1(a) $\begin{cases} W = mg & (g = 32\, ft/s) \\ W = \gamma v \end{cases}$

$$\Rightarrow \begin{cases} m = \frac{W}{g} = \frac{640}{32} = 20 \left(lb\, s^2/ft\right) \\ \gamma = \frac{W}{v} = \frac{200}{10} = 20 \left(lb\, s^2/ft\right) \end{cases}$$

1(b) $mh'' + \gamma h' + kh = 0$

we have $m = 20\ lb\, s^2/ft$

$\gamma = 20\ lb\, s^2/ft$

$k = 20\ lbs/ft$

$\Rightarrow$ By the characteristic eqn, we get

$$m\lambda^2 + \gamma\lambda + k = 0$$

$$\Rightarrow \lambda^2 + \lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$\Rightarrow h_h(t) = e^{-\frac{1}{2}t}\left(c_1 \sin\left(\frac{\sqrt{3}}{2}t\right) + c_2 \cos\left(\frac{\sqrt{3}}{2}t\right)\right)$$

Thus, $\omega = \frac{\sqrt{3}}{2}$

1(c)

$$C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (\gamma\omega)^2}}$$

$$\Rightarrow \quad 50 = \frac{F_0}{\sqrt{(20 - 20 \cdot \frac{3}{4})^2 + (20 \cdot \frac{\sqrt{3}}{2})^2}}$$

$$= \frac{F_0}{\sqrt{(5)^2 + (10\sqrt{3})^2}}$$

$$= \frac{F_0}{\sqrt{325}}$$

$$\Rightarrow \quad F_0 = 50\sqrt{325} = 250\sqrt{13}$$

1(d) $\gamma=0$

$$\Rightarrow \begin{cases} mh''+kh = F_0 \cos(\omega t) \\ h(0)=100, \quad h'(0)=0 \end{cases}$$

I) Homogeneous

$$mh''+kh=0$$

$$\Rightarrow \quad \lambda^2+1=0 \quad \Rightarrow \quad \lambda = \pm i$$

$$\Rightarrow \quad h_h(t) = c_1 \underbrace{\cos t}_{h_1(t)} + c_2 \underbrace{\sin t}_{h_2(t)}$$

II) Particular

$$mh''+kh = 250\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$\overset{(m=k=20)}{\Rightarrow} \quad h''+h = \frac{25\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

By using the variation of parameters,

we get

$$h_p(t) = -h_1(t)\int \frac{h_2(t)\cdot \frac{25\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)}{W(h_1,h_2)}\,dt$$

$$+ h_2(t)\int \frac{h_1(t)\cdot \frac{25\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)}{W(h_1,h_2)}\,dt$$

$$\Rightarrow W(h_1, h_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$\Rightarrow h_p(t) = \underbrace{-\cos t \int \sin t \cdot \frac{25\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2}t\right) dt}_{①}$$

$$\underbrace{+ \sin t \int \cos t \cdot \frac{25\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2}t\right) dt}_{②}$$

$$① \Rightarrow \frac{25\sqrt{3}}{2} \int \underbrace{\sin t \cos\left(\frac{\sqrt{3}}{2}t\right)}_{①'} dt$$

$$①' \Rightarrow \sin t \cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$= \frac{1}{2}\left(2 \sin t \cos\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$= \frac{1}{2}\left(\sin t \cos\left(\frac{\sqrt{3}}{2}t\right) + \cos t \sin\left(\frac{\sqrt{3}}{2}t\right) + \sin t \cos\left(\frac{\sqrt{3}}{2}t\right) - \cos t \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$= \frac{1}{2}\left(\sin\left(t + \frac{\sqrt{3}}{2}t\right) + \sin\left(t - \frac{\sqrt{3}}{2}t\right)\right)$$

$$① \Rightarrow \frac{25\sqrt{3}}{4}\int \sin\left(\frac{2+\sqrt{3}}{2}t\right) + \sin\left(\frac{2-\sqrt{3}}{2}t\right)\, dt$$

