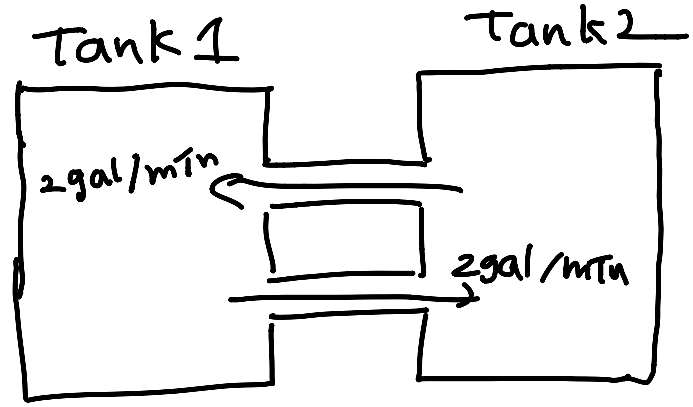
1



(a)

$$\begin{cases} \dfrac{dS_1(t)}{dt} = -\dfrac{2}{5}S_1 + \dfrac{2}{5}S_2 \\[2ex] \dfrac{dS_2(t)}{dt} = \dfrac{2}{5}S_1 - \dfrac{2}{5}S_2 \\[2ex] S(0) = \begin{bmatrix} S_1(0) \\ S_2(0) \end{bmatrix} = \begin{bmatrix} S_0 \\ 0 \end{bmatrix} \end{cases}$$

$$\Rightarrow \begin{cases} \begin{bmatrix} S_1' \\ S_2' \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \\[2ex] \begin{bmatrix} S_1(0) \\ S_2(0) \end{bmatrix} = \begin{bmatrix} S_0 \\ 0 \end{bmatrix} \end{cases}$$

Define $A := \begin{bmatrix} -\frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$

$\Rightarrow \det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} -\frac{2}{5}-\lambda & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5}-\lambda \end{vmatrix} = (\lambda+\frac{2}{5})^2 - \frac{4}{25}$$

$$= \lambda^2 + \frac{4}{5}\lambda = \lambda(\lambda + \frac{4}{5}) = 0$$

$\Rightarrow \lambda = 0$ or $-\frac{4}{5}$

i) $\lambda = 0$

$$\begin{bmatrix} -\frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ The corresponding eig. vec. is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

ii) $\lambda = -\frac{4}{5}$

$$\begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\not\Rightarrow$ The corresponding eig. vec. is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

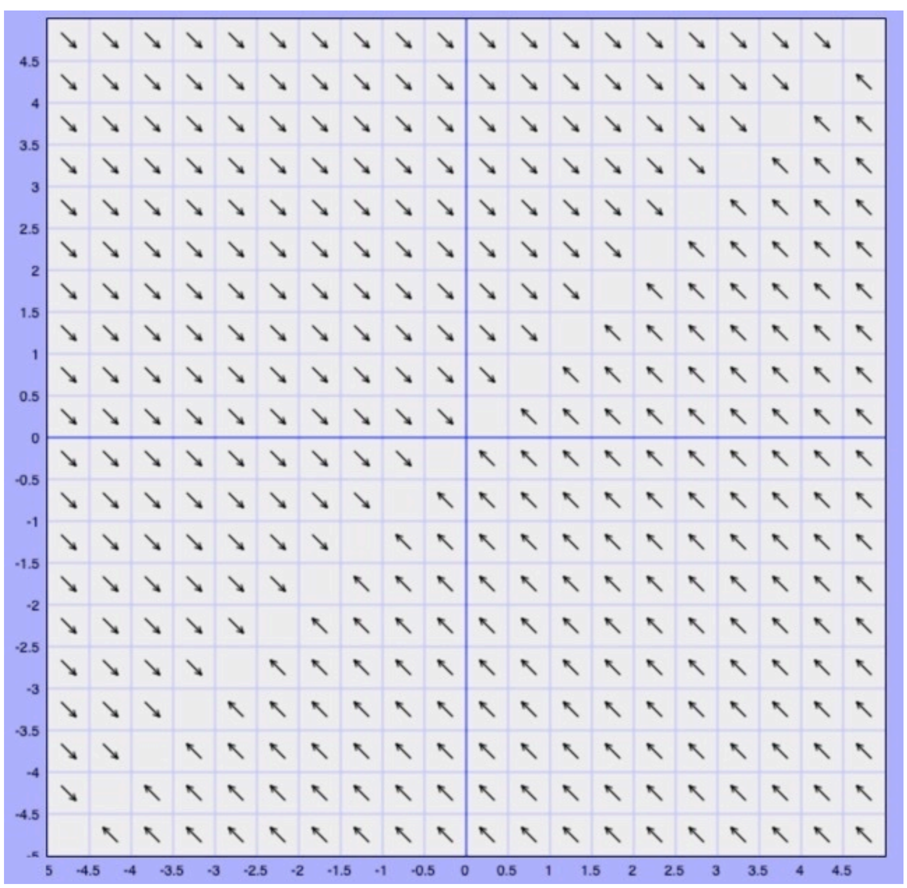
$$\Rightarrow S(t) = \begin{bmatrix} S_1(t) \\ S_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-\frac{4}{5}t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow S(0) = \begin{bmatrix} S_0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad c_1 = S_0/2 \qquad c_2 = S_0/2$$

$$\Rightarrow S(t) = \frac{S_0}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{S_0}{2} e^{-\frac{4}{5}t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

1(b)



1(c) $S_1(t) = S_2(t) = S_0/2$ as $t \to \infty$

1(d) $S_1(t) + S_2(t) = S_0$

We can easily verify (c) & (d) from the exact solution.

$$\ni \; S(t) = \frac{S_0}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{S_0}{2} e^{-\frac{4}{5}t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\Rightarrow \lim_{t\to\infty} S(t) = \frac{S_0}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\Longleftrightarrow \quad S_1(t) = S_2(t) = \frac{S_0}{2} \text{ as } t \to \infty$$

$$\Rightarrow \; S_1(t) + S_2(t) = \left(\frac{S_0}{2} + \frac{S_0}{2} e^{-\frac{4}{5}t}\right) + \left(\frac{S_0}{2} - \frac{S_0}{2} e^{-\frac{4}{5}t}\right) = S_0$$

2(a) $y'' + \gamma y' + y = 0$

Let $x = y'$

$\Rightarrow x' = y'' = -\gamma y' - y = -\gamma x - y$

$y' = x$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -\gamma x - y \\ x \end{bmatrix}$$

$$= \begin{bmatrix} -\gamma & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -\gamma & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $A = \begin{bmatrix} -\gamma & -1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow \operatorname{Det}(A) = 1$ & $\operatorname{tr}(A) = -\gamma$

$$\Rightarrow \lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}$$

① If $\gamma < -2$, $Re(\lambda) > 0, Im(\lambda) = 0 \Rightarrow$ source

② If $\gamma = -2$, $Re(\lambda) > 0$, $Im(\lambda) = 0 \Rightarrow$ source

③ If $0 > \gamma > -2$, $Re(\lambda) > 0$, $Im(\lambda) \neq 0$

$\Rightarrow$ spiral source

④ If $\gamma = 0$, $Re(\lambda) = 0$, $Im(\lambda) = 0 \Rightarrow$ center

⑤ If $0 < \gamma < 2$, $Re(\lambda) < 0$, $Im(\lambda) \neq 0$

$\Rightarrow$ spiral sink

⑥ If $\gamma = 2$, $Re(\lambda) > 0$, $Im(\lambda) = 0 \Rightarrow$ sink

⑦ If $\gamma > 2$, $Re(\lambda) < 0$, $Im(\lambda) = 0 \Rightarrow$ sink