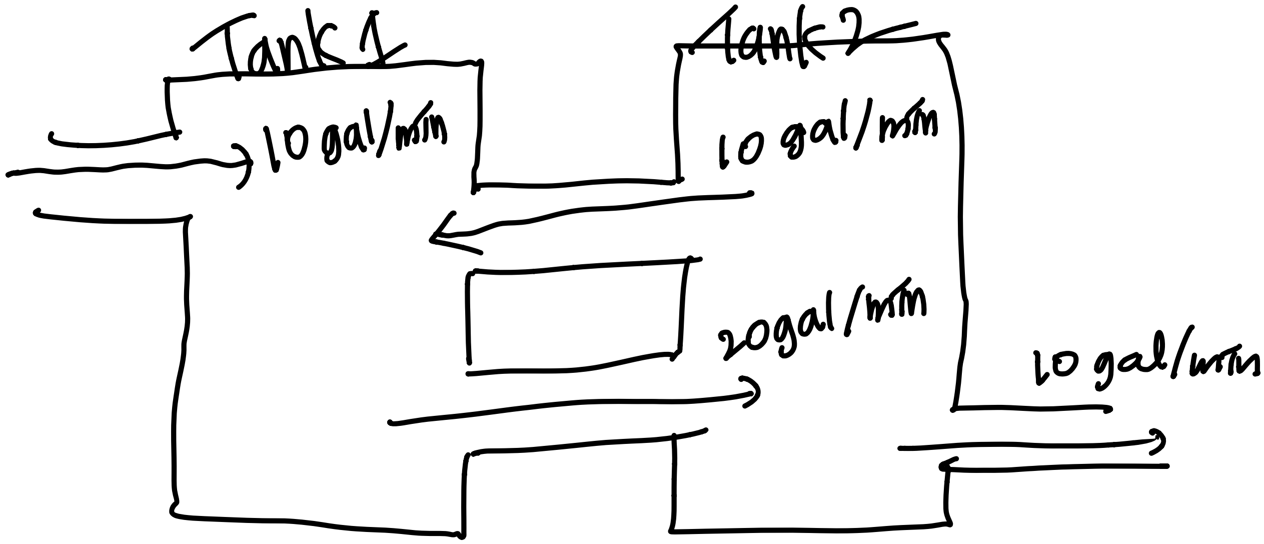

1(a)

$$\begin{cases} \dfrac{dS_1(t)}{dt} = \dfrac{1}{5}S_2(t) - \dfrac{2}{5}S_1(t) \\[2ex] \dfrac{dS_2(t)}{dt} = \dfrac{2}{5}S_1(t) - \dfrac{2}{5}S_2(t) \\[2ex] S_1(0) = 10 \ , \ S_2(0) = 0 \end{cases}$$

1(b)

$$\begin{cases} \begin{bmatrix} S_1' \\ S_2' \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{bmatrix}}_{A} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \\[2ex] \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \end{cases}$$

$$\det(A-\lambda I) = \begin{vmatrix} -\frac{2}{5}-\lambda & \frac{1}{5} \\ \frac{2}{5} & -\frac{2}{5}-\lambda \end{vmatrix}$$

$$= (\lambda+\tfrac{2}{5})^2 - \frac{2}{25}$$

$$= \lambda^2 + \frac{4}{5}\lambda + \frac{2}{25} = 0$$

$$\Rightarrow \quad \lambda = \frac{-\frac{4}{5} \pm \sqrt{\frac{16}{25} - \frac{8}{25}}}{2} = \frac{-\frac{4}{5} \pm \sqrt{\frac{8}{25}}}{2}$$

$$= \frac{-\frac{4}{5} \pm \frac{2}{5}\sqrt{2}}{2} = -\frac{2}{5} \pm \frac{\sqrt{2}}{5}$$

1) $\lambda_1 = -\frac{2}{5} + \frac{\sqrt{2}}{5}$

$$\begin{bmatrix} -\frac{2}{5}-\lambda_1 & \frac{1}{5} \\ \frac{2}{5} & -\frac{2}{5}-\lambda_1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{\sqrt{2}}{5} \end{bmatrix}$$

$\Rightarrow$ The corresponding eigenvector is

$$\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}.$$

(ii) $\lambda_2 = -\frac{2}{5} - \frac{\sqrt{2}}{5}$

$$\begin{bmatrix} -\frac{2}{5}-\lambda_2 & \frac{1}{5} \\ \frac{2}{5} & -\frac{2}{5}-\lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{\sqrt{2}}{5} \end{bmatrix}$$

$\Rightarrow$ The corresponding eigenvector is

$$\begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix}.$$

Thus, the solution $S(t) = \begin{bmatrix} S_1(t) \\ S_2(t) \end{bmatrix}$

is $$\underline{\underline{S(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix} e^{\lambda_2 t}}}$$ //

$$S(0) = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 - c_2 = 10 \\ c_1 + c_2 = 0 \end{cases} \Rightarrow \begin{array}{l} c_1 = 5 \\ \& \ c_2 = -5 \end{array}$$

$$\therefore \quad S(t) = 5\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{\lambda_1 t} - 5\begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix} e^{\lambda_2 t}$$

1(c)

$$\begin{cases} S_1(t) = 5e^{\lambda_1 t} + 5e^{\lambda_2 t} \\ S_2(t) = 5\sqrt{2}e^{\lambda_1 t} - 5\sqrt{2}e^{\lambda_2 t} \end{cases}$$

$\Rightarrow$ we want to find the time when $S_2 > S_1$.

$$\Rightarrow S_2(t) - S_1(t) = 5(\sqrt{2}-1)e^{\lambda_1 t} - 5(\sqrt{2}+1)e^{\lambda_2 t} > 0$$

$$\Rightarrow \quad e^{(\lambda_1 - \lambda_2)t} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\Rightarrow \quad (\lambda_1 - \lambda_2)t = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\Rightarrow \quad \frac{2\sqrt{2}}{5}t = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\Rightarrow \quad t = \frac{5\sqrt{2}}{4}\ln\left(3+2\sqrt{2}\right)$$