$$= \frac{25\sqrt{3}}{4}\left[-\frac{2}{2+\sqrt{3}}\cos\left(\frac{2+\sqrt{3}}{2}t\right) - \frac{2}{2-\sqrt{3}}\cos\left(\frac{2-\sqrt{3}}{2}t\right)\right]$$

$$= \frac{25\sqrt{3}}{\cancel{4}_{\,2}}\left[-\cancel{2}(2-\sqrt{3})\cos\left(\frac{2+\sqrt{3}}{2}t\right) - \cancel{2}(2+\sqrt{3})\cos\left(\frac{2-\sqrt{3}}{2}t\right)\right]$$

$$= -\frac{25\sqrt{3}}{2}\left[(2-\sqrt{3})\cos\left(\frac{2+\sqrt{3}}{2}t\right) + (2+\sqrt{3})\cos\left(\frac{2-\sqrt{3}}{2}t\right)\right]$$

$$② \Rightarrow \frac{25\sqrt{3}}{2}\int \underbrace{\cos t \ \cos\left(\frac{\sqrt{3}}{2}t\right)}_{②'} dt$$

$$②' \Rightarrow \cos t + \cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$= \frac{1}{2}\left(\cos\left(t+\frac{\sqrt{3}}{2}t\right) + \cos\left(t-\frac{\sqrt{3}}{2}t\right)\right]$$

$$② \Rightarrow \frac{25\sqrt{3}}{4}\int \cos\left(t+\frac{\sqrt{3}}{2}t\right) + \cos\left(t-\frac{\sqrt{3}}{2}t\right) dt$$

$$= \frac{25\sqrt{3}}{4}\left[\frac{2}{2+\sqrt{3}}\sin\left(\frac{2+\sqrt{3}}{2}t\right) + \frac{2}{2-\sqrt{3}}\sin\left(\frac{2-\sqrt{3}}{2}t\right)\right]$$

$$= \frac{25\sqrt{3}}{2}\left((2-\sqrt{3})\sin\left(\frac{2+\sqrt{3}}{2}t\right)+(2+\sqrt{3})\sin\left(\frac{2-\sqrt{3}}{2}t\right)\right]$$

$$h_p = -\cos t \cdot -\frac{25\sqrt{3}}{2}\left[(2-\sqrt{3})\cos\left(\frac{2+\sqrt{3}}{2}t\right)+(2+\sqrt{3})\cos\left(\frac{2-\sqrt{3}}{2}t\right)\right]$$

$$+\sin t \cdot \frac{25\sqrt{3}}{2}\left[(2-\sqrt{3})\sin\left(\frac{2+\sqrt{3}}{2}t\right)+(2+\sqrt{3})\sin\left(\frac{2-\sqrt{3}}{2}t\right)\right]$$

$$= 50\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$\Rightarrow h(t) = h_n(t) + h_p(t)$$

$$\Rightarrow \begin{cases} h(0) = c_1 + 50\sqrt{3} = 100 & \Rightarrow c_1 = 100 - 50\sqrt{3} \\ h'(0) = c_2 = 0 & \Rightarrow c_2 = 0 \end{cases}$$

$$\Rightarrow h(t) = (100-50\sqrt{3})\cos t + 50\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right).$$

1(e) $h(t) = 0$

$$\Rightarrow (100-50\sqrt{3})\cos t + 50\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right) = 0$$

$$\Rightarrow t = 1.777$$

$$h'(t) = -(100-50\sqrt{3})\sin t - 75\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$h'(1.777) = -88.0756$$

$$\Rightarrow \underline{88.0756 \text{ ft/s}}$$

Input interpretation:

solve	$(100 - 50\sqrt{3})\cos(t) + 50\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right) = 0$

Solution over the reals:

$t \approx -12.5175$

Numerical solutions:

$t \approx \pm 19.8561632988019\ldots$

$t \approx \pm 16.1632360741797\ldots$

$t \approx \pm 8.91279342301427\ldots$

$t \approx \pm 5.33682005817900\ldots$

$t \approx \pm 1.77718638363192\ldots$

$t \approx 12.5174586916843\ldots$

Root plot: