (a - b p_2) p_1 \\ \frac{dp_2}{dt} = k p_1 p_2 - l p_2 = (k p_1 - l) p_2 \end{cases}$$

$a=5$, $b=3$, $k=3$, $l=6$

$$\Rightarrow \quad \frac{dp_1}{dt} = (5 - 3p_2) p_1$$
$$\frac{dp_2}{dt} = (3p_1 - 6) p_2$$

$$\Rightarrow \begin{cases} (5-3p_2) p_1 = 0 \\ (3p_1 - 6) p_2 = 0 \end{cases} \Rightarrow (p_1, p_2) = \boxed{(0,0) \text{ or } (2, \tfrac{5}{3})}$$

1(b) Let $f(p_1,p_2) = (5-3p_2)p_1$ + $g(p_1,p_2) = (3p_1 - 6)p_2$

then
$$J = \begin{bmatrix} \frac{\partial f}{\partial p_1} & \frac{\partial f}{\partial p_2} \\ \frac{\partial g}{\partial p_1} & \frac{\partial g}{\partial p_2} \end{bmatrix} = \begin{bmatrix} 5-3p_2 & -3p_1 \\ 3p_2 & 3p_1 - 6 \end{bmatrix}$$

i) At $p^* = (0,0)$

the linearized equation is
$$p' = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}' = \begin{bmatrix} \frac{dp_1}{dt} \\ \frac{dp_2}{dt} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

ii) At $p^* = (2, \frac{5}{3})$

The linearized equation is
$$p' = \begin{bmatrix} \frac{dp_1}{dt} \\ \frac{dp_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

1(c) i) $p^* = (0,0)$

let $A = \begin{bmatrix} 5 & 0 \\ 0 & -6 \end{bmatrix}$

$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 0 \\ 0 & -6-\lambda \end{vmatrix} = (5-\lambda)(-6-\lambda) = 0$

$\Rightarrow \lambda = 5$ or -6

$\Rightarrow$ saddle

ii) $p^* = (2, \frac{5}{3})$

let $A = \begin{bmatrix} 0 & -6 \\ 5 & 0 \end{bmatrix}$

$\det(A-\lambda I) = \begin{vmatrix} -\lambda & -6 \\ 5 & -\lambda \end{vmatrix} = \lambda^2 + 30 = 0$

$\Rightarrow \lambda = \pm\sqrt{30}\, i$

$\Rightarrow$ center

2(a)
$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1^2 + 3x_1 - x_1x_2 = x_1(-2x_1 + 3 - x_2) \\ \dfrac{dx_2}{dt} = -2x_2^2 + 3x_2 - x_1x_2 = x_2(-2x_2 + 3 - x_1) \end{cases}$$

$$\Rightarrow \begin{cases} x_1(-2x_1 + 3 - x_2) = 0 \\ x_2(-2x_2 + 3 - x_1) = 0 \end{cases}$$

$$\Rightarrow (x_1, x_2) = (0,0) \text{ or } (0, \tfrac{3}{2}) \text{ or } (\tfrac{3}{2}, 0) \text{ or } (1,1)$$

2(b) every point in $(x_1, x_2) \in (0,\infty) \times (0,\infty)$ converges
(c) $(1,1)$.